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Application of Variational Iteration Method for Stefan Problem

Hossein Jafari [a,1], A. Golbabai[b], E. Salehpoor[c], Kh. Sayehvand[b]

[a]Department of Mathematics, University of Mazandaran, Babolsar, Iran
jafari_h@math.com

[b]Department of Mathematics, Iran University of Science and Technology, Tehran, Iran
golbabai@iust.ac.ir

[c] Department of Mathematics, Islamic Azad University, Ghaemshahr Branch, Iran
elham_salehpor61@yahoo.com

Abstract

In this paper, the variational iteration method (VIM) is employed to obtain approximate analytical solutions of the Stefan problem. The method is capable of reducing the size of calculation and easily overcomes the difficulty of the perturbation technique or Adomian polynomials.

Keywords: Variational iteration method; Stefan problem; Partial differential equation.

1 Introduction

The name Stefan problem encompasses a wide range of mathematical models describing thermal or diffusion processes which are characterized by a phase change in which the heat of the phase transition is emitted or absorbed,such as solidification of metals, freezing of the ground and water, freezing food, melting ice, crystal growth etc. The Stefan problem consists of temperature in the domain and the position of the moving interface(the freezing front)[5].

Nonlinear phenomena are of fundamental importance in various fields of science and engineering. Analytical methods commonly used to solve nonlinear equations are very restricted and numerical techniques involving discretization of the variables on the other hand gives rise to rounding off errors.

[1]Author for correspondence, jafari@umz.ac.ir, Tel:+98-1125342466

Recently introduced variational iteration method by He [1, 2, 3], which give rapidly convergent successive approximations of the exact solution if such a solution exists, has proved successful in deriving analytical solutions of linear and nonlinear differential Equation. This method is preferable over numerical methods as it is free from rounding off errors and neither requires large computer power/memory. The variational iteration method is a new approach searching for an analytical approximate solution of linear and nonlinear problems [1, 2, 3, 5].

In the present paper we employ VIM method for solving following equations. Consider the Stefan problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 \leq x \leq s(t), \quad t \geq 0 \tag{1}$$

with initial condition

$$\begin{aligned} &u(x,0) = -1, \quad t \geq 0 \\ &u(s(x),t) = 0, \quad t \geq 0 \\ &\frac{\partial u}{\partial x} = \frac{\partial s(x)}{\partial t}. \end{aligned}$$

2 He's variational iteration method

For the purpose of illustration of the methodology to the proposed method, using VIM, we begin by considering a differential equation in the formal form,

$$Lu + Nu = g(x,t), \tag{2}$$

where L is a linear operator, N a nonlinear operator and $g(x,t)$ is the source inhomogeneous term. According to the variational iteration method, we can construct a correction functional as follow

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(Lu_n(\xi) + N\tilde{u}(\xi) - g(\xi)\right) d\xi, \quad n \geq 0, \tag{3}$$

where λ is a general Lagrangian multiplier [4], which can be identified optimally via the variational theory, the subscript n denotes the nth order approximation, $\tilde{u}_n$ is considered as a restricted variation [3, 4] i.e., $\delta\tilde{u}_n = 0$.

So, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations $u_{n+1}(x,t)$, $n \geq 0$ of the solution $u(x,t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. Consequently, the solution

$$u(x,t) = \lim_{n \longrightarrow \infty} u_n(x,t). \tag{4}$$

For the convergence of VIM we refer the reader to Dehghan's work [3, 6].

3 Applying VIM for Stefan problem

In this section, we apply VIM to stefan prablem(Eq.(1)). Its correction variational functional in t-direction to obtain the solution of stefan prablem (1) by variational iteration method can be expressed as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^x \lambda(\xi,t)\left[\frac{\partial^2}{\partial\xi^2}(u_n) + (N(\tilde{u}_n))\right]d\xi, \quad n \geq 0, \quad (5)$$

where $N\widetilde{u}_n(\xi,t) = -\widetilde{u}_{nt}(\xi,t)$. taking variation with respect to the independent variable u_n noticing that $\delta N\widetilde{u}_n(\xi,t) = 0$

$$\begin{aligned} \delta u_{n+1}(x,t) &= \delta u_n(x,t) + \delta\int_0^x \lambda(\xi,t)\left[\frac{\partial^2}{\partial\xi^2}(u_n) + (N(\tilde{u}_n))\right]d\xi \\ &= \delta u_n(x,t) + \lambda(\xi,t)\,\delta u_{nx}|_{\xi=x} \quad (6)\\ &- \lambda'(\xi,t)\,\delta u_n|_{\xi=x} + \int_0^x \lambda''(\xi,t)\,\delta u_n d\xi = 0, \quad (7)\end{aligned}$$

This yields the stationary conditions

$$1 - \lambda'(\xi,t)_{\xi=x} = 0, \qquad \lambda(\xi,t)_{\xi=x} = 0, \qquad \lambda''(\xi,t)_{\xi=x} = 0. \quad (8)$$

This in turn gives $\lambda(\xi,t) = \xi - x$. Substituting this value of the Lagrange multiplier into the functional (5) gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^x (\xi - x)\left[\frac{\partial^2}{\partial\xi^2}(u_n) + (N(\tilde{u}_n))\right]d\xi, \quad (9)$$

thus, we can obtain approximation or exact solution for $u(x,t)$.

4 Illustrative Example

To demonstrate the effectiveness of the method we consider here Eqs. (1) with given initial condition.

Example 4.1 *Consider Stefan problem (1) with the I.C.*

$$u_0(x,t) = -x + e^{-t}. \quad (10)$$

Substituting (10) into Eq.(9) we obtain the following successive approximations

$$\begin{aligned} u_1(x,t) &= -x + e^t + \frac{1}{2}x^2e^t, \\ u_2(x,t) &= -x + e^t + \frac{1}{2}x^2e^t + \frac{1}{24}x^4e^t, \\ u_3(x,t) &= -x + e^t + \frac{1}{2}x^2e^t + \frac{1}{24}x^4e^t + \frac{1}{720}x^6e^t, \end{aligned}$$

finally, $u_n(x,t) = -x + e^t coshx$, then using Eq. (4) we have $u(x,t) = -x + e^t coshx$ which is exact solution.

5 Conclusion

The Variational iteration method is a powerful tool which is capable of handling linear/nonlinear partial differential equations. The method has been successfully applied to Stefan problem. This method dose not require small parameter in any equation as same as the perturbation approach. The results show that a correction functional can be easily constructed by a general Lagrange multiplier, and this multiplier can be optimally identified by variational theory. The application of restricted variations in correction functional makes it much easier to determine the multiplier.
Mathematica has been used for computations in this paper.

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