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Forward (r, s)-Difference Operator r, s and Solving Difference Equations

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Abstract

In this note we introduce a new operator that we call it forward (r, s)-difference operator $\Delta_{r,s}$, defined as follow

$$\Delta_{r,s}\ y_n = r\ y_{n+1} - s\ y_n.$$

Then, we investigate some properties of this new operator, we find a shift exponential formula and use it in solving of the nonhomogeneous difference equations with constant coefficients, may be written in the following form

$$(\prod_{j=1}^{m} \Delta_{r_j,s_j})y_n = f_n.$$

Keywords: Forward difference operator Δ, Forward (r, s)-difference operator $\Delta_{r,s}$, Difference equation, Shift exponential formula, Particular solution

1 Introduction

1.1. In Numerical Analysis, we use some linear operators: shift exponential operator E, "$Ef_j = f_{j+1}$", forward difference operator Δ,"$\Delta f_j = f_{j+1} - f_j$" and backward difference∇, "$\nabla f_j = f_j - f_{j-1}$". These operators are used in some topics of Numerical Analysis, particularly in interpolation, quadratures, difference equations, and so forth. [3], [4], [5].

In this paper we find the particular solution of the nonhomogeneous difference equations with constant coefficients. Under the forward difference operatorΔ, the linear difference equations are written in one of the following forms

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$$P(\Delta)y_n = 0, \qquad (homogeneous) \qquad (1)$$

$$P(\Delta)y_n = f_n. \qquad (nonhomogeneous) \qquad (2)$$

whereas P is polynomial.

In solving linear difference equations and finding general solution, we use the following theorems. [1], [2], [4], [5].

Theorem1 (superposition principle) Suppose that $y_1, y_2, ..., y_m$ are the (fundamental) solutions of the homogeneous difference equation(1), then any linear combinations of them is a solution for it too.

Theorem2 Suppose that the complex-valued function "$y_n = y_1 + i\ y_2$"be a solution of equation (1), then functions "y_1, y_2"also are solutions for it.

Theorem3 Let $y_h$be a solution for (1) and $y_p$be a particular solution for (2), then "$y_c = y_h + y_p$" is a solution for (2) too.

2 Solution of the difference equations

In this section, we discuss homogeneous and nonhomogeneous difference equations with constant coefficients.

Let "$y_n = r^n$" be a solution for equation (1), we have

$$P(r-1) = 0. \qquad (3)$$

Where (3) is called the corresponding characteristic equation to equation (1).

Remark 1 All roots of the characteristic equations may be distinct real values, either some of them equal or some of them are conjugate complex number.

(i) If $r_1, r_2, , r_k$be distinct real roots to the characteristic equations, then the functions "$r_1^n, r_2^n, ..., r_k^n$" will be solutions of the homogeneous equations, these functions are linearly independent [3].

These functions are said the fundamental solutions of the homogeneous equation.

(ii) If$r_1 = r_2 = ... = r_m = r$ be the repeated roots of the characteristic equation (3), then the fundamental solutions of the homogeneous equation are: "$r^n, n\ r^n, n^2 r^n, ..., n^{m-1} r^n$" that are linearly independent [3].

(iii) If $r_{1,2} = \alpha \pm i\beta$ be two conjugate complex roots, the fundamental solutions of the homogeneous equation are,
"$y_1 = (\alpha^2 + \beta^2)^{n/2} cosn\varphi, \quad y_2 = (\alpha^2 + \beta^2)^{n/2} sinn\varphi,$
where $\varphi = tan^{-1}(\beta/\alpha)$" [3].

Example1 Find the fundamental solutions of the following homogeneous difference equation:

$$(12\Delta^2 - 8\Delta + 1)y_n = 0.$$

Solution We have"$12(r-1)^2 - 8(r-1) + 1 = 0$" which yields,

$$"r_1 = \frac{7}{6},\ r_2 = \frac{3}{2}" \quad and \quad "y_1 = (\frac{7}{6})^n,\ y_2 = (\frac{3}{2})^n"$$

Example2 Solve the following difference equation and find the fundamental solutions

$$(5\Delta + 6)(32\Delta^2 + 56\Delta + 25)\ y_n = 0$$

Solution The roots of the corresponding characteristic equation are "$r_= \frac{-1}{5},\ r_{2,3} = \frac{1}{8}(1 \pm i)$" so the fundamental solutions will be written as follow

$$y_1 = (-\frac{1}{5})^n, \quad y_2 = 2^{-\frac{5}{2}n} cos\frac{n\pi}{4}, \quad y_3 = 2^{-\frac{5}{2}n} sin\frac{n\pi}{4}.$$

Example3 Find the fundamental solutions of the following homogeneous equation

$$(\Delta^4 + \Delta^2)\ y_n = 0.$$

Solution The characteristic equation is "$(r-1)^2\ (r^2 - 2r + 2) = 0$".This polynomial equation has one double root "$r = 1$"and two complex conjugate roots "$r = 1 \pm i$", therefore the fundamental solutions may be written as follow

$$y_1 = 1, \quad y_2 = n, \quad y_3 = 2^{\frac{n}{2}} cos\frac{n\pi}{4}, \quad y_4 = 2^{\frac{n}{2}} sin\frac{n\pi}{4}$$

Example4 Evaluate the fundamental solutions of the following DE

$$"(\Delta^6 - 6\Delta^4 + 9\Delta^2 - 4)\ y_n = 0".$$

Solution We have "$(r-1)^6 - 6(r-1)^4 + 9(r-1)^2 - 4 = 0$"which yields "$r_1 = r_2 = 2, \quad r_3 = -1, \quad r_4 = r_5 = 3, \quad r_6 = 0$"so the fundamental solutions may be written as follow "$y_1 = 2^n, \quad y_2 = n\ 2^n, \quad y_3 = (-1)^n, \quad y_4 = 3^n, \quad y_5 = n\ 3^n, \quad y_6 = 0$".

Lemma 1 Prove the accuracy of the following equalities.

$$\Delta \sum_{j=0}^{n-1} f_j = f_n \tag{4}$$

$$\frac{1}{\Delta}\ f_n = \sum_{j=0}^{n-1} f_j \tag{5}$$

Proof: The proof is easy, consider $\Delta \sum_{j=0}^{n-1} f_j = \sum_{j=0}^{n} f_j - \sum_{j=0}^{n-1} f_j = f_n$. Equality (5) is the inversion of (4).

Remark 2 Each of above identities are used for finding particular solution of the nonhomogeneous difference equations with constant coefficients therefore we can solve each of the following equations

$$\Delta y_n = f_n, \quad \Delta^m y_n = f_n$$

Example 5 Find the particular solution of the following difference equation

$$\Delta y_n = cosn\varphi$$

Solution We can write

$$y_p = \frac{1}{\Delta}(cosn\varphi) = \sum_{j=0}^{n-1} cosj\varphi = \frac{1}{2}sin(\frac{2n-1}{2}\ \varphi)cos\frac{\varphi}{2} - \frac{1}{2}$$

Example 6 Find the particular solution of the following difference equation

$$\Delta^3 y_n = 120n + 60$$

Solution By division operation we can write

$$y_p = \frac{1}{\Delta^2}(\frac{1}{\Delta}(120n+60)) = \frac{1}{\Delta^2}(\sum_{j=0}^{n-1}(120j+60)) = \frac{1}{\Delta}(\frac{1}{\Delta}(60n^2) = \frac{1}{\Delta}(\sum_{j=0}^{n-1} 60j^2)$$

$$= \frac{1}{\Delta}(10n(n-1)(2n-1) = 10\sum_{j=0}^{n-1}(2j^3 - 3j^2 + j) = 5n(n-1)^2(n-2).$$

Example 7 Find the particular solution of the following difference equation

$$\Delta^2 y_n = cos\frac{(n+1)\pi}{3}$$

Solution We can write

$$y_p = \frac{1}{\Delta^2}\, cos\frac{(n+1)\pi}{3} = \frac{1}{\Delta}\sum_{j=0}^{n-1} cos\frac{(j+1)\pi}{3} = \sum_{j=0}^{n-1}(-\frac{1}{2}+sin\frac{(2j+1)\pi}{6}) = 1-\frac{1}{2}\, n-cos\frac{n\pi}{3}$$

3 Main Results

Forward (r,s)-difference operator and the particular solution of the nonhomogeneous difference equations

DefinitionWe define the forward (r,s)-difference operator $\Delta_{r,s}$ as follow

$$\Delta_{r,s} y_n = r y_{n+1} - s y_n = (rE - s) y_n.$$

where y_n is the approximate value function $y(x)$ at point $x_n \in [x_0, x_m]$, then two operators "$\Delta_{r,s}$" and "$rE - s$" are equivalent.

Corollary 1 $\Delta_{r,s}$is a linear operator and $\Delta_{1,1} \equiv E - 1 \equiv \Delta$ and $\Delta_{r,r} \equiv r\Delta$.

Example 8

$$\Delta_{2,6}(3^n cos\frac{n\pi}{3} = 2\times 3^{n+1} cos\frac{(n+1)\pi}{3} - 6\times 3^n cos\frac{n\pi}{3} = -3^{n+1}(cos\frac{n\pi}{3} + \sqrt{3} sin\frac{n\pi}{3}).$$

Four principal operations in vector space of operator $\Delta_{r,s}$

we define

(i) $\Delta_{r_1,s} + \Delta_{r_2,s} \equiv \Delta_{r_1+r_2,s}$

(ii) $\Delta_{r,s_1} + \Delta_{r,s_2} \equiv \Delta_{r,s_1+s_2}$

(iii) $\Delta_{r_1,s} - \Delta_{r_2,s} \equiv \Delta_{r_1-r_2,s}$

(iv) $\Delta_{r,s_1} - \Delta_{r,s_2} \equiv \Delta_{r,s_1-s_2}$

(v) $\Delta_{r_1,s_1} \times \Delta_{r_2,s_2} \equiv \Delta_{r_2,s_2} \times \Delta_{r_1,s_1}$

$(vi)\quad \frac{\Delta_{r_1,s_1}}{\Delta_{r_2,s_2}} \equiv \Delta_{r_1,s_1}(\frac{1}{\Delta_{r_2,s_2}}) \equiv (\frac{1}{\Delta_{r_2,s_2}})\Delta_{r_1,s_1}$

We define order and inversion of the forward (r, s)-difference operator consider

(i) $\Delta_{r,s}^{-1} \equiv \frac{1}{\Delta_{r,s}}$ s.t. $\frac{1}{\Delta_{r,s}} f_n = g_n \Leftrightarrow \Delta_{r,s} g_n = f_n$

(ii) $\Delta_{r,s}\Delta_{r,s} \equiv \Delta_{r,s}^2, ..., \Delta_{r,s}(\Delta_{r,s}^m) \equiv \Delta_{r,s}^{m+1}$

Remark 3 Addition operation and multiplication operation are commutative and associative, namely

$$(\Delta_{r_1,s_1} + \Delta_{r_2,s_2}) + \Delta_{r_3,s_3} \equiv \Delta_{r_1,s_1} + (\Delta_{r_2,s_2} + \Delta_{r_3,s_3}) \equiv \Delta_{r_1+r_2+r_3,s_1+s_2+s_3},$$

$$\Delta_{r_1,s_1} \times (\Delta_{r_2,s_2} \times \Delta_{r_3,s_3}) \equiv (\Delta_{r_1,s_1} \times \Delta_{r_2,s_2}) \times \Delta_{r_3,s_3}$$

Theorem 4 The forward (r, s)-difference operator is linear operator, in addition to, every order of it and every polynomial of$\Delta_{r,s}$ and inversion $\Delta_{r,s^{-1}}$ are linear too.
Proof: The proof is easy and left to the readers.

Lemma 2 Prove that

$$\Delta_{r,s}(\sum_{j=0}^{n-1}((\frac{s}{r})^{n-j-1}y_j) = sy_n \tag{6}$$

$$\frac{1}{\Delta_{r,s}}y_n = \frac{1}{s}\sum_{j=0}^{n-1}((\frac{s}{r})^{n-j-1}y_j) \tag{7}$$

$$\Delta_{r,s}((\frac{s}{r})^n \sum_{j=0}^{n-1} y_j) = \frac{s^{n+1}}{r^n}y_n \tag{8}$$

$$\frac{1}{\Delta_{r,s}}((\frac{s}{r})^n y_n = \frac{1}{s}(\frac{s}{r})^n \sum_{j=0}^{n-1} y_j \tag{9}$$

Proof: The proof is easy, above equations are used in solving of NDE with constant coefficients.

Remark 4 Under the forward (r, s)-difference operator $\Delta_{r,s}$, the nonhomogeneous difference equation may be written as follow

$$(\prod_{j=1}^{m}\Delta_{r_j,s_j})y_n = f_n \quad (10)$$

Whereas "$r_j, j = 1, 2, ..., m$" can be real distinct, repeated or complex number.

Useful Results $(a \neq \frac{s}{r})$

(i) $\Delta_{r,s} = a^n(ra - s) \Rightarrow \frac{1}{\Delta_{r,s}} = \frac{a^n}{ra-s}$.

In general

(ii) $\Delta^k_{r,s}a^n = (ra - s)^k a^n \Rightarrow \frac{1}{\Delta^k_{r,s}}a^n = \frac{a^n}{(ra-s)^k}$

(iii) $\Delta_{r,s}(\frac{s}{r})^n = 0$

(iv) $\Delta_{r,s}(a^n y_n) = a^n\Delta_{ra,s}y_n$

(v) $\Delta^k_{r,s}(a^n y_n) = a^n\Delta^k_{ra,s}y_n$

Lemma 3 Prove that

$$\Delta^k_{r,s}((\tfrac{s}{r})^n y_n = (\tfrac{s}{r})^n(s\Delta)^k y_n \quad (11.1)$$

$$\frac{1}{\Delta^k_{r,s}}((\tfrac{s}{r})^n y_n = (\tfrac{s}{r})^n\frac{1}{(s\Delta)^k}y_n \quad (11.2)$$

Proof: Equality (11.1) is proved by the mathematical induction.

Particular case Suppose that $y_n = n^k$, then

$$\Delta^k_{r,s}(n^k(\tfrac{s}{r})^n = k!s^k(\tfrac{s}{r})^n \quad (12)$$

$$\frac{1}{\Delta^k_{r,s}}((\tfrac{s}{r})^n = \frac{n^k}{k!s^k}(\tfrac{s}{r})^n \quad (13)$$

Example 9 Evaluate $\frac{1}{\Delta^3_{1,2}}(2^n n)$

Solution $\frac{1}{\Delta^3_{1,2}}(2^n n) = 2^{n-3}\frac{1}{\Delta^3}(n) = 2^{n-3}\frac{1}{\Delta^2}\sum_{j=0}^{n-1} j = 2^{n-1}\frac{1}{\Delta}(\sum_{j=0}^{n-1}(j^2 + j)$

$$= \frac{2^{n-4}}{3}\sum_{j=0}^{n-1}(j^3 - 3j^2 + 2j) = \frac{2^n}{192}n(n-1)(n-2)(n-3)$$

Example 10 Find the particular solution of $\Delta^2_{1,3}y_n = 3^n sin(\frac{n\pi}{3})^n$

Solution Write $y_p = \frac{1}{\Delta_{1,3}}sin(\frac{n\pi}{3})$ and use (12), thus

$$y_p = 3^{n-2}\frac{1}{\Delta^2}(\frac{n\pi}{3}) = 3^{n-2}\frac{1}{\Delta}(\frac{\sqrt{3}}{2} - cos\frac{(2n-1)\pi}{6})$$

$$= 3^{n-2}(\frac{\sqrt{3}}{2} - \sum_{j=0}^{n-1}cos\frac{(2j-1)\pi}{6}) = 3^{n-2}(\frac{1}{2} + \frac{\sqrt{3}}{2}n) + \frac{3^{n-2}}{2}sin(\frac{(2n-1)\pi}{6})$$

Theorem 5 (shift exponential) Let P be a polynomial, then

$$P(\Delta_{r,s})((\tfrac{s}{r})^n y_n) = (\tfrac{s}{r})^n P(s\Delta)y_n \tag{14.1}$$
$$\frac{1}{P\Delta_{r,s}}((\tfrac{s}{r})^n y_n) = (\tfrac{s}{r})^n \frac{1}{P(s\Delta)} y_n \tag{14.2}$$

Proof: The proof is easy by using Lemma 3.

Example 11 Find the particular solution of the following $D.E.$

$$(E^4 - 10E^3 + 35E^2 - 50E + 24)y_n = (8n + 12)2^n$$

Solution This equation may be written as follows

$$(E - 1)(E - 2)(E - 3)(E - 4)y_n = \Delta\Delta_{1,2}\Delta_{1,3}\Delta_{1,4}y_n$$

$$(\Delta_{1,2} + 1)\Delta_{1,2}(\Delta_{1,2} - 1)(\Delta_{1,2} - 2)y_n = (8n + 12)2^n$$

Now divide two sides of this equality into coefficient of y_n, using the formula (11.2), we have

$$y_p = \frac{1}{(\Delta_{1,2}+1)\Delta_{1,2}(\Delta_{1,2}-1)(\Delta_{1,2}-2)}((n + 4)2^n)\ 2^n \frac{1}{(2\Delta+1)2\Delta(2\Delta-1)(2\Delta-2)}(8n + 12)$$
$$2^{n-4} \frac{1}{(\Delta+\frac{1}{2})\Delta(\Delta-\frac{1}{2})(\Delta-1)}(8n + 12) = 2^n(n + 4)$$

Solution of NDE with constant coefficients

We know that every nonhomogeneous difference equation with order$m$ can be written in the form (10).Therefore each of the following forms may be written in the form of (10).

$$P(E)y_n = f_n, \quad P(\Delta)y_n = f_n, \quad P(\nabla)y_n = f_n.$$

(10) is written as follows

$$y_p = \frac{1}{\prod_{j=1}^{m} \Delta_{r_j,s_j}} f_n = \frac{1}{\Delta_{r_m,s_m}}(\frac{1}{\Delta_{r_{m-1},s_{m-1}}}(\ldots\frac{1}{\Delta_{r_1,s_1}} f_n\ldots)) \tag{15}$$

Example12 Find the particular solution of the following NDE

$$\Delta_{2,1}\Delta_{2,3}y_n = 2^n n^2$$

Solution Write $y_p = \frac{1}{\Delta_{2,1}\Delta_{2,3}}(2^n n^2) = \frac{1}{\Delta_{2,1}}(\frac{1}{\Delta_{2,3}}(2^n n^2)) = \frac{1}{\Delta_{2,1}}(2^n(n^2 - 8n + 28))$

$$2^n(\tfrac{1}{3}n^2 - \tfrac{32}{9}n + \tfrac{368}{27})$$

Remark5 In using of the identity (15), we may use iterative divisions, in addition to, we can use the decomposition fraction, consider

$$\frac{1}{\Delta_{r_1,s_1}\Delta_{r_2,s_2}} \equiv \frac{A_1}{\Delta_{r_1,s_1}} + \frac{A_2}{\Delta_{r_2,s_2}}, \quad A_1 = \frac{r_1}{r_2s_1 - r_1s_2}, \quad A_2 = \frac{r_2}{r_1s_2 - r_2s_1}$$

$$\frac{1}{\Delta_{r_1,s_1}\Delta_{r_2,s_2}\Delta_{r_3,s_3}} \equiv \frac{A_1}{\Delta_{r_1,s_1}} + \frac{A_2}{\Delta_{r_2,s_2}} + \frac{A_3}{\Delta_{r_3,s_3}},$$

$$A_1 = \frac{r_1^2}{(r_2s_1-r_1s_2)(r_3s_1-r_1s_3)}, A_2 = \frac{r_2^2}{(r_1s_2-r_2s_1)(r_3s_2-r_2s_3)}, A_3 = \frac{r_3^2}{(r_1s_3-r_3s_1)(r_2s_1-r_1s_2)}$$

4 Discussion and results

The shift operator E method in solving of non-homogeneous difference equations with constant coefficients is a new method which we can solve all of NHDE with constant coefficients by using this method.

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