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Application of differential transformation method to linear sixth-order boundary value problems

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Abstract

Differential transformation method is applied to construct semi numerical-analytic solutions of linear sixth-order boundary value problems with two-point boundary value conditions. Two examples are considered for the numerical illustrations of this method. The results demonstrate reliability and efficiency of this method for such problems.

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1 Introduction

In this paper, we consider the following sixth-order boundary value problem of the form

$$\left.\begin{array}{c} y^{(6)}(x)+f(x)y(x)=g(x), \quad a<x<b, \\ y(a)=\alpha_0, \quad y(b)=\alpha_1, \\ y'(a)=\gamma_0, \quad y'(b)=\gamma_1, \\ y''(a)=\delta_0, \quad y''(b)=\delta_1, \end{array}\right\} \tag{1}$$

where α_i, γ_i and δ_i, $i=0,1$ are finite real constants while the functions $f(x)$ and $g(x)$ are continuous on $[a,b]$.

Sixth-order boundary value problems arise in astrophysics, i.e., the narrow convecting layers bounded by stable layers which are believed to surround A-type stars may be modelled by sixth-order boundary value problems [5]. Chandrasekhar [8] determined that when an infinite horizontal layer of fluid is heated from below and is under the action of rotation, instability sets in. When this instability is as ordinary convection, the ordinary differential equation is sixth-order.

Agarwal [7] presented the theorems stating the conditions for the existence and uniqueness of solutions of sixth-order boundary value problems, while no numerical methods are contained therein.A substantial amount of research work has been directed for such boundary value problems [2,3,6,9]. Several numerical techniques, such as Sinc-Galerkin method [6], decomposition method [2] and variational iteration method [3] have been implemented to solve such problems numerically. Also, in [9], Siddiqi and Twizell investigated the sixth-order boundary value problems using spline functions.

The technique that we used is the differential transformation method (in short DTM), which is based on Taylor series expansion. It is introduced by Zhou [4] in a study about electrical circuits. It gives exact values of the kth derivative of an analytical function at a point in terms of known and unknown boundary conditions in a fast manner.

2 Differential transformation method

The differential transformation of the kth derivative of the function $f(x)$ is defined as follows:

$$F(k) = \frac{1}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=x_0} \tag{2}$$

and the differential inverse transformation of $F(k)$ is defined as follows:

$$f(x) = \sum_{k=0}^{\infty} F(k)(x - x_0)^k. \tag{3}$$

In real applications, the function $f(x)$ is expressed by a finite series and Eq.(3) can be written as

$$f(x) = \sum_{k=0}^{n} F(k)(x - x_0)^k. \tag{4}$$

The following theorems that can be deduced from Eqs. (2) and (3) are given below [1]:

Theorem 1 *If* $f(x) = g(x) \pm h(x)$,*then* $F(k) = G(k) \pm H(k)$.

Theorem 2 *If* $f(x) = cg(x)$,*then* $F(k) = cG(k)$,*where* c *is a constant.*

Theorem 3 *If* $f(x) = \frac{d^n(x)}{dx^n}$,*then* $F(k) = \frac{(k+n)!}{k!}G(k+n)$.

Theorem 4 *If* $f(x) = g(x)h(x)$,*then* $F(k) = \sum_{k_1=0}^{k} G(k_1)H(k-k_1)$.

Theorem 5 *If* $f(x) = x^n$,*then* $F(k) = \delta(k-n)$ *where,* $\delta(k-n) = \begin{cases} 1, & k = n \\ 0, & k \neq n. \end{cases}$

Theorem 6 *If* $f(x) = g_1(x)g_2(x)\ldots g_{n-1}(x)g_n(x)$,*then*

$$F(k) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \ldots \sum_{k_2=0}^{k_3} \sum_{k1=0}^{k_2} G_2(k_1)G_1(k_2-k_1)\times$$

$$\ldots G_{n-1}(k_{n-1}-k_{n-2})G_n(k-k_{n-1}).$$

3 Numerical results

Example 1.

$$y^{(6)}(x) + y(x) = 12x\cos(x) + 30\sin(x), \ -1 < x < 1, \tag{5}$$

subject to the boundary conditions

$$\left.\begin{array}{c} y(-1) = y(1) = 0, \\ y'(-1) = y'(1) = 2\sin(1), \\ y''(-1) = -y''(1) = -4\cos(1) - 2\sin(1). \end{array}\right\} \tag{6}$$

By using the above theorems, the differential transformation of Eq.(5) is obtained as

$$Y(k+6) = \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)}\times \left(12\sum_{k_1=0}^{k} \delta(k_1-1)C(k-k_1) + 30S(k) - Y(k)\right). \tag{7}$$

The boundary conditions in Eq. (6) can be transformed at $x_0 = 0$ as

$$\left.\begin{array}{c}\sum_{k=0}^{n} Y(k)(-1)^k = 0,\ \sum_{k=0}^{n} Y(k) = 0,\\ \sum_{k=0}^{n} kY(k)(-1)^{k-1} = 2\sin(1),\\ \sum_{k=0}^{n} kY(k) = 2\sin(1),\\ \sum_{k=0}^{n} k(k-1)Y(k)(-1)^{k-2} = -4\cos(1) - 2\sin(1),\\ \sum_{k=0}^{n} k(k-1)Y(k) = 4\cos(1) + 2\sin(1),\end{array}\right\} \tag{8}$$

where $S(k)$ and $C(k)$ correspond to the differential transformation of $\sin(x)$ and $\cos(x)$ at $x_0 = 0$, respectively, which can be easily obtained from the definition of differential transformation in Eq. (2) as follows:

$$\begin{array}{l} S(k) = \begin{cases} \frac{(-1)^{\frac{k-1}{2}}}{k!}, & \text{if } k = odd,\\ 0, & \text{if } k = even, \end{cases}\\ C(k) = \begin{cases} \frac{(-1)^{\frac{k}{2}}}{k!}, & \text{if } k = even,\\ 0, & \text{if } k = odd. \end{cases} \end{array} \tag{9}$$

By using the recurrence relation in Eq.(7) and the transformed boundary conditions in Eq.(8), the following series solution up to $O(x^{10})$ is obtained:

$$\begin{aligned} y(x) = {} & a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 - \tfrac{a_0}{720} x^6 \\ & + \left(\tfrac{1}{120} - \tfrac{a_1}{5040}\right) x^7 - \tfrac{a_2}{20160} x^8 + \left(-\tfrac{11}{60480} - \tfrac{a_3}{60480}\right) x^9 + O(x^{10}). \end{aligned} \tag{10}$$

By taking $n = 9$, the following system of equations can be obtained from Eq. (8):

$$-\frac{493}{60480}+\frac{719a_0}{720}-\frac{5039a_1}{5040}+\frac{20159a_2}{20160}-\frac{60479a_3}{60480}+\frac{151199a_4}{151200}-a_5=0,$$

$$\frac{493}{60480}+\frac{719a_0}{720}+\frac{5039a_1}{5040}+\frac{20159a_2}{20160}+\frac{60479a_3}{60480}+\frac{151199a_4}{151200}+a_5=0,$$

$$\frac{127}{2240}+\frac{a_0}{120}+\frac{719a_1}{720}-\frac{5039a_2}{2520}+\frac{20159a_3}{6720}-\frac{60479a_4}{15120}+5a_5=2\sin(1),$$

$$\frac{127}{2240}-\frac{a_0}{120}+\frac{719a_1}{720}+\frac{5039a_2}{2520}+\frac{20159a_3}{6720}+\frac{60479a_4}{15120}+5a_5=2\sin(1),$$

$$-\frac{283}{840}-\frac{a_0}{24}+\frac{a_1}{120}+\frac{719a_2}{360}-\frac{5039a_3}{840}+\frac{20159a_4}{1680}-20a_5=-4\cos(1)-2\sin(1),$$

$$\frac{283}{840}-\frac{a_0}{24}-\frac{a_1}{120}+\frac{719a_2}{360}+\frac{5039a_3}{840}+\frac{20159a_4}{1680}+20a_5=4\cos(1)+2\sin(1), \tag{11}$$

where $a_0 = y(0) = Y(0)$, $a_1 = y'(0) = Y(1)$, $a_2 = y''(0)/2! = Y(2)$, $a_3 = y'''(0)/3! = Y(3)$, $a_4 = y^{(4)}(0)/4! = Y(4)$ and $a_5 = y^{(5)}(0)/5! = Y(5)$ are the missing boundary conditions.

We get from the system of equations (11)

$$\begin{aligned} &a_0 = 0,\ a_1 = -0.9999836,\ a_2 = 0,\\ &a_3 = 1.1666256,\ a_4 = 0,\ a_5 = -0.1749726. \end{aligned} \tag{12}$$

By taking 15 terms, the following results are obtained:

$$\begin{aligned} &a_0 = 0,\ a_1 = -1.000000000016041,\ a_2 = 0,\\ &a_3 = 1.166666666703337,\ a_4 = 0,\ a_5 = -0.175000000021396. \end{aligned} \tag{13}$$

It is clearly seen that as $n \to \infty$, $a_0 = 0$, $a_1 = -1$, $a_2 = 0$, $a_3 = 7/6$, $a_4 = 0$ and $a_5 = -7/40$. By using these values of the missing boundary conditions ,Eq. (10) becomes

$$y(x) = -x + \frac{7}{6}x^3 - \frac{7}{40}x^5 + \frac{43}{5040}x^7 - \frac{73}{362880}x^9 + \cdots \tag{14}$$

which can be written in the closed form as follows:

$$y(x) = (x^2 - 1)\sin(x). \tag{15}$$

Example 2. Consider the following boundary value problem

$$y^{(6)}(x) - y(x) = -6e^x,\ 0 < x < 1, \tag{16}$$

subject to the boundary conditions

$$\left.\begin{array}{c} y(0) = 1,\ y(1) = 0, \\ y'(0) = 0,\ y'(1) = -e, \\ y''(0) = -1,\ y''(1) = -2e. \end{array}\right\} \tag{17}$$

Knowing that the differential transformation of e^x is $1/k!$ and appliying the above theorems to Eq.(16), the following recurrence relation is obtained

$$\begin{array}{c} Y(k+6) = \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)} \times \\ \left(Y(k) - \frac{6}{k!}\right). \end{array} \tag{18}$$

The differential transformation of the boundary conditions in Eq.(17) at $x_0 = 0$ are

$$\begin{array}{c} Y(0) = 1,\ Y(1) = 0,\ Y(2) = -1/2,\ \sum_{k=0}^{n} Y(k) = 0 \\ \sum_{k=0}^{n} kY(k) = -e \text{ and } \sum_{k=0}^{n} k(k-1)Y(k) = -2e. \end{array} \tag{19}$$

By using the recurrence ralation in Eq.(18) and the transformed boundary conditions in Eq.(19), the following series solution up to $O(x^8)$ is obtained:

$$y(x) = 1 - \frac{x^2}{2} + a_3x^3 + a_4x^4 + a_5x^5 - \frac{x^6}{144} - \frac{x^7}{840} + O(x^8). \tag{20}$$

By taking $n = 7$, the following system of equations can be obtained from Eq. (19):

$$\begin{array}{c} \frac{2479}{5040} + a_3 + a_4 + a_5 = 0, \\ -\frac{21}{20} + +3a_3 + 4a_4 + 5a_5 = -e, \\ -\frac{151}{120} + 6a_3 + 12a_4 + 20a_5 = -2e, \end{array} \tag{21}$$

where $a_3 = y'''(0)/3! = Y(3)$, $a_4 = y^{(4)}(0)/4! = Y(4)$ and $a_5 = y^{(5)}(0)/5! = Y(5)$ are the missing boundary conditions. From the equation system(21), a_3, a_4 and a_5 can be obtained numerically as

$$a_3 = -0.334639,\ a_4 = -0.121766,\ a_5 = -0.035460. \tag{22}$$

By taking $n = 12$, the following results are obtained:

$$a_3 = -0.3333334101,\ a_4 = -0.1249998297,\ a_5 = -0.0333334290. \tag{23}$$

It is clear that in the limit case $n \to \infty$, a_3 converges to -1/3, a_4 converges to -1/8 and a_5 converges to -1/30. With these values of a_3, a_4 and a_5, Eq. (20) becomes

$$y(x) = 1 - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{8} - \frac{x^5}{30} - \frac{x^6}{144} - \frac{x^7}{840} + \cdots . \tag{24}$$

which is the series expansion of

$$y(x) = (1 - x)e^x. \tag{25}$$

4 Conclusions

In this work, we studied differential transformation method for solving linear sixth-order boundary value problems. This method was applied to solve two boundary value problems. In Examples 1 and 2, we obtained closed form exact series solutions. It is observed that the method is an effective and reliable tool for the solution of such problems.

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