Applied Mathematical Sciences, Vol. 1, 2007, no. 1, 41 - 50

# SOME RESULTS ON THE DIAMOND-$\alpha$ DYNAMIC DERIVATIVE ON TIME SCALES

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**Abstract**

Various dynamic derivative formulae have been proposed in the development of a time scales calculus, with the goal of unifying continuous and discrete analysis. Recent discussion of combined dynamic derivetives, in particular the $\diamond_\alpha$ derivative defined as a linear combination of the $\triangle$ and the $\bigtriangledown$ derivatives, have promised improved approximation formulae for computational aplication. In this article we study some result on the diamond-$\alpha$ dynamic derivative on time scales.

**Mathematics Subject Classification:** 34B10, 39A10, 65D25

**Keywords:** Dynamic derivatives, Combined formula, Linear combination

# 1 Introduction

Much of the development of time scales theory has focused on the unification of continuous and discrete analytical methods. Recent discussions have suggested that the theory and methods of time scales might also provide a means of integrating difference and differential methods for modeling nonlinear systems of dynamic equation on domains that are arbitrary nonempty closed subsets of the reals. To this end, the usefulness of various dynamic derivative formulae, including the standard $\triangle$ and $\nabla$ derivatives, in approximating functions and solution of nonlinear differential equations has been explored [2, 3, 5, 10]. It has been demonstrated in several recent papers [8-10] that a proposed dynamic derivative formula, called the $\diamond_\alpha$ derivative and defined as a linear combination, or the Broyden's formula [4, 12], of the $\triangle$ and the $\nabla$ dynamic derivatives, provides a more accurate approximation to the conventional derivative. The question remains, however, as to whether the $\diamond_\alpha$ derivative is a well-defined dynamic derivative upon which a calculus on time scale can be built.

James W. Profers Jr. and Qin Sheng redefined $\diamond_\alpha$ derivative independently of the standard $\triangle$ and $\nabla$ dynamic derivatives, and further examined its properties and relationship with the $\triangle$ and $\nabla$ formulae.

This paper give some results on the diamond-$\alpha$ dynamic derivative on time scales.

# 2 General Definitions

Here, first we mention several foundational definitions without proof and results from the calculus on time scales in an excellent introductory text by Bohner and Peterson [2, 3].

## 2.1 The delta and nabla derivatives

An one-dimensional time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$ and has the inherited topology. Let $a = \inf\ \mathbb{T}$ and $b = \sup \mathbb{T}$. For $t \in \mathbb{T}$

such that $a < t < b$, we define the *forward-jump operator,* $\sigma$, and *backward-jump operator*, $\rho$, a$\sigma(t) : \mathbb{T} \to \mathbb{T}$, is given by $\sigma(t) = \inf_{s\in\mathbb{T}} \{s > t\}$, $\rho(t) = \inf_{s\in\mathbb{T}} \{s < t\}$ respectively, and $\sigma(b) = b$, $\rho(a) = a$,

If $\mathbb{T}$ is bounded. The corresponding *forward-step* and *backward-step* functions $\mu, \eta$ are defined as $\mu(t) = \sigma(t) - t$, $\eta(t) = t - \rho(t)$, respectively. For a function $f$ defined on $\mathbb{T}$, to provide a shorthand notation we let

$f^{\sigma} = f(\sigma(t))$, $f^{\rho} = f(\rho(t))$.We say that a point $t \in \mathbb{T}$ is *right-scattered* if $\sigma(t) > t$ and *left-scattered* if $\rho(t) < t$. A point $t \in \mathbb{T}$ that is both right-scattered and left-scattered is called *scattered.* Also, we say that a point $t \in \mathbb{T}$ is *right-dense* if $\sigma(t) = t$, *left-scattered* if $\rho(t) = t$, and *dense* if it is both right-dense and left-dense.

We define $\mathbb{T}^k = \mathbb{T}\backslash \{b\}$ if $\mathbb{T}$ is bounded above and $b$ is left-scattered; otherwise $\mathbb{T}^k = \mathbb{T}$. Similarly, we define $\mathbb{T}_k = \mathbb{T}\backslash \{a\}$ if $\mathbb{T}$ is bounded below and $a$ is right-scattered; otherwise $\mathbb{T}^k = \mathbb{T}$. We denote $\mathbb{T}^k \cap \mathbb{T}_k$ by $\mathbb{T}^k_k$, $\mu(t) = \eta(t)$. A uniform time scale is an interval if $\mu(t) = 0$, and is a uniform difference grid if $\mu(t) > 0$.

We say a function $f$ defined on $\mathbb{T}$ is right continuous at $t \in \mathbb{T}$ if for all $\varepsilon > 0$ there is some $\delta > 0$ such that for all $s \in [t, t+\delta)$, $|f(t) - f(s)| < \varepsilon$. Similarly, we say that $f$ is left continuous at $t \in \mathbb{T}$ if for all $\varepsilon > 0$ there is some $\delta > 0$ such that for all $s \in [t-\delta, t)$, $|f(t) - f(s)| < \varepsilon$. The function $f(t)$ is said to be continuous if it is both right and left continuous.

For the sake of readability of subsequent formulas, we introduce the following notation. Let $t, s \in \mathbb{T}$ and define $\mu_{ts} = \sigma(t) - s$, $\eta_{ts} = \rho(t) - s$.

Let $f : \mathbb{T} \to \mathbb{R}$ be a function on a time scale. Then for $t \in \mathbb{T}^k$ we define $f^{\triangle}(t)$ to be the value, if one exists, such that for all $\varepsilon > 0$ there is a neighborhood $U$ of $t$ (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that for all $s \in \mathbb{U}$

$$\left|[f^{\sigma}(t) - f(s))] - f^{\triangle}(t)\left[\sigma(t) - s\right]\right| \le \varepsilon \left|\sigma(t) - s\right|.$$

We say that $f$ is *delta differentiable* on $\mathbb{T}^k$ provided $f^{\triangle}(t)$ exists for all $t \in \mathbb{T}^k$. Similarly, for $t \in \mathbb{T}^k$ we define $f^{\nabla}(t)$ to be the number, if one exists, such that for all $\varepsilon > 0$ there is a neighborhood $V$ of $t$ such that for all $s \in V$

$$\left|[f^{\rho}(t) - f(s))] - f^{\nabla}(t)\left[\sigma(t) - s\right]\right| \le \varepsilon \left|\rho(t) - s\right|.$$

We say that $f$ is *nabla differentiable* on $\mathbb{T}_k$ provided $f^{\nabla}(t)$ exists for all $t \in \mathbb{T}_k$.

In subsequence proofs, we will wish to make use of the following theorem due to Hilger [6], and the analogous theorem for the nabla case which can be found in [1, 2]:

**Theorem 1** *Assume $f:\mathbb{T}\to\mathbb{R}$ is a function and let $t\in\mathbb{T}^k$. Then we have the following:*

*(i) If $f$ is delta differentiable at $t$, then $f$ is continuous at $t$.*

*(ii) If $f$ is left continuous at $t$ and $t$ is right-scattered, then $f$ is delta differentiable at $t$ with*

$$f^{\triangle}(t)=\frac{f^{\sigma}(t)-f(t)}{\sigma(t)-t}.$$

*(iii) If $t$ is right-dense, then $f$ is delta differentiable at $t$ iff the limit*

$$\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$$

*exists as a finite number. In this case*

$$f^{\triangle}(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}.$$

**Theorem 2** *Asssume $f,g:\mathbb{T}\to\mathbb{R}$ are differentiable at $t\in\mathbb{T}^k$. then:*

*(i) The sum $f+g:\mathbb{T}\to\mathbb{R}$ is differentiable at $t$ with*

$$(f+g)^{\triangle}(t)=f^{\triangle}(t)+g^{\triangle}(t).$$

*(ii) For any constant $\alpha$, $\alpha f:\mathbb{T}\to\mathbb{R}$ is differentiable at $t$ with*

$$(\alpha f)^{\triangle}(t)=\alpha f^{\triangle}(t).$$

*(iii) The product $fg:\mathbb{T}\to\mathbb{R}$ is differentiable at $t$ with*

$$(fg)^{\triangle}(t)=f^{\triangle}(t)g(t)+f(\sigma(t))g^{\triangle}(t)=f(t)g^{\triangle}(t)+f^{\triangle}(t)g(\sigma(t).$$

*(iv) If $f(t)f(\sigma(t))\neq 0$, then $\frac{1}{f}$ is differentiable at $t$ with*

$$\left(\frac{1}{f}\right)^{\triangle}=-\frac{f^{\triangle}(t)}{f(t)f(\sigma(t))}.$$

*(v) If $g(t)g(\sigma(t))\neq 0$, then $\frac{f}{g}$ is differentiable at $t$ and*

$$\left(\frac{f}{g}\right)^{\triangle}=\frac{f^{\triangle}(t)g(t)-f(t)g^{\triangle}(t)}{f(t)f(\sigma(t))}.$$

***Example 1*** *(i) If $f:\mathbb{T}\to\mathbb{R}$ is defined by $f(t)=\alpha$ for all $t\in\mathbb{T}$, where $\alpha\in\mathbb{R}$ is constant, then $f^{\triangle}(t)\equiv 0$.*

*(ii) If $f:\mathbb{T}\to\mathbb{R}$ is defined by $f(t)=t$ for all $t\in\mathbb{T}$, then $f^{\triangle}(t)\equiv 1$*

***Example 2*** *The derivative of $t^2$ is $t+\sigma(t)$.*

**Theorem 3** *Assume $f:\mathbb{T}\to\mathbb{R}$ is a function and let $t\in\mathbb{T}_k$. Then we have the following:*

*(i) If $f$ is nabla differentiable at $t$, then $f$ is continuous at $t$.*

*(ii) If $f$ is right continuous at $t$ and $t$ is left-scattered, then $f$ is nabla differentiable at $t$ with*

$$f^{\nabla}(t)=\frac{f(t)-f^{\rho}(t)}{t-\rho(t)}.$$

*(iii) If $t$ is left-dense, then $f$ is nabla differentiable at $t$ iff the limit*

$$\lim_{s\to t}\frac{f(t)-f(s)}{t-s}.$$

*exists as a finite number. In this case*

$f^{\nabla}(t) = \lim_{s \to t} \frac{f(t)-f(s)}{t-s}$.

**Theorem 4** *Asssume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$. then:*

*(i) The sum $f+g : \mathbb{T} \to \mathbb{R}$ is differentiable at $t$ with*

$(f+g)^{\nabla}(t) = f^{\nabla}(t) + g^{\nabla}(t)$.

*(ii) For any constant $\alpha$, $\alpha f : \mathbb{T} \to \mathbb{R}$ is differentiable at $t$ with*

$(\alpha f)^{\nabla}(t) = \alpha f^{\nabla}(t)$.

*(iii) The product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at $t$ with*

$(fg)^{\nabla}(t) = f^{\nabla}(t)g(t) + f(\sigma(t))g^{\nabla}(t) = f(t)g^{\nabla}(t) + f^{\nabla}(t)g(\sigma(t)$.

*(iv) If $f(t)f(\sigma(t)) \neq 0$, then $\frac{1}{f}$ is differentiable at $t$ with*

$\left(\frac{1}{f}\right)^{\nabla} = -\frac{f^{\nabla}(t)}{f(t)f(\rho(t))}$.

*(v) If $g(t)g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is differentiable at $t$ and*

$\left(\frac{f}{g}\right)^{\nabla} = \frac{f^{\nabla}(t)g(t) - f(t)g^{\nabla}(t)}{f(t)f(\rho(t))}$.

**Theorem 5** *Let $c$ be constant and $m \in \mathbb{N}$.*

*(i) For $f$ defined by $f(t) = (t-c)^m$ we have*

$$f^{\triangle}(t) = \sum_{\nu=0}^{m-1} (\sigma(t)-c)^{\nu}(t-c)^{m-1-\nu}.$$

*(ii) For $g$ defined by $g(t) = \frac{1}{(t-c)^m}$ we have*

$$g^{\triangle}(t) = -\sum_{\nu=0}^{m-1} \frac{1}{(\sigma(t)-c)^{m-\nu}(t-c)^{\nu+1}},$$

*provided $(\sigma(t)-c)(t-c) \neq 0$.*

**Lemma 6** *[11] If $f_1, f_2, ..f_n$ are $\triangle$ differentiable on $\mathbb{T}^i$, then the product $f_1 f_2 ... f_n$ is $\triangle$ differentiable on $\mathbb{T}^i$ and*

$(f_1 f_2 ... f_n)^{\triangle} = f_1^{\triangle} f_2 ... f_n + f_1^{\sigma} f_2^{\triangle} ... f_n + f_1^{\sigma} f_2^{\sigma} f_3^{\triangle} ... f_n + f_1^{\sigma} f_2^{\sigma} ... f_{n-1}^{\sigma} f_n^{\triangle}$.

## 2.2 The diamond-$\alpha$ dynamic derivate

**Definition 1** *[7] Let $\mathbb{T}$ be a time scale. We define $f^{\diamond_{\alpha}}(t)$ to be the value, if one exists, such that for all $\varepsilon > 0$ there is a neighborhood $U$ of $t$ (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$ ) such that for all $s \in U$*

$|\alpha [f^{\sigma}(t) - f(s)] \eta_{ts} + (1-\alpha) [f^{\rho}(t) - f(s)] \mu_{ts} - f^{\diamond_{\alpha}}(t) \eta_{ts} \mu_{ts}| \leq \varepsilon |\eta_{ts} \mu_{ts}|$

*we say that $f$ is diamond-$\alpha$ differentiable on $\mathbb{T}_k^k$ provided $f^{\diamond_{\alpha}}(t)$ exists for all $t \in \mathbb{T}^k$.*

**Theorem 7** *[7] Let* $0 \leq \alpha \leq 1$. *If* $f$ *is both* $\triangle$ *and* $\triangledown$ *differentiable at* $t \in \mathbb{T}$, *then* $f$ *is* $\diamond_\alpha$ *differentiable at* $t$ *and* $f^{\diamond_\alpha}(t) = \alpha f^{\triangle}(t) + (1-\alpha) f^{\triangledown}(t)$.

**Theorem 8** *[10] Let* $f, g : \mathbb{T} \rightarrow \mathbb{R}$ *be diamond -*$\alpha$ *differentiable at* $t \in \mathbb{T}$. *Then*

*(i)* $f+g : \mathbb{T} \rightarrow \mathbb{R}$ *is diamond-*$\alpha$ *differentiable at* $t \in \mathbb{T}$ *with*

$$(f+g)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t) + g^{\diamond_\alpha}(t).$$

*(ii) For any constant* $c$, $cf : \mathbb{T} \rightarrow \mathbb{R}$ *is diamond-*$\alpha$ *differentiable at* $t \in \mathbb{T}$ *with*

$$(cf)^{\diamond_\alpha}(t) = c f^{\diamond_\alpha}(t).$$

*(iii)* $fg : \mathbb{T} \rightarrow \mathbb{R}$ *is diamond-*$\alpha$ *differentiable at* $t \in \mathbb{T}$ *with*

$$(fg)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t) g(t) + \alpha f^{\sigma}(t) g^{\triangle}(t) + (1-\alpha) f^{\rho}(t) g^{\triangledown}(t).$$

*(iv) For* $g(t)g^{\sigma}(t)g^{\rho}(t) \neq 0$, $\frac{1}{g} : \mathbb{T} \rightarrow \mathbb{R}$ *is diamond-*$\alpha$ *differentiable at* $t \in \mathbb{T}$ *with*

$$\left(\frac{1}{g}\right)^{\diamond_\alpha}(t) = -\frac{1}{g(t)g^{\sigma}(t)g^{\rho}(t)}\big((g^{\sigma}(t) + g^{\rho}(t)) g^{\diamond_\alpha}(t) - \\ -\alpha g^{\triangle}(t) g^{\sigma}(t) - (1-\alpha) g^{\triangledown}(t) g^{\rho}(t)\big).$$

*(v) For* $g(t)g^{\sigma}(t)g^{\rho}(t) \neq 0$, $\frac{f}{g} : \mathbb{T} \rightarrow \mathbb{R}$ *is diamond-*$\alpha$ *differentiable at* $t \in \mathbb{T}$ *with*

$$\left(\frac{f}{g}\right)^{\diamond_\alpha}(t) = \frac{1}{g(t)g^{\sigma}(t)g^{\rho}(t)}\big(f^{\diamond_\alpha}(t) g^{\sigma}(t) g^{\rho}(t) - \alpha f^{\sigma}(t) g^{\rho}(t) g^{\triangle}(t) - \\ -(1-\alpha) f^{\rho}(t) g^{\sigma}(t) g^{\triangledown}(t)\big).$$

# 3 Main Results

**Theorem 9** *Let* $c$ *be constant and* $m \in \mathbb{N}$.

*(i) For* $f$ *defined by* $f(t) = (t-c)^m$ *we have*

$$f^{\triangledown}(t) = \sum_{\nu=0}^{m-1} (\rho(t) - c)^{\nu} (t-c)^{m-1-\nu}.$$

*(ii) For* $g$ *defined by* $g(t) = \frac{1}{(t-c)^m}$ *we have*

$$g^{\triangledown}(t) = -\sum_{\nu=0}^{m-1} \frac{1}{(\rho(t)-c)^{m-\nu}(t-c)^{\nu+1}},$$

*provided* $(\rho(t) - c)(t-c) \neq 0$.

**Proof.** We will prove the first formula by induction. If $m = 1$, then $f(t) = (t-c)$, and clearly $f^{\triangledown}(t) = 1$ holds by Example 1 (i), (ii), and Theorem 2.(i). Now we assume that

$$f^{\triangledown}(t) = \sum_{\nu=0}^{m-1} (\rho(t) - c)^{\nu} (t-c)^{m-1-\nu}.$$

holds for $f(t) = (t-c)^m$ and let $F(t) = (t-c)^{m+1}$=$(t-c)f(t)$. We use the product rule (Theorem 2. (iii) ) to obtain

$$
\begin{aligned}
F^{\nabla}(t) = f(\rho(t)) + (t-c)\ f^{\nabla}(t)\\
&=(\rho(t)-c)^m + (t-c)\sum_{\nu=0}^{m-1}(\rho(t)-c)^{\nu}(t-c)^{m-1-\nu}\\
&=(\rho(t)-c)^m + \sum_{\nu=0}^{m-1}(\rho(t)-c)^{\nu}(t-c)^{m-\nu}\\
&=\sum_{\nu=0}^{m}(\rho(t)-c)^{\nu}(t-c)^{m-\nu}.
\end{aligned}
$$

Hence, by mathematical induction, part (i) holds.

(ii) Next, for $g(t) = \frac{1}{(t-c)^m} = \frac{1}{f(t)}$ we apply Theorem 4. (iv) to obtain

$$
\begin{aligned}
\left(\frac{1}{g}\right)^{\nabla}(t) &= -\frac{g^{\nabla}(t)}{g(t)g(\rho(t))}\\
&= -\frac{\sum_{\nu=0}^{m}(\rho(t)-c)^{\nu}(t-c)^{m-\nu}}{(t-c)^m(\rho(t)-c)^m}\\
&= -\sum_{\nu=0}^{m-1}\frac{1}{(\rho(t)-c)^{m-\nu}(t-c)^{\nu+1}}. \blacksquare
\end{aligned}
$$

**Theorem 10** *Let $c$ be constant and $m \in \mathbb{N}$.*

*(i) For $f$ defined by $f(t) = (t-c)^m$ we have*

$$f^{\diamond_\alpha}(t) = \alpha\sum_{\nu=0}^{m-1}(\sigma(t)-c)^{\nu}(t-c)^{m-1-\nu} + (1-\alpha)\sum_{\nu=0}^{m-1}(\ \rho(t)-c)^{\nu}(t-c)^{m-1-\nu}.$$

*(ii) For $g$ defined by $g(t) = \frac{1}{(t-c)^m}$ we have*

$$g^{\diamond_\alpha}(t) = -\alpha\sum_{\nu=0}^{m-1}\frac{1}{(\sigma(t)-c)^{m-\nu}(t-c)^{\nu+1}} - (1-\alpha)\sum_{\nu=0}^{m-1}\frac{1}{(\rho(t)-c)^{m-\nu}(t-c)^{\nu+1}}.$$

*provided $(\rho(t)-c)(t-c) \neq 0$.*

**Proof.** We will prove the first formula by induction. If $m = 1$, then $f(t) = (t-c)$, and clearly $f^{\triangle}(t) = 1$ and $f^{\nabla}(t) = 1$ holds by Example 1. (i), (ii), and Theorem 2. (i).

$$f^{\diamond_\alpha}(t) = \alpha(t-c)^{\triangle} + (1-\alpha)(t-c)^{\nabla} = 1$$

If $m = 2$, then $f(t) = t^2 - 2ct + c^2$, and clearly $f^{\diamond_\alpha}(t) = \alpha\sigma(t) + (1-\alpha)\rho(t) + t - 2c$ holds by Example 1. (i), (ii), Example 2., and Theorem 2. (i).

Now we assume that

$$f^{\diamond_\alpha}(t) = \alpha\sum_{\nu=0}^{m-1}(\sigma(t)-c)^{\nu}(t-c)^{m-1-\nu} + (1-\alpha)\sum_{\nu=0}^{m-1}(\rho(t)-c)^{\nu}(t-c)^{m-1-\nu}.$$

holds for $f(t) = (t-c)^m$ and let $F(t) = (t-c)^{m+1}$=$(t-c)f(t)$

$$\begin{aligned} F^{\diamond_\alpha}(t) &= \alpha\left[f(\sigma(t)) + (t-c)\ f^{\triangle}(t)\right] + (1-\alpha)\left[f(\rho(t)) + (t-c)\ f^{\nabla}(t)\right] \\ &= \alpha\left[(\sigma(t)-c)^m + \sum_{\nu=0}^{m-1}(\sigma(t)-c)^{\nu}(t-c)^{m-1-\nu}\right] + \\ &\quad +(1-\alpha)\left[(\rho(t)-c)^m + \sum_{\nu=0}^{m-1}(\rho(t)-c)^{\nu}(t-c)^{m-\nu}\right] \\ &= \alpha\sum_{\nu=0}^{m}(\sigma(t)-c)^{\nu}(t-c)^{m-\nu} + (1-\alpha)\sum_{\nu=0}^{m}(\rho(t)-c)^{\nu}(t-c)^{m-\nu}. \end{aligned}$$

(ii) Next, for $g(t) = \frac{1}{(t-c)^m} = \frac{1}{f(t)}$ we apply Theorem 8. (iv) to obtain

$$\begin{aligned} \left(\frac{1}{g}\right)^{\diamond_\alpha}(t) &= -\alpha\frac{g^{\nabla}(t)}{g(t)g(\sigma(t))} - (1-\alpha)\frac{g^{\nabla}(t)}{g(t)g(\rho(t))} \\ &= -\alpha\frac{\sum_{\nu=0}^{m-1}(\sigma(t)-c)^{\nu}(t-c)^{m-1-\nu}}{(t-c)^m(\sigma(t)-c)^m} - (1-\alpha)\frac{\sum_{\nu=0}^{m}(\rho(t)-c)^{\nu}(t-c)^{m-\nu}}{(t-c)^m(\rho(t)-c)^m} \\ &= -\alpha\sum_{\nu=0}^{m-1}\frac{1}{(\sigma(t)-c)^{m-\nu}(t-c)^{\nu+1}} - (1-\alpha)\sum_{\nu=0}^{m-1}\frac{1}{(\rho(t)-c)^{m-\nu}(t-c)^{\nu+1}}. \end{aligned}$$

■

**Corollary 11** *It is clear that $f^{\diamond_\alpha}(t)$ reduces to $f^{\triangle}(t)$ for $\alpha = 1$ and $f^{\nabla}(t)$ for $\alpha = 0$.*

**Lemma 12** *If $f_1, f_2, ..f_n$ are $\nabla$ differentiable on $\mathbb{T}^i$, then the product $f_1 f_2 ... f_n$ is $\nabla$ differentiable on $\mathbb{T}^i$ and*

$$(f_1 f_2 ... f_n)^{\nabla} = f_1^{\nabla} f_2 ... f_n + f_1^{\rho} f_2^{\nabla} ... f_n + f_1^{\rho} f_2^{\rho} f_3^{\nabla} ... f_n + f_1^{\rho} f_2^{\rho} ... f_{n-1}^{\rho} f_n^{\nabla} \qquad (1)$$

***Proof.*** *The proof is by induction. If $n = 2$, then $(f_1 f_2)^{\nabla} = f_1^{\nabla} f_2 + f_1^{\rho} f_2^{\nabla}$ by Theorem 2. (iii). Assume that formula is true for $n$. Then*

$$\begin{aligned} (f_1 f_2 ... f_n f_{n+1})^{\nabla} &= f_1^{\nabla} f_2 ... f_n f_{n+1} + f_1^{\rho}\,(f_2 ... f_n f_{n+1})^{\nabla} \\ &= f_1^{\nabla} f_2 ... f_n f_{n+1} + f_1^{\rho} f_2^{\nabla} f_3 ... f_n f_{n+1} + \\ &\quad + f_1^{\rho} f_2^{\rho} f_3^{\nabla} .... f_n f_{n+1} + f_1^{\rho} f_2^{\rho} f_3^{\rho} ... f_n^{\sigma} f_{n+1}^{\nabla} \end{aligned}$$

*and formula (1) is true for $n$ replaced by $n + 1$.* ■

**Theorem 13** *If $f_1, f_2, ..f_n$ are both $\nabla$ and $\nabla$ differentiable on $\mathbb{T}^i$, then the product $f_1 f_2 ... f_n$ is both $\nabla$ and $\nabla$ differentiable on $\mathbb{T}^i$ and*

$$\begin{aligned} (f_1 f_2 ... f_n)^{\diamond_\alpha} &= \alpha\left[f_1^{\triangle} f_2 ... f_n + f_1^{\sigma} f_2^{\triangle} ... f_n + f_1^{\sigma} f_2^{\sigma} f_3^{\triangle} ... f_n + f_1^{\sigma} f_2^{\sigma} ... f_{n-1}^{\sigma} f_n^{\triangle}\right] + \\ &\quad +(1-\alpha)\left[f_1^{\nabla} f_2 ... f_n + f_1^{\rho} f_2^{\nabla} ... f_n + f_1^{\rho} f_2^{\rho} f_3^{\nabla} ... f_n + f_1^{\rho} f_2^{\rho} ... f_{n-1}^{\rho} f_n^{\nabla}\right] \end{aligned}$$

**Proof.** The proof is by induction. If $n = 2$, then

$(f_1 f_2)^{\diamond_\alpha} = \alpha\left[f_1^{\triangle} f_2 + f_1^{\sigma} f_2^{\triangle}\right] + (1-\alpha)\left[f_1^{\nabla} f_2 + f_1^{\rho} f_2^{\nabla}\right]$ by Theorem 8. (iii) and Theorem 2. (iii). Assume that formula is true for $n$. Then

$$(f_1 f_2 ... f_n f_{n+1})^{\diamond_\alpha} = \alpha\left[f_1 f_2 ... f_n f_{n+1}\right]^{\triangle} + (1-\alpha)\left[f_1 f_2 ... f_n f_{n+1}\right]^{\nabla}$$
$$= \alpha[f_1^{\nabla} f_2 ... f_n f_{n+1} + f_1^{\sigma} f_2^{\nabla} f_3 ... f_n f_{n+1} + f_1^{\sigma} f_2^{\sigma} f_3^{\nabla} .... f_n f_{n+1} +$$
$$+ f_1^{\sigma} f_2^{\sigma} f_3^{\sigma} ... f_n^{\sigma} f_{n+1}^{\nabla}] + (1-\alpha)[f_1^{\nabla} f_2 ... f_n f_{n+1} + f_1^{\rho} f_2^{\nabla} f_3 ... f_n f_{n+1} +$$
$$+ f_1^{\rho} f_2^{\rho} f_3^{\nabla} .... f_n f_{n+1} + f_1^{\rho} f_2^{\rho} f_3^{\rho} ... f_n^{\sigma} f_{n+1}^{\nabla}].$$

and formula is true for $n$ replaced by $n+1$. ■

**Corollary 14** *It is clear that $f^{\diamond_\alpha}(t)$ reduces to $f^{\triangle}(t)$ for $\alpha = 1$ and $f^{\nabla}(t)$ for $\alpha = 0$.*

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**Received: May 6, 2006**