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AN APPLICATION OF THE DOUBLE SUMUDU TRANSFORM

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Abstract

The *double Sumudu transform* of functions expressible as polynomials or convergent infinite series are derived. The relationship of the former to the double Laplace transform is obtained, and it turns out that they are theoretical dual to each other. The applicability of this relatively new transform is demonstrated using some special functions, which arise in the solution of evolution equations of population dynamics as well as partial differential equations.

Keywords: Laplace transform, Sumudu transform, population dynamics, partial differential equations

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1. Introduction

A lot of work has been done on the theory and applications of transforms such as Laplace, Fourier, Mellin, Hankel, to name a few, but very little on the power series transformation or Sumudu transform, probably because it is little known, and not widely used [3]. An ingenious solution to visualizing the Sumudu transform was proposed originally by Watugala [6], and subsequently exploited by Weerakoon [8,9] and Asiru [1,2]. Nevertheless, this new transform rivals the Laplace transform in problem solving [3]. Its main advantage

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is the fact that it may be used to solve problems without resorting to a new frequency domain, because it preserves scales and units properties. Indeed, The Sumudu transform, is still not widely known, nor used. Having scale and unit-preserving properties, the Sumudu transform may be used to solve intricate problems in engineering mathematics and applied sciences [10] without resorting to a new frequency domain.

As shown in [6], the theory of Sumudu transform, defined for functions of exponential order, is adequate for many applications in mathematics (ordinary and partial differential equations), and control engineering problems. Belgacem *et al.,* [3], showed that the Sumudu transform has deeper connections with the Laplace transform. Our approach is somewhat different as we consider its double transform counterpart. Watugala [7] extended the transform to functions of two variables with emphasis on solutions to partial differential equations, but his approach is slightly different from ours. Thus, the aim of this paper is to derive, we believe for the first time, the double Sumudu transform. Motivated by the study carried in [5], where the double Laplace transform is used to solve an age-physiology dependent population model, and taking cognisance of the fact that the Sumudu transform is the theoretical dual of the Laplace transform, we ask: - *Will the double Sumudu transform also rivals its double Laplace dual in common differential equations problem solving*? Our interest also comes from the fact that this new transform can certainly treat most problems (may they be of integral, differential, or engineering control nature) that are usually treated by the well-known and extensively used Laplace transform [3]. Since this new transform has very special and useful properties, and can help with intricate applications in sciences and engineering, we do believe its double transform will also be a natural choice in solving problems with scale and units preserving requirements. Therefore, our goal is to apply the double transform to the age- and physiology-dependent population dynamic problem studied in [5].

We recall that the Laplace transform is defined by,

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)dt, \quad Re(s) > 0, \tag{1.1}$$

while the Sumudu transform is defined over the set of functions

$$A = \{f(t)|\exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{\frac{t}{\tau_j}}, if t \in (-1)^j \times [0, \infty), \tag{1.2}$$

by

$$F(u) = S[f(t)] = \int_0^\infty f(ut)e^{-t}dt, \quad u \in (-\tau_1, \tau_2). \tag{1.3}$$

In the sequel, and by analogy with the double Laplace transform, we shall denote the double Sumudu transform by S_2.

2. The Double Sumudu Transform

The Sumudu transform offers a straightforward and relatively uncluttered way of introducing the double Sumudu transform, provided the function has a power series transformation with respect to its variables. The double Laplace transform of a function of two variables defined in the positive quadrant of the $xy-$plane [see 4] is given by

$$\mathcal{L}_2[f(x,y);(p,q)] = \int_0^\infty \int_0^\infty f(x,y)e^{-(px+qy)}dxdy,$$

where p and q are the transform variables for x and y, respectively.

Definition 2.1 Let $f(t,x);\, t,x \in \mathbb{R}_+$ be a function which can be expressed as a convergent infinite series, then, its double Sumudu transform is given by

$$\begin{aligned} F(u,v) = S_2[f(t,x);(u,v)] &= S[S\{f(t,x);t \to u\};x \to v] \\ &= S\left[\left\{\frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}f(t,x)dt\right\};x \to v\right] \\ &= \frac{1}{uv}\int_0^\infty \int_0^\infty e^{-\left(\frac{t}{u}+\frac{x}{v}\right)}f(t,x)dtdx. \end{aligned} \tag{2.1}$$

We present the applications of the double Sumudu transform to some special functions, similar to those obtained in the solution of equations of population dynamics with age structure. However, it is a trivial exercise to show that the double Sumudu and Laplace transforms are also theoretical dual. That is;

$$uvF(u,v) = \mathcal{L}_2\left[f(x,y);\left(\frac{1}{u},\,\frac{1}{v}\right)\right], \tag{2.2}$$

where $\mathcal{L}_2$ represents the operation of double Laplace transform.

Theorem 2.2 Let $f(x,y), x, y \in \mathbb{R}_+$ be a real-valued function, then,

$$S_2[f(x+y);(u,v)] = \frac{1}{u-v}\{uF(u) - vF(v)\}. \tag{2.3}$$

The case $f(x-y)$ is more interesting from the biological point of view where such functions are frequently encountered in mathematical biology, with f representing the population density, x the age, and y the time, or vice-versa. The proof for the case $x \geq y$ is simple and robust enough, but with a tedious manipulation.

We limit ourselves to the first quadrant because negative populations are biologically irrelevant. Thus, geometrically, if the line separating the first quadrant into two equal parts represents the η-axis (the lower part being represented by Q_1, and the upper part Q_2), while that separating both the second and fourth quadrants represents the ζ-axis (arrow pointing upwards) and ξ-axis (arrow from origin into the fourth quadrant), respectively, then the proof is as follows:

Let f be an even function, then

$$\begin{aligned} S_2[f(x-y);(u,v)] &= \frac{1}{uv}\int\int_{Q_1} f(x-y)e^{-(\frac{x}{u}+\frac{y}{v})}dxdy \\ &+\frac{1}{uv}\int\int_{Q_2} f(x-y)e^{-(\frac{x}{u}+\frac{y}{v})}dxdy, \end{aligned} \tag{2.4}$$

while for $f(\cdot)$ odd, we have

$$\begin{aligned} S_2[f(x-y);(u,v)] &= \frac{1}{uv}\int\int_{Q_1} f(x-y)e^{-(\frac{x}{u}+\frac{y}{v})}dxdy \\ &-\frac{1}{uv}\int\int_{Q_2} f(x-y)e^{-(\frac{x}{u}+\frac{y}{v})}dxdy. \end{aligned} \tag{2.5}$$

Let $x = \frac{1}{2}(\xi+\eta);\ y = \frac{1}{2}(\xi-\eta)$, then,

$$\begin{aligned} \int\int_{Q_1} f(x-y)e^{-(\frac{x}{u}+\frac{y}{v})}dxdy &= \frac{1}{2}\int_0^\infty f(\xi)d\xi\int_\xi^\infty e^{-\frac{1}{2}(\frac{1}{u}+\frac{1}{v})\xi-\frac{1}{2}(\frac{1}{u}-\frac{1}{v})\eta}d\eta \\ &= \frac{uv}{u-v}\int_0^\infty e^{-\frac{\xi}{v}}f(\xi)d\xi \\ &= \frac{vu^2}{u-v}F(u). \end{aligned} \tag{2.6}$$

Similarly,

$$\int\int_{Q_2} f(x-y)e^{-(\frac{x}{u}+\frac{y}{v})}dxdy = \frac{v^2u}{u-v}F(v). \tag{2.7}$$

Hence, for f even,

$$S_2[f(x-y);(u,v)] = \frac{uF(u)+vF(v)}{u-v}, \tag{2.8}$$

and for odd functions

$$S_2[f(x-y);(u,v)] = \frac{uF(u)-vF(v)}{u+v}. \tag{2.9}$$

From equations (2.3) and (2.9), it is obvious that if f is an even function, then

$$(u+v)S_2[f(x-y)] = (u-v)S_2[f(x+y)]. \tag{2.10}$$

We state without proof the following Lemmas.

Lemma 2.3 Let f and g be two real-valued functions satisfying (2.1), then

(i) $S_2[f(ax)g(by);(u,v)] = F(au)G(bv)$,

(ii) $S_2[f(ax,by);(u,v)] = F(au,bv)$,

where a and b are positive constants (the double Sumudu transform like its single counterpart is scale preserving).

Lemma 2.4 Let $f(x)$, $x \in \mathbb{R}_+$ and $H(\cdot)$ represents the Heaviside function then

(i) $S_2[f(x)H(x-y);(u,v) = F(u) + (\frac{w}{v}-1)F(w)$, where $w = \dfrac{uv}{u+v}$, *and*

(ii) $S_2[f(x)H(y-x);(u,v) = \left(1-\frac{w}{v}\right)F(w)$.

The proof is simple, for instance, by rewriting the left-hand side of the equation in (ii) as

$$\frac{1}{uv}\int_0^\infty f(x)e^{-\frac{x}{u}}\int_x^\infty e^{-\frac{y}{v}}dxdy,$$

and performing the integrations, bearing in mind that f satisfies Fubini's Theorem, the result follows. □

Corollary 2.5

(i) $S_2[H(x-y);(u,v)] = \dfrac{v}{u+v}$.

(ii) $S_2[H(y-x);(u,v)] = \dfrac{u}{u+v}$.

Consequently,

$$S_2[f(x)H(y-x)] + S_2[f(x)H(x-y)] = F(u). \tag{2.11}$$

The *application of the double Sumudu transform to partial derivatives* is as follows: Let $f(0,a) = F_0(a)$, then

$$\begin{aligned} S_2\left[\frac{\partial f}{\partial t} f(t,a);(u,v)\right] &= \frac{1}{uv}\int_0^\infty \int_0^\infty e^{-\left(\frac{t}{u}+\frac{a}{v}\right)} \frac{\partial}{\partial t} f(t,a)dtda \\ &= \frac{1}{v}\int_0^\infty e^{-\frac{a}{v}} \left\{\frac{1}{u}\int_0^\infty e^{-\frac{t}{u}} \frac{\partial}{\partial t} f(t,a)dt\right\} da. \end{aligned}$$

The inner integral gives (see [6]),

$$\frac{F(u,a) - f(0,a)}{u}, \tag{2.12}$$

$$\begin{aligned} \therefore\ S_2\left[\frac{\partial f(t,a)}{\partial t};(u,v)\right] &= \frac{1}{u}\left\{\frac{1}{v}\int_0^\infty e^{-\frac{a}{v}} F(u,a)da - \frac{1}{v}\int_0^\infty e^{-\frac{a}{v}} f_0(a)da\right\} \\ &= \frac{1}{u}\{F(u,v) - F_0(v)\}. \end{aligned} \tag{2.13}$$

Also,

$$\begin{aligned} S_2\left[\frac{\partial f(t,a)}{\partial a};(u,v)\right] &= \frac{1}{v}\int_0^\infty e^{-\frac{a}{v}} \left\{\frac{1}{u}\int_0^\infty e^{-\frac{t}{u}} \frac{\partial}{\partial a} f(t,a)dt\right\} da \\ &= \frac{1}{v}\int_0^\infty e^{-\frac{a}{v}} \frac{d}{da} F(u,a)da \\ &= F_v(u,v), \end{aligned} \tag{2.14}$$

alternatively,

$$\begin{aligned} S_2\left[\frac{\partial f(t,a)}{\partial a};(u,v)\right] &= \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}} \left(\frac{1}{v}\int_0^\infty e^{-\frac{a}{v}} \frac{\partial f}{\partial a} da\right) dt \\ &= \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}} \frac{1}{v}[F(t,v) - f(t,0)]dt \\ &= \frac{1}{v}(F(u,v) - F_0(u)), \end{aligned} \tag{2.15}$$

where $F(u,0) = F_0(u)$ and $F(0,v) = F_0(v)$. It is obvious from equations (2.14) and (2.15) that,

$$F_v(u,v) = \frac{F(u,v) - F_0(u)}{v}.$$

If u and v are equal, we obtain a special case of the double Sumudu transform, herein referred to as the iterated Sumudu transform, by analogy to the iterated Laplace transform [3]. Thus, the iterated Sumudu transform of any given function of two variables $f(x,y)$, say, is defined by

$$J[f(x,y);(u,u)] = S_2[f(x,y);(u,u)] = \frac{1}{u^2}\int_0^\infty\int_0^\infty e^{-\left(\frac{x+y}{u}\right)} f(x,y)dxdy. \tag{2.16}$$

Therefore, the Sumudu transform of the general convolution function $f^{(2)}(x) := \int_0^x f(x-y,y)dy$ is related to the iterated Sumudu transform as follows:

$$S[f^{(2)}(x);u] = \frac{1}{u}\int_0^\infty e^{-\frac{x}{u}} f^{(2)}(x)dx = \frac{1}{u}\int_0^\infty e^{-\frac{x}{u}}\int_0^x f(x-y,y)dxdy. \tag{2.17}$$

By letting $x = s+t$ and $y = t$ in equation (2.17), we obtain

$$S[f^{(2)}(x)] = \frac{1}{u}\int\int_\Omega f(s,t)e^{-\left(\frac{s+t}{u}\right)}dsdt = uS_2[f(s,t);(u,u)] = uJ[f(s,t);(u,u)], \tag{2.18}$$

with the convolution integral taken in the classical sense, and $x, y, s, t \in \mathbb{R}^+$.

3. Applications

In this section, we establish the validity of the double Sumudu transform by applying it to an evolution equation of population dynamic, namely the famous Kermack-Mackendrick Von Foerster type model. Let f be the population density of individuals aged a at time t, λ the death modulus. Then population evolves according to the following system

$$\begin{aligned} &f_t + f_a + \lambda(a)f = 0, \\ &f(0,a) = f_0(a), \\ &f(t,0) = B(t). \end{aligned} \tag{4.1}$$

Taking the double Sumudu transform of equation (4.1) with with u, v as the transform variables for t, a, respectively, after some little arrangements, we obtain

$$[F(u,v);(t,a)] = \left[\frac{vF_0(v) + uF_0(u)}{u+v+\lambda uv}\right] \tag{4.2}$$

In order to find the inverse double Sumudu transform of equation (4.2), which we assume it exists and satisfy conditions of existence of the double Laplace transform, we proceed as follows: Let the right-hand side of equation (4.2) be written as $\frac{vF_0(v)+uF_0(u)}{u+v+\lambda uv} = \frac{vF_0(v)+uF_0(u)}{(u+v)(1+\frac{\lambda uv}{u+v})}$, then, taking the inverse double Sumudu transform of (4.2) using Corollary (2.5) and Lemma (1.3(i)), we have:

$$\begin{aligned} S_2^{-1}[F(u,v);(t,a)] &= S_2^{-1}\left[\frac{vF_0(v)+uF_0(u)}{(u+v)(1+\frac{\lambda uv}{u+v})}\right] \\ &= S_2^{-1}[\frac{v}{u+v}F_0(v)+\frac{u}{u+v}F_0(u)]S_2^{-1}(1+\frac{\lambda uv}{u+v})^{-1} \\ &\leq H(t-a)B(t-a)e^{(-\lambda a)}+H(a-t)f_0(a-t)e^{(-\lambda t)} \quad (4.3) \end{aligned}$$

We obtain an approximate solution, but it is important to not here that the survival function $exp(-\lambda\cdot)$ does not disappear as in [5]. Thus, in order to obtain for instance $e^{(-\lambda a)}$, we assume without loss of reality that $u=1$ in the expansion, which gives us an approximation, whence the inequality in (4.3). Comparing the above result with that obtain in [5], we conclude that the double Sumudu transform also rivals his double Laplace counterpart.

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