Applied Mathematical Sciences, Vol. 1, 2007, no. 2, 69 - 76

Some New Dynamic Inequalities for First Order Linear Dynamic Equations on Time Scales

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Abstract. We study some new dynamic inequalities for first order linear dynamic equations on time scales.

Mathematics Subject Classification: 39A11, 26D10

Keywords: Time scales, Dynamic inequality, Linear dynamic equation

1. Introduction

Hilger and Aulbach [2, 5] generalized the definition of a derivative and of an integral to time scales in order to unify results from the calculus of real numbers with results from the difference calculus. So after that time scales calculus created. In this way many paper time scales were written by Agarwal, Bohner, Hilscher, Peterson and joint professors.

A time scale is an arbitrary nonempty closed subset of the real numbers. The calculus of time scales was initiated by B. Aulbach and S. Hilger [2, 5] in order to create a theory that can unify discrete and continuous analysis. For a treatment of the single variable calculus of time scales see [3, 4, 8] and the references given therein. After that many theories in real numbers and integer numbers are extended to time scales.

In this paper, we study some new dynamic inequalities for first order linear dynamic equations on time scales.

We consider nonhomogeneous linear dynamical equation of first order

$$y^{\triangle}(t) + g(t)y^{\sigma}(t) + e_g(\sigma(t), t)h(t) = 0, \quad y(a) = x \tag{1.1}$$

We assume that $\mathbb{T} = [a, b]$ is an arbitrary interval. We moreover that $g : \mathbb{T} \to \mathbb{R}$, $h : \mathbb{T} \to \mathbb{R}$, and $\varphi : \mathbb{T} \to [0, \infty)$ are functions such that for arbitrary $c \in$

$\mathbb{T}$, $g(t)$ and $e_g(\sigma(t),a)h(t)$ are integrable on $[a,c]$, futher such that $\varphi(t)e_g(t,a)$ is integrable on $\mathbb{T}$, and $g(t)$ is regressive (i.e.,$g \in \Re$).

We prove in Theorem.1 that if a rd-continuously differentiable function $y : \mathbb{T} \to \mathbb{R}$ satisfies the differential inequality

$$\left|y^{\triangle}(t) + g(t)y^{\sigma}(t) + e_g(\sigma(t),t)h(t)\right| \leq \varphi(t) \tag{1.2}$$

for all $t \in \mathbb{T}^k$, then exists a unique solution $y_0(t)$ of the dynamical equation (1.1) such that

$$|y(t) - y_0(t)| \leq e_{\ominus g}(t,a) \int_t^b \varphi(\upsilon)e_g(\upsilon,a) \triangle \upsilon$$

for any $t \in \mathbb{T}$.

Here, first we mention several foundational definitions without proof and results from the calculus on time scales in an excellent introductory text by Bohner and Peterson [3, 4].

2. General Definitions

Definition 1. *A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. We assume throughout that $\mathbb{T}$ has the topology that is inherited from the standard topology on $\mathbb{R}$. It also assumed throughout that in $\mathbb{T}$ the interval $[a,b]$ means the set $\{t \in \mathbb{T} : \ s < t\}$ for the points $a < b$ in $\mathbb{T}$. Since a time scale may not be connected, we need the following concept of jump operators.*

Definition 2. *The mappings $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined by $\sigma(t) = \inf\{s \in \mathbb{T} : \ s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : \ s < t\}$ are called the jump operators.*

The jump operators σ and ρ allow the classification of points in $\mathbb{T}$ in the following way:

Definition 3. *A nonmaximal element $t \in \mathbb{T}$ is said to be right-dense if $\sigma(t) = t$, right-scattered if $\sigma(t) > t$, left-dense if $\rho(t) = t$,left-scattered if $\rho(t) < t$.*

In the case $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = t$, and if $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then $\sigma(t) = t + h$.

Definition 4. *The mapping $\mu : \mathbb{T} \to \mathbb{R}^+$ defined by $\mu(t) = \sigma(t) - t$ is called the graininess function. If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$, and when $\mathbb{T} = \mathbb{Z}$, we have $\mu(t) = 1$.*

Definition 5. *Let $f : \mathbb{T} \to \mathbb{R}$. f is called differentiable at $t \in \mathbb{T}^k$, with (delta) derivative $f^{\triangle}(t) \in \mathbb{R}$ if given $\varepsilon > 0$ there exists a neighborhood U of t such that, for all $s \in U$,*

$$\left\|f^{\sigma}(t) - f(s) - f^{\triangle}(t)[\sigma(t) - s]\right\| \leq \varepsilon \left\|\sigma(t) - s\right\|,$$

where $f^{\sigma} = f \circ \sigma$. If $\mathbb{T} = \mathbb{R}$, then $f^{\triangle}(t) = \frac{df(t)}{dt}$,and if $\mathbb{T} = \mathbb{Z}$, then $f^{\triangle}(t) = f(t+1) - f(t)$.

Some basic properties of delta derivatives are the following [3].

Theorem 1. *Assume that $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^k$.*

(i) If f is differentiable at t, then f is continuous at t.

(ii) If f is differentiable at t and t is right-scattered, then f is differentiable at t with

$$f^{\triangle}(t) = \frac{f^{\sigma}(t) - f(t)}{\sigma(t) - t}.$$

(iii) If f is differentiable at t and t is right-dense, then

$$f^{\triangle}(t) = \lim_{t \to s} \frac{f(t) - f(s)}{t - s}.$$

(iv) If f is differentiable at t, then

$$f^{\sigma}(t) = f(t) + \mu(t) f^{\triangle}(t)$$

Theorem 2. *Asssume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$. then:*

(i) The sum $f + g : \mathbb{T} \to \mathbb{R}$ is differentiable at t with
$(f + g)^{\triangle}(t) = f^{\triangle}(t) + g^{\triangle}(t)$.

(ii) For any constant α, $\alpha f : \mathbb{T} \to \mathbb{R}$ is differentiable at t with
$(\alpha f)^{\triangle}(t) = \alpha f^{\triangle}(t)$.

(iii) The product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with
$(fg)^{\triangle}(t) = f^{\triangle}(t) g(t) + f(\sigma(t)) g^{\triangle}(t) = f(t) g^{\triangle}(t) + f^{\triangle}(t) g(\sigma(t)$.

(iv) If $f(t) f(\sigma(t)) \neq 0$, then $\frac{1}{f}$ is differentiable at t with
$\left(\frac{1}{f}\right)^{\triangle} = -\frac{f^{\triangle}(t)}{f(t) f(\sigma(t))}$.

(v) If $g(t) g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is differentiable at t and
$\left(\frac{f}{g}\right)^{\triangle} = \frac{f^{\triangle}(t) g(t) - f(t) g^{\triangle}(t)}{f(t) f(\sigma(t))}$.

Definition 6. *The function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous (denoted by $f \in C_{rd}(\mathbb{T}, \mathbb{R})$) if, at all $t \in \mathbb{T}$,*

(i) f is continuous at every right-dense point $t \in \mathbb{T}$,

(ii) $lim_{s \to t^-} f(s)$ exists and is finite at every left-dense point $t \in \mathbb{T}$.

Definition 7. *Let $f \in C_{rd}(\mathbb{T}, \mathbb{R})$. Then $g : \mathbb{T} \to \mathbb{R}$ is called the antiderivative of f on $\mathbb{T}$ if it is differentiable on $\mathbb{T}$ and satisfies $g^{\triangle}(t) = f(t)$ for any $t \in \mathbb{T}^k$. In this case, we define*

$$\int_a^t f(s) \triangle s = g(t) - g(a), \quad t \in \mathbb{T}.$$

2.1. **The Hilger complex plane.** For $h>0$, define the *Hilger complex numbers*, the *Hilger real axis*, the *Hilger alternating axis*, and the *Hilger imaginary circle* by

$\mathbb{C}_h = \left\{z \in \mathbb{C} : z \neq -\frac{1}{h}\right\}$, $\mathbb{R}_h = \left\{z \in \mathbb{R} : z > -\frac{1}{h}\right\}$

$\mathbb{A}_h = \left\{z \in \mathbb{R} : z < -\frac{1}{h}\right\}$, $\mathbb{I}_h = \left\{z \in \mathbb{C} : \left|z + \frac{1}{h}\right| = \frac{1}{h}\right\}$

respectively. For $h=0$, let $\mathbb{C}_0 := \mathbb{C}$, $\mathbb{R}_0 := \mathbb{R}$, $\mathbb{A}_0 := \emptyset$, and $\mathbb{I}_0 := i\mathbb{R}$.

Let $h>0$ and $z \in \mathbb{C}_h$. The *Hilger real part of* z is defined by $\mathrm{Re}_h(z) := \frac{|zh+1|}{h}$, and the *Hilger imaginary part of* z is defined by $\mathrm{Im}_h(z) := \frac{Arg(xh+1)}{h}$, where $Arg(z)$ denotes the principle argument of z (i.e.,$-\pi < Arg z \leq \pi$).

For $h>0$, define the strip $\mathbb{Z}_h := \left\{z \in \mathbb{C} : \frac{-\pi}{h} < Arg z \leq \frac{\pi}{h}\right\}$, and for $h=0$, set $\mathbb{Z}_0 := \mathbb{C}$. Then we can define the *cylinder transformation* $\xi_h = \mathbb{C}_h \to \mathbb{Z}_h$ by

$$\xi_h(z) = \frac{1}{h} Log(1+zh), \quad h>0$$

where Log is the principle logarithm function. When $h=0$, we define $\xi_0(z) = z$, for all $z \in \mathbb{C}$. It then follows that the *inverse cylinder transformation* $\xi_h^{-1} : \mathbb{Z}_h \to \mathbb{C}_h$ is given by

$$\xi_h^{-1}(z) = \frac{e^{zh}-1}{h}.$$

Since the graininess may not be constant for a given time scale, we will interchangeably subscript various quantities (such as ξ and ξ^{-1}) with $\mu = \mu(t)$ instead of h to reflect this.

2.2. **Generalized exponential Functions.** The function $p : \mathbb{T} \to \mathbb{R}$ is *regressive* if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$, and this concept motivates the definition of the following sets:

$\Re = \left\{p : \mathbb{T} \to \mathbb{R} : p \in C_{rd}(\mathbb{T}) \text{ and } 1 + \mu(t)p(t) \neq 0 \ \forall t \in \mathbb{T}^k\right\}$,

$\Re^+ = \left\{p \in \Re : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}^k\right\}$.

The function $p : \mathbb{T} \to \mathbb{R}$ is *uniformly regressive* on $\mathbb{T}$ there exists a positive constant δ such that $0 < \delta^{-1} \leq |1 + \mu(t)p(t)|$, $t \in \mathbb{T}^k$.

If $p \in \Re$, then we define the *generalized time scale exponential function* by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau)) \triangle \tau\right) \quad \text{for all } s,t \in \mathbb{T}$$

The following theorem is a compilation of properties of $e_p(t,s)$ (some of which are counterintuitive) that we need in the main body of the paper.

Theorem 3. *The function $e_p(t,s)$ has the following properties:*

(i) If $p \in \Re$, then $e_p(t,r)e_p(r,s) = e_p(t,s)$ for all $r,s,t \in \mathbb{T}$.

(ii) $e_p(\sigma(t),s) = (1 + \mu(t)p(t))e_p(t,s)$.

(iii) If $p \in \Re^+$, then $e_p(t,t_0) > 0$ for all $t \in \mathbb{T}$.

(iv) If $1 + \mu(t)p(t) < 0$ for some $t \in \mathbb{T}^k$, then $e_p(t,t_0)e_p(\sigma(t),t_0) < 0$.

(v) If $\mathbb{T}=\mathbb{R}$, *then* $e_p(t,s)=e^{\int_s^t p(\tau)d\tau}$. *Morever, If* p *is constant, then* $e_p(t,s)=e^{p(t-s)}$.

(vi) If $\mathbb{T}=\mathbb{Z}$, *then* $e_p(t,s)=\Pi_{\tau=s}^{t-1}(1+p(\tau))$. *Morever, If* $\mathbb{T}=h\mathbb{Z}$, *with* $h>0$ *and* p *is constant, then* $e_p(t,s)=(1+hp)^{\frac{(t-s)}{h}}$.

Definition 8. *If* $p\in\Re$ *and* $f:\mathbb{T}\to\mathbb{R}$ *is rd-continuous, then the dynamic equation*

$$y^{\triangle}(t)=p(t)y(t)+f(t) \tag{2.1}$$

is called regressive.

Theorem 4. *If* $p,q\in\Re$, *then*
(i) $e_0(t,s)\equiv 1$ *and* $e_p(t,t)\equiv 1$;
(ii) $e_p(\sigma(t),s)=(1+\mu(t)p(t))e_p(t,s)$;
(iii) $\frac{1}{e_p(t,s)}=e_{\ominus p}(t,s)$;
(iv) $e_p(t,s)=\frac{1}{e_p(s,t)}=e_{\ominus p}(s,t)$;
(v) $e_p(t,s)e_p(s,\tau)=e_p(t,\tau)$;
(vi) $e_p(t,s)e_q(t,s)=e_{p\oplus q}(t,s)$;
(vii) $\frac{e_p(t,s)}{e_q(t,s)}=e_{p\ominus q}(t,s)$;

Theorem 5. *(Variation of constants). Let* $t_0\in\mathbb{T}$ *and* $y(t_0)=y_0\in\mathbb{R}$.*Then the regressive IVP (2.1) has a unique solution* $y:\mathbb{T}\to\mathbb{R}$ *given by*

$$y(t)=y_0e_p(t,t_0)+\int_{t_0}^{t}e_p(t,\sigma(\tau))f(\tau)\triangle\tau.$$

Theorem 6. *(Variation of constants). Suppose (2.1) is regressive. Let* $t_0\in\mathbb{T}$ *and* $x(t_0)=x_0\in\mathbb{R}$. *The unique solution of the initial value problem*

$$x^{\triangle}(t)=-p(t)x^{\sigma}+f(t),\quad x(t_0)=x_0$$

is given by

$$x(t)=e_{\ominus p}(t,t_0)x_0+\int_{t_0}^{t}e_{\ominus p}(t,\tau)f(\tau)\triangle\tau.$$

3. Main Results

Theorem 7. *Let* $\mathbb{T}=[a,b]$ *is an arbitrary interval, where* $a,b\in\mathbb{R}\cup\{\pm\infty\}$ *are arbitrarily given with* $a<b$. *Assume that* $g:\mathbb{T}\to\mathbb{R}$ *with* $g(t)\in\Re$, $h:\mathbb{T}\to\mathbb{R}$, *are rd-continuous functions such that* $g(t)$ *and* $e_g(\sigma(t),a)h(t)$ *are integrable* $[a,c]$ *for each* $c\in\mathbb{T}$. *Moreover, suppose* $\varphi:\mathbb{T}\to[0,\infty)$ *is a functions such that* $\varphi(t)e_g(t,a)$ *is integrable on* $\mathbb{T}$. *If a rd-continuously differentiable function*

$y : \mathbb{T} \to \mathbb{R}$ satisfies the differential inequality (1.2) for all $t \in \mathbb{T}$, then there exists a unique $x \in \mathbb{R}$ such that

$$\left| y(t) - e_{\ominus g}(t,a)(x - \int_a^t e_g(\sigma(\upsilon),a)h(\upsilon) \bigtriangleup \upsilon) \right| \tag{3.1}$$

$$\leq \ e_{\ominus g}(t,a) \int_t^b \varphi(\upsilon) e_g(\upsilon,a) \bigtriangleup \upsilon$$

for every $\mathbb{T}$.

Proof. For simplicity, we use the following notation:

$z(t) := e_g(t,a)y(t) + \int_a^t e_g(\sigma(\upsilon),a)h(\upsilon) \bigtriangleup \upsilon$

for each $t \in \mathbb{T}$. By making use of this notation and by (1.2), we get

$$|z(t) - z(s)| = \left| e_g(t,a)y(t) - e_g(s,a)y(s) + \int_s^t e_g(\sigma(\upsilon),a)h(\upsilon) \bigtriangleup \upsilon \right|$$

$$= \left| \int_s^t [e_g(\upsilon,a)y(\upsilon)]^{\bigtriangleup} \bigtriangleup \upsilon + \int_s^t e_g(\sigma(\upsilon),a)h(\upsilon) \bigtriangleup \upsilon \right|$$

$$= \left| \int_s^t \Big([e_g(\upsilon,a)y(\upsilon)]^{\bigtriangleup} + e_g(\sigma(\upsilon),a)h(\upsilon) \Big) \bigtriangleup \upsilon \right|$$

$$= \left| \int_s^t \big[e_g(\upsilon,a)y^{\bigtriangleup}(\upsilon) + g(\upsilon)e_g(\upsilon,a)y^{\sigma}(\upsilon) + e_g(\sigma(\upsilon),a)h(\upsilon) \big] \bigtriangleup \upsilon \right|$$

$$= \left| \int_s^t e_g(\upsilon,a) \big[y^{\bigtriangleup}(\upsilon) + g(\upsilon)y^{\sigma}(\upsilon) + e_g(\sigma(\upsilon),\upsilon)h(\upsilon) \big] \bigtriangleup \upsilon \right|$$

$$\leq \left| \int_s^t \varphi(\upsilon) e_g(\upsilon,a) \bigtriangleup \upsilon \right| \tag{3.2}$$

for any $s, t \in \mathbb{T}$.

Finally, it follows from (3.2) and the above argument that for any $t \in \mathbb{T}$,

$\left| y(t) - e_{\ominus g}(t,a)(x - \int_a^t e_g(\sigma(\upsilon),a)h(\upsilon) \bigtriangleup \upsilon) \right|$

$= |e_{\ominus g}(t,a)(z(t) - x)|$

$\leq |e_{\ominus g}(t,a)(z(t) - z(s))| + |e_{\ominus g}(t,a)(z(s) - x)|$

$$\leq e_{\ominus g}(t,a)\,|z(t)-z(s)| + e_{\ominus g}(t,a)\,|z(s)-x|$$
$$\leq e_{\ominus g}(t,a)\left|\int_s^t \varphi(\upsilon)e_g(\upsilon,a)\triangle \upsilon\right| + e_{\ominus g}(t,a)\,|z(s)-x|$$
$$\rightarrow e_{\ominus g}(t,a)\int_t^b \varphi(\upsilon)e_g(\upsilon,a)\triangle \upsilon$$

as $s \rightarrow b$, since $z(s) \rightarrow x$ as $s \rightarrow b$.

It now remains to prove the uniquenass of $x \in \mathbb{R}$. Assume that $x_1 \in \mathbb{R}$ also satisfies the inequality (3.1) in place of x. Then, we have

$$|e_{\ominus g}(t,a)(x_1 - x)| \leq 2e_{\ominus g}(t,a)\int_t^b \varphi(\upsilon)e_g(\upsilon,a)\triangle \upsilon$$

for any $t \in \mathbb{T}$. It follows from the itegrability hypotheses that

$$|x_1 - x| \leq 2\int_t^b \varphi(\upsilon)e_g(\upsilon,a)\triangle \upsilon \rightarrow 0$$

as $t \rightarrow b$. This implies the uniqueness of $x \in \mathbb{R}$. ∎

Remark 1. *we may now remark that*

$$y(t) = e_{\ominus g}(t,a)(x - \int_a^t e_g(\sigma(\upsilon),a)h(\upsilon)\triangle \upsilon)$$

is the general solution of the differential equations (1.1), where $x \in \mathbb{R}$ is an arbitrary element.

3.1. **Examples.** In this section, we will introduce some examples for linear differential equations of first order whenever $\mathbb{T} = \mathbb{R}$ as follows.

Example 1. *If we take $\mathbb{T} = [a,b] \subset \mathbb{R}$ is an arbitrary interval in $\mathbb{R}$, and we set $h(t) \equiv 0$, $\varphi(t) \equiv \varepsilon$ in Theorem.1, we obtain the following result:*

Let $\mathbb{T} = [a,b]$ is an arbitrary interval in $\mathbb{R}$, where $a,b \in \mathbb{R} \cup \{\pm\infty\}$ are arbitrarily given with $a < b$. It is clear that when $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$, $y^{\sigma}(t) = y(t)$ and $\mu = 0$. Also, when $\mathbb{T} = \mathbb{R}$, then from (1.1) equation we get

$$y'(t) + g(t)y(t) + h(t) = 0, \ldots y(a) = x$$

for all $t \in \mathbb{T}$, and (1.2) inequality

$$|y'(t) + g(t)y(t) + h(t)| \leq \varphi(t)$$

for any $t \in \mathbb{T}$. Assume that $g : \mathbb{T} \rightarrow \mathbb{R}$ is a continuous and integrable function on $[a,c]$ for each $c \in \mathbb{T}$ such that $\exp\left\{\int_a^t g(u)du\right\}$ is integrable on $\mathbb{T}$. If a continuously differentiable function $y : \mathbb{T} \rightarrow \mathbb{R}$ satisfies the differential inequality

$$|y'(t) + g(t)y(t)| \leq \varepsilon$$

for all $t \in \mathbb{T}$, then there exists a unique $x \in \mathbb{R}$ such that

$$\left|y(t) - \exp\left\{-\int_a^t g(u)du\right\}x\right|$$
$$\leq \varepsilon \exp\left\{-\int_a^t g(u)du\right\}\int_t^b \exp\left\{\int_a^v g(u)du\right\}\triangle v$$

for each $t \in \mathbb{T}$.

Example 2. *Let $g < 0$ nad h be fixed real numbers, let $\mathbb{T} = [a, \infty)$ be an interval with $a \in \mathbb{R}$, and $\varphi : \mathbb{T} \to \mathbb{R}$ satisfies the differential inequality*

$$|y'(t) + g(t)y(t) + h(t)| \leq \varphi(t)$$

for all $t \in \mathbb{T}$.

We can easily verify that the choices of g, h, φ and $\mathbb{T}$ are consistent with the hypotheses of Theorm.1. Hence, there exists a unique $c_0 \in \mathbb{R}$ such that

$$\left| y(t) - c_0 e^{-gt} + \frac{h}{g}(1 - e^{-g(t-a)}) \right| \leq e^{-gt} \int_t^{\infty} \varphi(\upsilon) e^{g\upsilon} d\upsilon$$

for any $t \in \mathbb{T}$. Further, we know that $y_0(t) = c_0 e^{-gt} - \frac{h}{g}(1 - e^{-g(t-a)})$ is a (particular) solution of the differential equation $y'(t) + g(t)y(t) + h(t) = 0$.

If we set $\varphi(t) \equiv \varepsilon$ and $\mathbb{T} = [a, \infty)$ with $a \geq 0$ in the above statement, then there exists a unique solution $y_0(t)$ of the differential equation $y'(t) + g(t)y(t) + h(t) = 0$ such that

$$|y(t) - y_0(t)| \leq \frac{\varepsilon}{g}$$

for all $t \in \mathbb{T}$. (We may compare this with [1] or [10].)

Example 3. *If we take $\mathbb{T} = [a, b]$ in real interval in $\mathbb{R}$, it is clear that we can have the same Theorem 1. in ([7]).*

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Received: July 1, 2006