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Uniform blow-up profiles for a nonlocal degenerate parabolic system

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Abstract

This paper deals with the blow-up profiles of the nonnegative solutions to a degenerate reaction-diffusion system with nonlinear nonlocal sources involved in a product with local terms, subject to the homogeneous Dirichlet boundary conditions. It will be proved that If $p_1,\ p_2 \leq 1$ and $q_1 q_2 > (m - p_1)(n - p_2)$ the nonlocal terms play a leading role in the blow-up profiles, i.e. the system has global blow-up and the uniform blow-up profiles are obtained. This extends a recent work of [10], which considered the uniform blow-up profile of the single equation of the same system.

Keywords: Degenerate reaction-diffusion system, Uniform blow-up profiles, Global blow-up

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1 Introduction

In this paper, we study the following coupled degenerate parabolic system with nonlinear nonlocal sources

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$$\begin{cases} u_t = \triangle u^m + u^{p_1}\displaystyle\int_\Omega v^{q_1}(x,t)d\,x, & x\in\Omega, t>0,\\ v_t = \triangle v^n + v^{p_2}\displaystyle\int_\Omega u^{q_2}(x,t)d\,x, & x\in\Omega, t>0,\\ u(x,t)=v(x,t)=0, & x\in\partial\Omega, t>0,\\ u(x,0)=u_0(x), v(x,0)=v_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $m,\ n > 1,\ p_1,\ p_2,\ q_1,\ q_2 > 0$. The initial data $u_0,\ v_0$ are nontrivial nonnegative bounded smooth functions and vanish on $\partial\Omega$. Many physical phenomena have formulated into nonlocal mathematical models(see [2, 5, 6, 10, 11] and the reference therein).

In recent years, a numbers of works have contributed to the study of the blow-up profiles of the semilinear parabolic system. Souplet's elegant work [11] plays a critical role in this area. The method (or modified method) in [11] was extensively used in many other works, we refer the readers to [6, 8, 9, 10] and the references therein.

In [10], Liu *et al* have considered the following single equation

$$u_t = u^p(\triangle u + au^r\int_\Omega u^s d\,x), \qquad x\in\Omega,\ t>0, \tag{1.2}$$

with null Dirichlet boundary condition. When $p+r\le 1$, they have obtained the following limit under some hypotheses

$$\lim_{t\to T^*} u(x,t)(T^*-t)^{1/(p+r+s-1)} = (a\mid\Omega\mid(p+r+s-1))^{1/(1-p-r-s)}.$$

In [4], Du considered the global existence and non-existence of system (1.1), and obtained that

Theorem A. If $m<p_1$ or $n<p_2$ or $q_1q_2>(m-p_1)(n-p_2)$, then the nonnegative solution of (1.1) blows up in finite time for sufficiently large values and exists globally for sufficiently small initial values.

Furthermore, if $m>p_1$, $n<p_2$, $q_1>m-p_1$ and $q_2>n-p_2$, they yielded the blow-up rates of system (1.1) under some appropriate hypotheses. But for problem (1.1), it seems that the blow-up solutions have global blow-up and the blow-up is uniformly in any compact subset of the domain Ω provided that $p_1,\ p_2\le 1$ and $q_1q_2>(m-p_1)(n-p_2)$. Motivated by this result, we will prove it in this paper.

Throughout this paper we assume that $q_1q_2>(m-p_1)(n-p_2)$ and the initial data u_0 and v_0 satisfy the conditions as follows:

$(H_1)\triangle u_0^m(x)+u_0^{p_1}(x)\int_\Omega v_0^{q_1}(x)d\,x>0,\ \triangle v_0^n(x)+v_0^{p_2}(x)\int_\Omega u_0^{q_2}(x)d\,x>0\ \ for\ \ x\in\Omega.$

$(H_2)\triangle u_0^m(x)\le 0,\ \triangle v_0^n(x)\le 0\ \ for\ \ x\in\ \Omega.$

Now let us state our main results.

Theorem 1.1 *Assume $(H_1)-(H_2)$ hold and (u,v) is a classical solution of (1.1) which blows up in finite time T^*. Let $p_1,\ p_2\leq 1$, then the following statements hold uniformly on any compact subset of Ω.*
(i) If $p_1,\ p_2<1$ and $q_1q_2>(m-p_1)(n-p_2)$, then

$$\lim_{t\to T^*} u(x,t)(T^*-t)^{\theta}=\mid\Omega\mid^{-\theta}\sigma^{\frac{q_1}{q_1q_2-(1-p_1)(1-p_2)}}\theta^{\frac{1-p_2}{q_1q_2-(1-p_1)(1-p_2)}},$$

$$\lim_{t\to T^*} v(x,t)(T^*-t)^{\sigma}=\mid\Omega\mid^{-\sigma}\theta^{\frac{q_2}{q_1q_2-(1-p_1)(1-p_2)}}\sigma^{\frac{1-p_1}{q_1q_2-(1-p_1)(1-p_2)}},$$

where $\theta=\frac{1+q_1-p_2}{q_1q_2-(1-p_1)(1-p_2)},\quad \sigma=\frac{1+q_2-p_1}{q_1q_2-(1-p_1)(1-p_2)}$.
(ii) If $p_1=1$ or $p_2=1$, then

$$\lim_{t\to T^*}\log u(x,t)\mid\log(T^*-t)\mid^{-1}=\frac{1+q_1-p_2}{q_1q_2},$$

$$\lim_{t\to T^*}\log v(x,t)\mid\log(T^*-t)\mid^{-1}=\frac{1+q_2-p_1}{q_1q_2}.$$

2 Proof of the Theorem 1.1

In this section we will give the proof of Theorem 1.1. We first introduce some transformations. Let $U(x,\tau)=u^m(x,t),\ V(x,\tau)=(n/m)^{n/n-1}v^n(x,t),\ \tau=mt$, then (1.1) becomes the following system not in divergence form:

$$\begin{cases} U_\tau=U^{r_1}(\triangle U+aU^{p_3}\int_\Omega V^{q_3}(x,\tau)dx), & x\in\Omega,\tau>0,\\ V_\tau=V^{r_2}(\triangle V+bV^{p_4}\int_\Omega U^{q_4}(x,\tau)dx), & x\in\Omega,\tau>0,\\ U(x,\tau)=V(x,\tau)=0, & x\in\partial\Omega,\tau>0,\\ U(x,0)=U_0(x),V(x,0)=V_0(x), & x\in\Omega, \end{cases}\tag{2.1}$$

where $r_1=(m-1)/m,\ r_2=(n-1)/n,\ p_3=p_1/m,\ q_3=q_1/n,\ p_4=p_2/n,\ q_4=q_2/m,\ a=(m/n)^{q_1/(n-1)},\ b=(m/n)^{(p_2-n)/(n-1)},\ U_0(x)=u_0^m(x),\ V_0(x)=(m/n)^{n/(n-1)}v_0^n(x)$.
Remark. Clearly, when $m=n,\ p_1=p_2,\ q_1=q_2,\ u_0(x)=v_0(x)$, system (2.1) is reduced to a single equation (1.2), the uniform blow-up profile of which has been considered by Liu *et al* in [10]. And the uniform profile of the special case $p_3=p_4=0$ of system (2.1) have been considered by Duan *et al* in [6].

Under these transformations, the assumptions $(H_1)-(H_2)$ become
$(H_1')\triangle U_0(x)+aU_0^{p_3}(x)\int_\Omega V_0^{q_3}(x)dx>0,\ \triangle V_0(x)+bV_0^{p_4}(x)\int_\Omega U_0^{q_4}(x)dx>0,\ \ for\ x\in\ \Omega.$
$(H_2')\triangle U_0(x)\leq 0,\ \triangle V_0(x)\leq 0,\ \ for\ x\in\ \Omega.$

In section 4 of [4], Du have given the existence of the classical solution $(U,\ V)$ of (2.1) under the hypothesis (H_1').

Before we prove Theorem 1.1, we give the following Lemmas.

For convenience, we denote

$$f(\tau)=\int_\Omega U^{q_4}(x,\tau)d\,x,\ \ F(\tau)=\int_0^\tau f(s)d\,s,$$

$$g(\tau)=\int_\Omega V^{q_3}(x,\tau)d\,x,\ \ G(\tau)=\int_0^\tau g(s)d\,s.$$

Lemma 2.1 *Assume that* (U,V) *is a classical solution of* (2.1) *which blows up in finite time* $T_*\equiv mT^*$. *If* $p_1\le 1$, $p_2\le 1$, *then*

$$\lim_{\tau\to T_*}g(\tau)=\lim_{\tau\to T_*}G(\tau)=\infty,\quad \lim_{\tau\to T_*}f(\tau)=\lim_{\tau\to T_*}F(\tau)=\infty.$$

Moreover, U *and* V *blow up simultaneously.*

Proof. In view of (U,V) blows up in finite time T_*, We have $\parallel U(\cdot,\tau)\parallel_\infty+\parallel V(\cdot,\tau)\parallel_\infty\to\infty,\ as\ \ \tau\to T_*$. Without loss of generality we may assume that $\parallel U(\cdot,\tau)\parallel_\infty\to\infty,\ as\ \ \tau\to T_*$. Suppose on the contrary that $\lim_{\tau\to T_*}g(\tau)<\infty$. So, from the equation of U in system (2.1), we know that U exists globally for any $U_0(x)$(see [12]), since $0<p_3=p_1/m\le 1$. This leads to a contradiction. Therefore $\lim_{\tau\to T_*}g(\tau)=\infty$. It can be deduced that $\lim_{\tau\to T_*}\parallel V(\cdot,\tau)\parallel_\infty=\infty$ from $g(\tau)=\int_\Omega V^{q_3}(x,\tau)d\,x$ and $\lim_{\tau\to T_*}g(\tau)=\infty$. Then we conclude that U and V blow up simultaneously.

Next we infer that $\lim_{\tau\to T_*}G(\tau)=\infty$. Set $\widetilde{U}(\tau)=\max_{x\in\overline{\Omega}}U(x,\tau)$. By Theorem 4.5 of [7] we know that $\widetilde{U}(\tau)$ is Lipschitz continuous and

$$\widetilde{U}'(\tau)\le\widetilde{U}^{r_1+p_3}(\tau)g(\tau)\qquad a.e.\ in\ [0,T_*). \tag{2.2}$$

In view of $r_1+p_3=1+(p_1-1)/m$, integrating (2.2) over $(0,\tau)$, we obtain

$$\begin{cases}\dfrac{1}{1-r_1-p_3}\widetilde{U}^{1-r_1-p_3}(\tau)\le aG(\tau)+\dfrac{1}{1-r_1-p_3}\widetilde{U}^{1-r_1-p_3}(0), & if\ p_1<1,\\ \log\widetilde{U}(\tau)\le aG(\tau)+\log\widetilde{U}(0), & if\ p_1=1.\end{cases} \tag{2.3}$$

From $\lim_{\tau\to T_*}\widetilde{U}(\tau)=\infty$, it follows that $\lim_{\tau\to T_*}G(\tau)=\infty$.

Furthermore, from $\lim_{\tau\to T_*}\parallel V(\cdot,\tau)\parallel_\infty=\infty$, applying the similar arguments as above to the equation of V in system (2.1), we have $\lim_{\tau\to T_*}f(\tau)=\lim_{\tau\to T_*}F(\tau)=\infty$. □

To prove Theorem 1.1, we try to show the relationships among U, V, $F(\tau)$ and $G(\tau)$. We use the notation $f(\tau)\sim g(\tau)$ for $\lim_{\tau\to T_*}\frac{f(\tau)}{g(\tau)}=1$.

Lemma 2.2 *Under the conditions of Theorem 1.1, the following statements hold uniformly on any compact subset of* Ω.
(i). If $p_1 < 1$ *and* $p_2 < 1$*, then*

$$U^{1-r_1-p_3}(x,\tau) \sim a(1-r_1-p_3)G(\tau), \quad V^{1-r_2-p_4}(x,\tau) \sim b(1-r_2-p_4)F(\tau).$$

(ii). If $p_1 < 1$ *and* $p_2 = 1$*, then*

$$U^{1-r_1-p_3}(x,\tau) \sim a(1-r_1-p_3)G(\tau), \quad \log V(x,\tau) \sim bF(\tau).$$

(iii). If $p_1 = 1$ *and* $p_2 < 1$*, then*

$$\log U(x,\tau) \sim aG(\tau), \quad V^{1-r_2-p_4}(x,\tau) \sim b(1-r_2-p_4)F(\tau).$$

(iv). If $p_1 = p_2 = 1$*, then*

$$\log U(x,\tau) \sim aG(\tau), \quad \log V(x,\tau) \sim bF(\tau).$$

Proof. (i). From $p_1 < 1$, we have $1 - r_1 - p_3 > 0$, then a direct computation yields

$$\begin{aligned} \frac{\partial U^{1-p_3}}{\partial \tau} &= U^{r_1}(\triangle U^{1-p_3} + p_3(1-p_3)U^{1-p_3} \mid \nabla U \mid^2 + a(1-p_3)g(\tau)) \\ &\geq U^{r_1}(\triangle U^{1-p_3} + a(1-p_3)g(\tau)), \end{aligned}$$

which shows that $U^{1-p_3}(x,\tau)$ is a supersolution of the following problem

$$\begin{cases} w_t = w^{\frac{r_1}{1-p_3}}(\triangle w + a(1-p_3)g(\tau),) & x \in \Omega, \quad 0 < \tau < T_*, \\ w(x,\tau) = 0, & x \in \partial\Omega, \quad 0 < \tau < T_*, \\ w(x,0) = U_0^{1-p_3}(x), & x \in \Omega. \end{cases}$$

In view of $0 < r_1/(1-p_3) < 1$, under the assumptions $(H_1') - (H_2')$, it follows from (4.15) in [3] that

$$\lim_{\tau \to T_*} \frac{w^{\frac{r_1}{1-p_3}}(x,\tau)}{a(1-r_1-p_3)G(\tau)} = \lim_{\tau \to T_*} \frac{\| w(\cdot,\tau) \|_\infty^{\frac{r_1}{1-p_3}}}{a(1-r_1-p_3)G(\tau)} = 1 \tag{2.4}$$

holds uniformly on any compact subset of Ω. By comparison methods (see [1]), we obtain

$$U^{1-p_3}(x,\tau) \geq w(x,\tau), \qquad for\ (x,\tau) \in \Omega \times [0,T_*).$$

Hence from (2.4), the following limit holds on any compact subset of Ω

$$\liminf_{\tau \to T_*} \frac{U^{1-r_1-p_3}(x,\tau)}{a(1-r_1-p_3)G(\tau)} \geq 1\ ,\ \liminf_{\tau \to T_*} \frac{\| U(\cdot,\tau) \|_\infty^{1-r_1-p_3}}{a(1-r_1-p_3)G(\tau)} \geq 1. \tag{2.5}$$

On the other hand, it follows from the case $p_1 < 1$ in (2.3) that

$$\limsup_{\tau\to T_*} \frac{\widetilde{U}^{1-r_1-p_3}(\tau)}{a(1-r_1-p_3)G(\tau)} \le 1. \tag{2.6}$$

From $\widetilde{U}(\tau) = \max_{x\in\overline{\Omega}} U(x,\tau)$, (2.5) and (2.6) guarantee that

$$\lim_{\tau\to T_*} \frac{U^{1-r_1-p_3}(x,\tau)}{a(1-r_1-p_3)G(\tau)} = \lim_{\tau\to T_*} \frac{\| U(\cdot,\tau) \|_\infty^{1-r_1-p_3}}{a(1-r_1-p_3)G(\tau)} = 1$$

holds uniformly on any compact subset of Ω.

If $p_2 < 1$, by the similar arguments, we have

$$\lim_{\tau\to T_*} \frac{V^{1-r_2-p_4}(x,\tau)}{b(1-r_2-p_4)F(\tau)} = \lim_{\tau\to T_*} \frac{\| V(\cdot,\tau) \|_\infty^{1-r_2-p_4}}{b(1-r_2-p_4)F(\tau)} = 1$$

holds uniformly on any compact subset of Ω.

(ii). If $p_1 < 1$, analogous to case (i), we have

$$U^{1-r_1-p_3}(x,\tau) \sim a(1-r_1-p_3)G(\tau).$$

If $p_2 = 1$, i.e. $1-r_2-p_4 = 0$, using the similar computations as case (i), we obtain

$$V_\tau \ge V\left(\frac{\triangle V^{1-p_4}}{1-p_4} + bf(\tau)\right).$$

Hence V^{1-p_4} is a supersolution of the following problem:

$$\begin{cases} z_\tau = z(\triangle z + b(1-p_4)f(\tau)), & x\in\Omega, \quad 0<\tau<T_*, \\ z(x,\tau) = 0, & x\in\partial\Omega, \quad 0<\tau<T_*, \\ z(x,0) = V_0^{1-p_4}(x), & x\in\Omega. \end{cases}$$

Set

$$\alpha(x,\tau) = b(1-p_4)F(\tau) - \log z, \quad \beta(\tau) = \int_\Omega \alpha(y,\tau)\varphi(y)d\,y,$$

where $\varphi(y) > 0$ is the eigenfunction corresponding to the first eigenvalue λ_1 of $-\triangle$ in Ω with $\int_\Omega \varphi(y)d\,y = 1$. Under the assumptions $(H_1') - (H_2')$, using the similar methods in [3], we have the following statement holds uniformly on any compact subset of Ω:

$$\lim_{\tau\to T_*} \frac{\log z(x,\tau)}{b(1-r_2-p_4)F(\tau)} = \lim_{\tau\to T_*} \frac{\| \log z(\cdot,\tau) \|_\infty}{b(1-r_2-p_4)F(\tau)} = 1.$$

Proceeding as case (i), we arrive at the corresponding conclusion.

Case (iii) and (iv) can be treated similarly. □

Lemma 2.3 *Under the assumptions of the Theorem 1.1, for any given positive constants $0<\delta,\ \epsilon<1,\ \ \gamma>1$, there exists $\widetilde{T}$ such that for all $\tau\in[\widetilde{T},T_*)$, the following statements hold.*
(i). If $p_1<1$ and $p_2<1$, then

$$\begin{aligned}
&\epsilon\delta a(1+q_4-r_1-p_3)(b(1-r_2-p_4)F(\tau))^{\frac{1+q_3-r_2-p_4}{1-r_2-p_4}}\\
\le{}& \gamma b(1+q_3-r_2-p_4)(a(1-r_1-p_3)G(\tau))^{\frac{1+q_4-r_1-p_3}{1-r_1-p_3}},\\
&\epsilon\delta b(1+q_3-r_2-p_4)(a(1-r_1-p_3)G(\tau))^{\frac{1+q_4-r_1-p_3}{1-r_1-p_3}}\\
\le{}& \gamma a(1+q_4-r_1-p_3)(b(1-r_2-p_4)F(\tau))^{\frac{1+q_3-r_2-p_4}{1-r_2-p_4}}.
\end{aligned}$$

(ii) If $p_1<1$ and $p_2=1$, then

$$\frac{1+q_4-r_1-p_3}{1-r_1-p_3}\log(a(1-r_1-p_3)G(\tau))$$

$$+\log(\epsilon\delta\gamma)+\log\frac{bq_3}{a(1+q_4-r_1-p_3)}\le bq_3\gamma F(\tau),$$

$$bq_3\delta F(\tau)\le\frac{1+q_4-r_1-p_3}{1-r_1-p_3}\log(a(1-r_1-p_3)G(\tau))$$

$$+\log\frac{\gamma\delta}{\epsilon}+\log\frac{bq_3}{a(1+q_4-r_1-p_3)}.$$

(iii). If $p_1=1$ and $p_2<1$, then

$$\frac{1+q_3-r_2-p_4}{1-r_2-p_4}\log(b(1-r_2-p_4)F(\tau))$$

$$+\log(\epsilon\delta\gamma)+\log\frac{aq_4}{b(1+q_3-r_2-p_4)}\le aq_4\gamma G(\tau),$$

$$aq_4\delta G(\tau)\le\frac{1+q_3-r_2-p_4}{1-r_2-p_4}\log(b(1-r_2-p_4)F(\tau))$$

$$+\log\frac{\gamma\delta}{\epsilon}+\log\frac{aq_4}{b(1+q_3-r_2-p_4)}.$$

(iv). If $p_1=p_2=1$, then

$$\begin{aligned}
&\log\frac{aq_4\epsilon\gamma}{bq_3\delta}+bq_3\delta F(\tau)\le aq_4\gamma G(\tau),\\
&aq_4\delta G(\tau)\le\log\frac{aq_4\delta}{bq_3\epsilon\gamma}+bq_3\gamma F(\tau).
\end{aligned}$$

Proof. (i). $p_1 < 1$ *and* $p_2 < 1$. In view of $F'(\tau) = f(\tau) = \int_\Omega U^{q_4}(x,\tau)dx$, $G'(\tau) = g(\tau) = \int_\Omega V^{q_3}(x,\tau)dx$, from case (i) of Lemma 2.2, we have

$$F'(\tau) \sim \mid \Omega \mid (a(1-r_1-p_3)G(\tau))^{\frac{q_4}{1-r_1-p_3}}, \quad G'(\tau) \sim \mid \Omega \mid (b(1-r_2-p_4)F(\tau))^{\frac{q_3}{1-r_2-p_4}}$$

as $\tau \to T_*$. Then, for chosen positive constants $\delta < 1 < \gamma$, there exists $t_0 < T_*$ such that for all $t_0 \leq \tau < T_*$

$$\delta \mid \Omega \mid (a(1-r_1-p_3)G(\tau))^{\frac{q_4}{1-r_1-p_3}} \leq F'(t) \leq \gamma \mid \Omega \mid (a(1-r_1-p_3)G(\tau))^{\frac{q_4}{1-r_1-p_3}},$$
$$\delta \mid \Omega \mid (b(1-r_2-p_4)F(\tau))^{\frac{q_3}{1-r_2-p_4}} \leq G'(\tau) \leq \gamma \mid \Omega \mid (b(1-r_2-p_4)F(\tau))^{\frac{q_3}{1-r_2-p_4}}.$$

And thus, for any $\tau \in [t_0, T_*)$

$$\frac{\delta(a(1-r_1-p_3)G(\tau))^{\frac{q_4}{1-r_1-p_3}}}{\gamma(b(1-r_2-p_4)F(\tau))^{\frac{q_3}{1-r_2-p_4}}} \leq \frac{dF}{dG} \leq \frac{\gamma(a(1-r_1-p_3)G(\tau))^{\frac{q_4}{1-r_1-p_3}}}{\delta(b(1-r_2-p_4)F(\tau))^{\frac{q_3}{1-r_2-p_4}}}. \quad (2.7)$$

From the right side of (2.7), we get

$$\delta(b(1-r_2-p_4)F(\tau))^{\frac{q_3}{1-r_2-p_4}}\, dF \leq \gamma(a(1-r_1-p_3)G(\tau))^{\frac{q_4}{1-r_1-p_3}}\, dG, \quad for\ \tau \in [t_0, T_*).$$

Integrating above from t_0 to τ, it follows that

$$\frac{\delta(b(1-r_2-p_4)F(s))^{\frac{1+q_3-r_2-p_4}{1-r_2-p_4}}}{b(1+q_3-r_2-p_4)} \Big|_{t_0}^{\tau} \leq \frac{\gamma(a(1-r_1-p_3)G(\tau))^{\frac{1+q_4-r_1-p_3}{1-r_1-p_3}}}{a(1+q_4-r_1-p_3)}. \quad (2.8)$$

Due to $\lim_{\tau\to T_*} F(\tau) = \infty$ and $1-r_2-p_4 = (1-p_2)/n > 0$, for given constant $0 < \epsilon < 1$, there exists $\widetilde{t_0} : t_0 \leq \widetilde{t_0} < T_*$ such that

$$F^{\frac{1+q_3-r_2-p_4}{1-r_2-p_4}}(t_0) \leq (1-\epsilon)F^{\frac{1+q_3-r_2-p_4}{1-r_2-p_4}}(\tau), \qquad for\ \tau \in [\widetilde{t_0}, T_*).$$

Hence, from (2.8), it can be deduced that for all $\tau \in [\widetilde{t_0}, T_*)$

$$\epsilon\delta a(1+q_4-r_1-p_3)(b(1-r_2-p_4)F(\tau))^{\frac{1+q_3-r_2-p_4}{1-r_2-p_4}}$$
$$\leq \gamma b(1+q_3-r_2-p_4)(a(1-r_1-p_3)G(\tau))^{\frac{1+q_4-r_1-p_3}{1-r_1-p_3}}. \quad (2.9)$$

Application of the similar analysis as above to the left side of (2.7) guarantees that there exists $t_0^* < T_*$, such that for all $\tau \in [t_0^*, T_*)$

$$\epsilon\delta b(1+q_3-r_2-p_4)(a(1-r_1-p_3)G(\tau))^{\frac{1+q_4-r_1-p_3}{1-r_1-p_3}}$$
$$\leq \gamma a(1+q_4-r_1-p_3)(b(1-r_2-p_4)F(\tau))^{\frac{1+q_3-r_2-p_4}{1-r_2-p_4}}. \quad (2.10)$$

Set $\widetilde{T} = \max\{\widetilde{t_0}, t_0^*\}$, then (2.9) and (2.10) ensure case (i) of Lemma 2.3.

Analogous to case (i), we can draw the other conclusions of Lemma 2.3. $\square$

Proof of Theorem 1.1. Choose $\{\delta_i\}_{i=1}^{\infty}$, $\{\epsilon_i\}_{i=1}^{\infty}$, $\{\gamma_i\}_{i=1}^{\infty}$, satisfying $0 < \delta_i,\ \epsilon_i < 1$ and $\gamma_i > 1$, $i = 1 \dots \infty$, with $\delta_i,\ \epsilon_i,\ \gamma_i \to 1,\ as\ i \to \infty$. Putting $(\delta,\ \epsilon,\ \gamma) = (\delta_i,\ \epsilon_i,\ \gamma_i)$ in Lemma 2.3, we get $\widetilde{T}_i < T_*$ such that the corresponding $(i) - (iv)$ of Lemma 2.3 hold for all $\widetilde{T}_i \leq \tau < T_*$.

(i) $p_1 < 1,\ p_2 < 1$. From case (i) of Lemma 2.2 it follows that for such sequences $\{\delta_i\}_{i=1}^{\infty}, \{\gamma_i\}_{i=1}^{\infty}$, there exists $\{t_i\}_{i=1}^{\infty} :\ t_i\ <\ T_*$, with $t_i \to T_*,\ as\ i \to \infty$, such that for all $\tau \in [t_i, T_*)$

$$\delta_i(a(1-r_1-p_3)G(\tau))^{\frac{q_4}{1-r_1-p_3}} \leq U^{q_4}(x,t) \leq \gamma_i(a(1-r_1-p_3)G(\tau))^{\frac{q_4}{1-r_1-p_3}}. \quad (2.11)$$

Denote $T_i^* = \max\{t_i, \widetilde{T}_i\}$, then (2.11) and case (i) of Lemma 2.3 assert that for all $T_i^* \leq \tau < T_*$

$$F'(\tau) \geq \delta_i \mid \Omega \mid (a(1-r_1-p_3)G(\tau))^{\frac{q_4}{1-r_1-p_3}}$$

$$\geq \delta_i \mid \Omega \mid (\frac{\epsilon_i\delta_i}{\gamma_i})^{\frac{q_4}{1+q_4-r_1-p_3}} (\frac{a(1+q_4-r_1-p_3)}{b(1+q_3-r_2-p_4)})^{\frac{q_4}{1+q_4-r_1-p_3}} (b(1-r_2-p_4)F(\tau))^{\frac{q_4(1+q_3-r_2-p_4)}{(1+q_4-r_1-p_3)(1-r_2-p_4)}}, \quad (2.12)$$

$$F'(\tau) \leq$$

$$\gamma_i \mid \Omega \mid (\frac{\gamma_i}{\epsilon_i\delta_i})^{\frac{q_4}{1+q_4-r_1-p_3}} (\frac{a(1+q_4-r_1-p_3)}{b(1+q_3-r_2-p_4)})^{\frac{q_4}{1+q_4-r_1-p_3}} (b(1-r_2-p_4)F(\tau))^{\frac{q_4(1+q_3-r_2-p_4)}{(1+q_4-r_1-p_3)(1-r_2-p_4)}}. \quad (2.13)$$

Notice that $1-\frac{q_4(1+q_3-r_2-p_4)}{(1+q_4-r_1-p_3)(1-r_2-p_4)} = -\frac{1}{\sigma(1-p_2)} < 0$, where $\sigma = \frac{1+q_2-p_1}{q_1q_2-(1-p_1)(1-p_2)}$ is defined in Theorem 1.1. Integrating (2.12) and (2.13), we obtain that for all $T_i^* \leq \tau < T_*$

$$\begin{aligned} c_i \mid \Omega \mid \tfrac{b}{n\sigma}(\tfrac{a(1+q_4-r_1-p_3)}{b(1+q_3-r_2-p_4)})^{\frac{q_4}{1+q_4-r_1-p_3}} &\leq (T_*-\tau)^{-1}(b(1-r_2-p_4)F(\tau))^{\frac{-1}{\sigma(1-p_2)}} \\ &\leq C_i \mid \Omega \mid \tfrac{b}{n\sigma}(\tfrac{a(1+q_4-r_1-p_3)}{b(1+q_3-r_2-p_4)})^{\frac{q_4}{1+q_4-r_1-p_3}}, \end{aligned} \quad (2.14)$$

where $c_i = \delta_i(\frac{\epsilon_i\delta_i}{\gamma_i})^{\frac{q_4}{1+q_4-r_1-p_3}},\ C_i = \gamma_i(\frac{\gamma_i}{\epsilon_i\delta_i})^{\frac{q_4}{1+q_4-r_1-p_3}}$.

By letting $i \to \infty$ in (2.14), we can deduce that

$$(b(1-r_2-p_4)F(\tau))^{\frac{-1}{\sigma(1-p_2)}} \sim \frac{b}{n\sigma}(\frac{a(1+q_4-r_1-p_3)}{b(1+q_3-r_2-p_4)})^{\frac{q_4}{1+q_4-r_1-p_3}}(T_*-\tau).$$

In view of $1-r_2-p_4 = (1-p_2)/n$, it follows from case (i) of Lemma (2.2) that

$$(T_*-\tau)^{\sigma}V^{1/n}(x,\tau) \sim (\frac{b}{n\sigma} \mid \Omega \mid)^{-\sigma}(\frac{a(1+q_4-r_1-p_3)}{b(1+q_3-r_2-p_4)})^{\frac{-q_4\sigma}{1+q_4-r_1-p_3}} \quad (2.15)$$

holds uniformly on any compact subset of Ω.

Similarly as above, it can be inferred that

$$(T_*-\tau)^\theta U^{1/m}(x,\tau)\sim(\frac{a}{m\theta}\mid\Omega\mid)^{-\theta}(\frac{b(1+q_3-r_2-p_4)}{a(1+q_4-r_1-p_3)})^{\frac{-q_3\theta}{1+q_3-r_2-p_4}} \qquad (2.16)$$

holds uniformly on any compact subset of Ω, where $\theta=\frac{1+q_1-p_2}{q_1q_2-(1-p_1)(1-p_2)}$.

Combining (2.15), (2.16) with the transform about $(u(x,t),\ v(x,t))$, We can draw the conclusion of case (i) of Theorem 1.1.

(ii) $p_1=1$ *or* $p_2=1$. we divide this case into three subcases (a) $p_1<1,\ p_2=1$, (b) $p_1=1,\ p_2<1$, (c) $p_1=p_2=1$. We first discuss subcase (a).

Analogous to the beginning of the proof of case (i), it follows from case (ii) of Lemma 2.2 and case (ii) of Lemma 2.3 that for all $T_i^*\le\tau<T_*$

$$\begin{aligned} F'(\tau) &\ge \delta_i\mid\Omega\mid(a(1-r_1-p_3)G(\tau))^{\frac{q_4}{1-r_1-p_3}}\\ &\ge \delta_i\mid\Omega\mid(\frac{\epsilon_i}{\tau_i\delta_i})^{\frac{q_4}{1-r_1-p_3}}(\frac{a(1+q_4-r_1-p_3)}{bq_3})^{\frac{q_1}{1+q_1-p_2}}\exp(\frac{bq_3q_4\delta_i}{1+q_4-r_1-p_3}F(\tau)), \end{aligned}$$

$$F'(\tau)\le\gamma_i\mid\Omega\mid(\gamma_i\epsilon_i\delta_i)^{-\frac{q_4}{1-r_1-p_3}}(\frac{a(1+q_4-r_1-p_3)}{bq_3})^{\frac{q_1}{1+q_1-p_2}}\exp(\frac{bq_3q_4\delta_i}{1+q_4-r_1-p_3}F(\tau)).$$

Application of similar analysis as in case (i), we get

$$\lim_{\tau\to T_*}bF(\tau)\mid\log(T^*-t)\mid^{-1}=\frac{1+q_4-r_1-p_3}{q_3q_4}. \qquad (2.17)$$

Since $\delta_i\ ,\epsilon_i,\gamma_i\to1,\ as\ i\to\infty$ and $G(\tau),\ F(\tau)\to\infty,\ as\ \tau\to T_*$, then by case (ii) of Lemma 2.3

$$\lim_{\tau\to T_*}\frac{bF(\tau)}{\log(a(1-r_1-p_3)G(\tau))}=\frac{1+q_4-r_1-p_3}{(1-r_1-p_3)q_3}.$$

Hence

$$\lim_{\tau\to T_*}\log(a(1-r_1-p_3)G(\tau))\mid\log(T_*-\tau)\mid^{-1}=\frac{1-r_1-p_3}{q_4}. \qquad (2.18)$$

By joining (2.17), (2.18) and case (ii) of Lemma 2.2, we have

$$\log V(x,\tau)\sim bF(\tau)\sim\frac{1+q_4-r_1-p_3}{q_3q_4}\mid\log(T_*-\tau)\mid, \qquad (2.19)$$

$$\log U(x,t)\sim\frac{1}{1-r_1-p_3}\log(a(1-r_1-p_3)G(\tau))\sim\frac{1}{q_4}\mid\log(T_*-\tau)\mid \qquad (2.20)$$

uniformly on any compact subset of Ω.

The corresponding conclusion in Theorem 1.1 of subcase (a) can be directly drawn by combining (2.19), (2.20) with the transformation about $(u(x,t),\ v(x,t))$.

Finally, we can verify subcase (b) and (c) by similar means of subcase (a) and case (i). So we complete the proof of Theorem 1.1. □

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