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Harmonic Morphisms from Conformally Flat Spaces[1]

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Abstract

In this note we give a method to construct non-trivial harmonic morphisms via conformal change of the metric of the domain generalizing a theorem previously only known in the case of start manifold to be an open subset of $\mathbb{C}^2$. As its application, we manufacture harmonic morphisms from conformally flat spaces.

Keywords: horizontally conformal map, harmonic morphism, conformally flat space

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1 Preliminaries

Harmonic morphisms between Riemannian manifolds are mappings, which preserve solutions of Laplace's equation. They form a special class of harmonic maps, namely those that are horizontally conformal.

Call a smooth map $\phi : (M, g) \to (N, h)$ between Riemannian manifolds is *horizontally (weakly) conformal* if for any point $x \in M$ which is not contained in the critical set $C_\phi = \{x \in M \mid d\phi_x = 0\}$ of ϕ, the restriction of $d\phi_x$ to the orthogonal complement

$$\mathcal{H}_x = \{X \in T_x M \mid g(X, Y) = 0 \text{ for all } Y \in \operatorname{Ker} d\phi_x\}$$

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of Ker $d\phi_x$ is surjective and conformal onto the tangent space $T_{\phi(x)}N$.

Recall that a smooth map $f : M \to N$ between Riemannian manifold is *harmonic* if and only if it has vanishing tension field, equivalently, it is a critical point of its energy functional [1].

A smooth map $f : M \to N$ between Riemannian manifold is called a *harmonic morphism* if for any harmonic function $\psi : U \to \mathrm{R}$ defined on an open subset U of N with $f^{-1}(U)$ non-empty, $\psi \circ f : f^{-1}(U) \to \mathrm{R}$ is a harmonic function. The reader is referred to [2] for a detailed account of harmonic morphisms. Harmonic morphisms can be characterized as follows:

Theorem 1.1([2, 3]) *A map $\phi : M \to N$ between Riemannian manifolds is a harmonic morphism if and only if it is a horizontally (weakly) conformal harmonic map.*

2 Harmonic morphisms with respect to a conformally altered metric

In this section we extract a sufficient condition for $\varphi : (M^{2n}, e^{2\eta}g) \to (N^2, h)$ to be harmonic.

Recall that an almost Hermitian manifold (M, g, J) with Kähler form ω, is said to be *cosymplectic* if $d^*\omega = 0$ or equivalently, $\mathrm{div}J = 0$.

Theorem 2.1. *Assume that (M^{2n}, g, J) is a cosymplectic manifold and that η is a real valued function defined in M^{2n}. If φ is a holomorphic map from M^{2n} into some Riemann surface (N^2, h, J^N) satisfying $d\varphi(\mathrm{grad}\eta) = 0$, then*

$$\varphi : (M^{2n}, e^{2\eta}g) \to (N^2, h)$$

is a harmonic morphism.

Proof. The well-known result by Lichnerowicz tells us that holomorphic map from a cosymplectic manifold to a (1, 2)-symplectic manifold is harmonic [5]. Note that an almost Hermitian manifold with Kähler form ω, is said to be (1, 2)-*symplectic* if the (1, 2)-part of $d\omega$ vanishes, and any Riemann surface is automatically (1, 2)-symplectic.

Recall from the Cauchy-Riemann equations that any holomorphic map from an almost Hermitian manifold to a Riemann surface is horizontally weakly conformal. Combining this with Lichnerowicz's result and Fuglede-Ishihara' characterization [2, 3], we obtain that a holomorphic map φ from cosymplectic manifold (M^{2n}, g, J) to Riemann surface (N^2, h, J^N) is a harmonic morphism. Setting $\tilde{g} = e^{2\eta}g$. It is easy to verify that $\varphi : (M^{2n}, \tilde{g}) \to (N^2, h)$ is horizontally weakly conformal. Moreover, by Theorem 5.1 of [8], $\varphi : (M^{2m}, \tilde{g}) \to$

$(N^2,\, h)$ is harmonic if and only if *grad* $\left[(e^{-\eta})^{n-2}\right]$ is vertical on $M\backslash C_\varphi$ where grad denotes the gradient of function. This is equivalent to $d\varphi(\text{grad}\eta) = 0$. On the other hand, if x is a interior point of C_φ, then there is an open subset U of M, such that $x \in U \subset C_\varphi$, and $\tau(\varphi)(x) = Trace_{\tilde{g}} \tilde{\nabla} d\varphi(x) = 0$ where $\tau(\varphi)$ is the tension field with respect to $\tilde{g}$. Suppose that x is a condensation point of C_φ. Then there exists a sequence x_j $(j = 1,\, 2,\, \cdots)$ on $M\backslash C_\varphi$ such that $\lim_{j\to+\infty} x_j = x$. Because $\tau(\varphi)$ is a smooth field along φ, we get

$$0 = \lim \tau(\varphi)(x_j) = \tau(\varphi)(x)$$

To sum up, we have $\tau(\varphi) = 0$, hence, φ is harmonic if $d\varphi(\text{grad}\eta) = 0$. Therefore φ is a harmonic morphism by Fuglede-Ishihara' result [2, 3]. □

Remark 2.2 Theorem 2.1 is a natural extension of Theorem 3.1 of [10].

3 Harmonic morphisms from conformaly flat spaces

Two Riemannian metrics g and $\bar{g}$ on M are said to be *conformally equivalent*, if there exists a function ψ on M such that $\bar{g} = e^{2\psi} g$. A map $\varphi : (M,\, g) \to (N,\, h)$ between Riemannian manifolds is said to be *conformal* if there exists a function ψ on M such that $\varphi^* h = e^{2\psi} g$. Two Riemannian manifolds $(M,\, g)$ and $(N,\, h)$ are said to be *conformally diffeomorphic*, if there exists a conformal diffeomorphism $\varphi : (M,\, g) \to (N,\, h)$. An n-dimensional Riemannian manifold $(M,\, g)$ is called a *conformaly flat space* if for any point of M there is a neighborhood which is conformally diffeomorphic to the Euclidean space R^n.

We shall construct a harmonic morphism from $\mathbb{R}^{2m}$, with a suitable conformally flat metric, to $\mathbb{R}^2$. Let $k_1,\, \cdots,\, k_m$ be non-negative integers which are not all zero, and let $\varphi : \mathbb{R}^{2m} \to \mathbb{R}^2$ be the polynomial map, homogeneous of degree $k_1 + \cdots + k_m$, defined in complex coordinates by

$$\varphi(z) = z_1^{k_1} z_2^{k_2} \cdots z_m^{k_m} \tag{3.1}$$

$$(z = (z_1,\, \cdots,\, z_m) \in \mathbb{C} \times \cdots \times \mathbb{C} = \mathbb{R}^{2m})$$

For any i, $\dfrac{\partial \varphi}{\partial \bar{z}_i} = 0$ implies that φ is holomorphic.

Consider real valued function in $\mathbb{C}^m$. Then

$$\text{grad}\eta = \Sigma_i \left(\frac{\partial \eta}{\partial z_i} \frac{\partial}{\partial \bar{z}_i} + \frac{\partial \eta}{\partial \bar{z}_i} \frac{\partial}{\partial z_i} \right).$$

Note that φ is holomorphic,

$$\begin{aligned} d\varphi(\mathrm{grad}\eta) &= \Sigma_i\left(\frac{\partial\eta}{\partial z_i}\frac{\partial\varphi}{\partial\bar{z}_i}+\frac{\partial\eta}{\partial\bar{z}_i}\frac{\partial\varphi}{\partial z_i}\right)\\ &= \Sigma_i\frac{\partial\eta}{\partial\bar{z}_i}\frac{\partial\varphi}{\partial z_i}\\ &= \Sigma_i\frac{\partial\eta}{\partial\bar{z}_i}k_i z_i^{k_i-1}z_1^{k_1}\cdots\hat{z}_i^{k_i}\cdots z_m^{k_m}\\ &= \left(\prod_{i=1}^m z_i^{k_i-1}\right)\left(\Sigma_{j=1}^m k_j z_1\cdots\hat{z}_j\cdots z_m\frac{\partial\eta}{\partial\bar{z}_j}\right). \end{aligned}$$

Thus the equation $d\varphi(\mathrm{grad}\eta)=0$ is equivalent to

$$\left(\prod_{i=1}^m z_i^{k_i-1}\right)\left(\Sigma_{j=1}^m k_j z_1\cdots\hat{z}_j\cdots z_m\frac{\partial\eta}{\partial\bar{z}_j}\right)=0.$$

A solution to this equation is given by

$$\eta(z)=\begin{cases} \Sigma_{i=1}^{m/2}(k_{2i}|z_{2i-1}|^2-k_{2i-1}|z_{2i}|^2) & \text{if } m \text{ is even}\\ k_2k_3|z_1|^2-\frac{k_1}{2}(k_3|z_2|^2+k_2|z_3|^2) & \text{if } m \text{ is odd.}\\ +\Sigma_{i=2}^{(m-1)/2}(k_{2i+1}|z_{2i}|^2-k_{2i}|z_{2i+1}|^2) & \end{cases} \tag{3.2}$$

In fact, when m is even, then

$$\eta(z_1,\cdots,z_m)=\Sigma_{i=1}^{m/2}(k_{2i}z_{2i-1}\bar{z}_{2i-1}-k_{2i-1}z_{2i}\bar{z}_{2i}).$$

It follows that
$$\frac{\partial\eta}{\partial\bar{z}_{2i-1}}=k_{2i}z_{2i-1},\qquad \frac{\partial\eta}{\partial\bar{z}_{2i}}=-k_{2i-1}z_{2i}.$$

Thus we have

$$\begin{aligned} &\Sigma_{j=1}^m k_j z_1\cdots\hat{z}_j\cdots z_m\frac{\partial\eta}{\partial\bar{z}_j}\\ =\ &\Sigma_{j=1}^{m/2}k_{2j}z_1\cdots\hat{z}_{2j}\cdots z_m\frac{\partial\eta}{\partial\bar{z}_{2j}}+\Sigma_{j=1}^{m/2}k_{2j-1}z_1\cdots\hat{z}_{2j-1}\cdots z_m\frac{\partial\eta}{\partial\bar{z}_{2j-1}}\\ =\ &\Sigma_{j=1}^{m/2}k_{2j}z_1\cdots\hat{z}_{2j}\cdots z_m(-k_{2j-1}z_{2j})\\ &+\Sigma_{j=1}^{m/2}k_{2j-1}z_1\cdots\hat{z}_{2j-1}\cdots z_m(k_{2j}z_{2j-1})\\ =\ &-\Sigma_{j=1}^{m/2}k_{2j}k_{2j-1}\prod_{i=1}^m z_i+\Sigma_{j=1}^{m/2}k_{2j}k_{2j-1}\prod_{i=1}^m z_i=0. \end{aligned}$$

It follows that $d\varphi(\mathrm{grad}\eta)=0$. If m is odd, then

$$\begin{aligned} \eta(z_1,\cdots,z_m)=\ &k_2k_3z_1\bar{z}_1-\frac{k_1}{2}(k_3z_2\bar{z}_2+k_2z_3\bar{z}_3)\\ &+\Sigma_{i=2}^{(m-1)/2}(k_{2i+1}z_{2i}\bar{z}_{2i}-k_{2i}z_{2i+1}\bar{z}_{2i+1}). \end{aligned}$$

It follows that

$$\frac{\partial\eta}{\partial\bar{z}_1} = -k_2k_3z_1, \quad \frac{\partial\eta}{\partial\bar{z}_2} = -\frac{k_1k_3}{2}z_2, \quad \frac{\partial\eta}{\partial\bar{z}_3} = -\frac{k_1k_2}{2}z_3$$

and when $i \geq 2$,

$$\frac{\partial\eta}{\partial\bar{z}_{2i}} = k_{2i+1}z_{2i}, \quad \frac{\partial\eta}{\partial\bar{z}_{2i+1}} = -k_{2i}z_{2i+1}.$$

Thus we have

$$\begin{aligned}
& \Sigma_{j=1}^{m} k_j z_1 \cdots \hat{z}_j \cdots z_m \frac{\partial\eta}{\partial\bar{z}_j} \\
= \; & k_1z_2 \, \cdots \, z_m \frac{\partial\eta}{\partial\bar{z}_1} + k_2z_1z_3 \, \cdots \, z_m \frac{\partial\eta}{\partial\bar{z}_2} + k_3z_1z_2z_4 \, \cdots \, z_m \frac{\partial\eta}{\partial\bar{z}_3} \\
& + \Sigma_{j=2}^{(m-1)/2} k_{2j} z_1 \cdots \hat{z}_{2j} \cdots z_m \frac{\partial\eta}{\partial\bar{z}_{2j}} + \Sigma_{j=2}^{(m-1)/2} k_{2j+1} z_1 \cdots \hat{z}_{2j+1} \cdots z_m \frac{\partial\eta}{\partial\bar{z}_{2j+1}} \\
= \; & k_1z_2 \, \cdots \, z_m k_2k_3z_1 + k_2z_1z_3 \, \cdots \, z_m(-\frac{k_1k_3}{2}z_2) + k_3z_1z_2z_4 \, \cdots \, z_m(-\frac{k_1k_2}{2}z_3) \\
& + \Sigma_{j=2}^{(m-1)/2} k_{2j} z_1 \cdots \hat{z}_{2j} \cdots z_m k_{2j+1} z_{2j} \\
& + \Sigma_{j=2}^{(m-)/2} k_{2j+1} z_1 \cdots \hat{z}_{2j+1} \cdots z_m(-k_{2j}z_{2j+1}) = 0.
\end{aligned}$$

We obtain that $d\varphi(\text{grad}\eta) = 0$ if m is odd.

By using Theorem 2.1, we obtain the following

Proposition 3.1 *Let $\varphi : (\mathbb{R}^{2m}, g_0) \to \mathbb{R}^2$ be the polynomial map defined in (3.1) where g_0 is the standard Riemannain metric on $\mathbb{R}^{2m}$. Then φ is a harmonic morphism from conformally flat space $(\mathbb{R}^{2m}, e^{\eta}g_0)$ to $\mathbb{R}^2$ where η is defined in (3.2).*

Remark 3.2 For a different approach to the same problem where $m = 2$, using isoparametric functions, see [1], page 404.

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