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The Analytical Solution of Singular Linear Periodic Boundary Value Problem

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Abstract

In this paper, a new method is presented to obtain the analytical solution of singular linear periodic boundary value problems in the reproducing kernel space. The analytical solution is represented in the form of series. An example is given to demonstrate the reliability and validity of the presented algorithm.

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1 Introduction

In this paper, we consider the following singular second-order periodic boundary value problems in the reproducing kernel space

$$\begin{cases} a(x)u''(x) + b(x)u'(x) + c(x)u(x) = f(x), \ \ 0 \leq x \leq 1, \\ u(0) = u(1), \\ u'(0) = u'(1), \end{cases} \tag{1.1}$$

where $f(x) \in W_1[0,1], u(x) \in W_3[0,1]$, $a(x), b(x), c(x)$ are continuous, $a(0) = 0$ or/and $a(1) = 0$. The singular boundary value problem arises in a variety of

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applied mathematics and physics such as gas dynamics, nuclear physics, chemical reaction, studies of atomic structures and atomic calculations. The study of the periodic boundary value problem is one of the main classical topics in the theory of ordinary differential equations. The periodic boundary value problems have been widely investigated by a number of authors in recent years[1-5]. However, the singular periodic boundary value problems are rarely considered. In recent paper, Wang and jiang have established the existence and uniqueness of results for the singular second-order periodic boundary value problems[6]. In [7-9], the authors discussed the existence of solutions to the singular second-order periodic boundary value problems. However, there are few valid methods for solving the singular second-order periodic boundary value problems.

In this paper, we will give the representation of analytical solution to $Eq.(1.1)$ and approximate solution in the reproducing kernel space under the assumption that the solution to $Eq.(1.1)$ is unique.
Let $Lu \equiv a(x)u''(x)+b(x)u'(x)+c(x)u(x)$. Then the $Eq.(1.1)$ can be converted into the following form:

$$\begin{cases} Lu(x) = f(x), \quad 0 \le x \le 1, \\ u(0) = u(1), \\ u'(0) = u'(1), \end{cases} \tag{1.2}$$

where $f(x) \in W_1[0,1], u(x) \in W_3[0,1]$. $W_1[0,1], W_3[0,1]$ are defined in the following section.

2 Preliminary

In this section, some reproducing kernel spaces are defined to solve $Eq.(1.2)$.

Space $W_3[0,1]$ is defined as $W_3[0,1] = \{u|u''$ is absolutely continuous real valued function and $u^{(3)} \in L^2[0,1], u(0) = u(1), u'(0) = u'(1)\}$. The inner product $(\cdot,\cdot)$ and the the norm $\|\cdot\|_{W_3}$ are taken to be

$$(u(y), v(y)) \stackrel{\text{def}}{=} \int_0^1 (36uv + 49u'v' + 14u''v'' + u^{(3)}v^{(3)})dy, u, v \in W_3[0,1] \tag{2.1}$$

$$\|u\|_{W_3} \stackrel{\text{def}}{=} \sqrt{(u,u)}$$

respectively .

Theorem 2.1. *The space $W_3[0,1]$is a reproducing kernel space, that is, for any $u(y) \in W_3[0,1]$ and each fixed $x \in [0,1]$, there exists $R_x(y) \in W_3[0,1], y \in [0,1]$, such that $(u(y), R_x(y)) = u(x)$, the reproducing kernel $R_x(y)$ can be denoted by*

$$R_x(y) = \begin{cases} c_1e^{y} + c_2e^{-y} + c_3e^{2y} + c_4e^{-2y} + c_5e^{3y} + c_6e^{-3y}, & y \le x, \\ d_1e^{y} + d_2e^{-y} + d_3e^{2y} + d_4e^{-2y} + d_5e^{3y} + d_6e^{-3y}, & y > x, \end{cases} \tag{2.2}$$

The coefficients of the reproducing kernel $R_x(y)$ and the proof of Theorem 2.1 are given in appendix A,B.

Space $W_1[0,1]$ is defined by $W_1[0,1] = \{u|u$ is absolutely continuous real valued function and $u' \in L^2[0,1]\}$ equipped with the inner product

$$(u,v)_{W_1} \stackrel{\text{def}}{=} \int_0^1 uvdx + \int_0^1 u'v'dx, \qquad u,v \in W_1[0,1]$$

and norm

$$\|u\|_{W_1} \stackrel{\text{def}}{=} \sqrt{(u,u)_{W_1}}$$

respectively. In Ref.[10], the authors proved that $W_1[0,1]$ is a reproducing kernel space with the kernel

$$\overline{R}_x(y) = \frac{1}{2\sinh(1)}[\cosh(x+y-1) + \cosh(|x-y|-1)].$$

3 The method for solving $Eq.(1.2)$

In order to prove the main theorem of this paper, we give some lemmas first of all.

Lemma 3.1. *Put* $\varphi_0 = f \in W_1[0,1], \varphi_i(x) = \overline{R}_{x_i}(x), i = 1,2,\ldots,$ *where* $\overline{R}_{x_i}(x)$ *is the reproducing kernel of* $W_1[0,1]$*. If* $\{x_i\}_{i=1}^{\infty}$ *is dense on* $[0,1]$*, then* $\{\varphi_i(x)\}_{i=0}^{\infty}$ *is the complete system of* $W_1[0,1]$*.*

Proof. For $\forall u(x) \in W_1[0,1]$, let $(u(x),\varphi_i(x))_{W_1} = 0, i = 1,2,\ldots,$ then $(u(x),\varphi_i(x))_{W_1} = (u(x),\overline{R}_{x_i}(x))_{W_1} = u(x_i) = 0, i = 1,2,\ldots.$ Due to the density of $\{x_i\}_{i=1}^{\infty}$ and the continuity of $u(x)$, we have $u(x) = 0$, thus $\{\varphi_i(x)\}_{i=1}^{\infty}$ is the complete system of $W_1[0,1]$. Therefore $\{\varphi_i(x)\}_{i=0}^{\infty}$ is also the complete system of $W_1[0,1]$. □

Now, we orthogonalize the function system $\{\varphi_i(x)\}_{i=0}^{\infty}$ and obtain an orthogonal system $\{\overline{\varphi}_i(x)\}_{i=0}^{\infty}$, i.e. ,

$$\overline{\varphi}_i(x) = \sum_{k=0}^{i} \beta_{ik}\varphi_k(x) \tag{3.1}$$

where $\beta_{ik}, i,k = 0,1,2,\ldots$ are the coefficients of orthogonalization.

Let $\psi_i(x) = L^*\overline{\varphi}_i(x), i = 0,1,2,\ldots$ $L^*$is the conjugate operator of L. The orthonormal system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ of $W_3[0,1]$ can be derived from Gram-Schmidt orthogonalization process of

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \alpha_{ik}\psi_k(x)(i = 1,2,\cdots). \tag{3.2}$$

Lemma 3.2. *L is a bounded operator from $W_3[0,1]$ to $W_1[0,1]$.*

Proof. Let $R_x(\xi)$ be the reproducing kernel of $W_3[0,1]$, by the property of $R_x(\xi)$, we obtain

$$|u(x)| = |(u(\xi), R_x(\xi))_{W_3}| \leq \| R_x(\xi) \|_{W_3} \| u(x) \|_{W_3} \leq M_0 \| u(x) \|_{W_3} . \quad (3.3)$$

From the representation of $R_x(\xi)$, we can get

$$|u^{(i)}(x)| = |(u(\xi), \frac{d^{(i)}}{dx^{(i)}} R_x(\xi))_{W_3}| \leq \| \frac{d^{(i)}}{dx^{(i)}} R_x(\xi) \|_{W_3} \| u(x) \|_{W_3} . \quad (3.4)$$

Since $\frac{d^{(i)}}{dx^{(i)}} R_x(\xi), i = 1, 2, 3$ is uniformly bounded about x and ξ, we have

$$|u^{(i)}(x)| = M_i \| u(x) \|_{W_3}, i = 0, 1, 2, 3. \quad (3.5)$$

Hence

$$\begin{aligned} \| Lu(x) \|^2_{W_1} &= \| a(x)u''(x) + b(x)u'(x) + c(x)u(x) \|^2_{W_1} \\ &= \int_0^1 (a(x)u''(x) + b(x)u'(x) + c(x)u(x))dx \\ &\quad + \int_0^1 (a(x)u''(x) + b(x)u'(x) + c(x)u(x))'^2 dx \\ &\leq M \| u(x) \|_{W_3} \end{aligned}$$

by the continuity of $a(x), b(x), c(x)$ and Eq. (3.5). □

From the proof of above lemma, we have the following corollary.

Corollary 3.1. *$W_3[0,1]$ is imbeded to space $C[0,1]$.*

Lemma 3.3. *A arbitrary bounded set of $W_3[0,1]$ is a compact set of $C[0,1]$.*

Proof. Let $\{u_n(x)\}_{n=1}^{\infty}$ be a bounded set of $W_3[0,1]$. Assume $\|u_n(x)\| \leq M$, by the (3.5), we known $\|u_n(x)\|_C \leq M$, where $\| \, . \, \|_C$ denote the norm of $C[0,1]$. In order to prove that $\{u_n(x)\}_{n=1}^{\infty}$ is a compact set of $C[0,1]$, we only need to prove $\{u_n(x)\}_{n=1}^{\infty}$ are equicontinuous functions. In fact, from the property of $R_x(\xi)$, it follows that

$$\begin{aligned} |u_n(x_1) - u_n(x_2)| &= |(u_n(\xi), R_{x_1}(\xi) - R_{x_2}(\xi))| \\ &\leq \|u_n(\xi)\|_{W_3} \|R_{x_1}(\xi) - R_{x_2}(\xi)\| \\ &\leq M \|R_{x_1}(\xi) - R_{x_2}(\xi)\|. \end{aligned} \quad (3.6)$$

Using the symmetry of $R_x(\xi)$ and the mean value theorem of differentials, we have

$$\begin{aligned} | R_{x_1}(\xi) - R_{x_2}(\xi) | &= | R_\xi(x_1) - R_\xi(x_2) | = [\tfrac{d}{dx} R_\xi(x)] |_{x=\eta} | x_1 - x_2 | \\ &\leq M_1 | x_1 - x_2 | . \end{aligned} \quad (3.7)$$

By the (3.6), (3.7), when $\delta \leq | x_1 - x_2 | \leq \frac{\varepsilon}{MM_1}$, we have $| u_n(x_1) - u_n(x_2) | < \varepsilon$. □

Lemma 3.4. *$\psi_i(x) = \sum_{k=0}^{i} \beta_{ik} LR_x(x_k), i = 0, 1, 2, \ldots,$ where $\{x_k\}_{k=0}^{\infty}$ is a dense set of $[0, 1]$.*

Proof.

$$\begin{aligned}\psi_i(x) &= L^*\overline{\varphi}_i(x) = (L^*\overline{\varphi}_i(x), R_x(\xi))_{W_3} = (\overline{\varphi}_i(x), LR_x(\xi))_{W_1} \\ &= (\textstyle\sum_{k=0}^{i} \beta_{ik}\varphi_k, LR_x(\xi))_{W_1} = \sum_{k=0}^{i} \beta_{ik}(\varphi_k, LR_x(\xi)) \\ &= \textstyle\sum_{k=0}^{i} \beta_{ik} LR_x(x_k), i = 0, 1, 2, \ldots\end{aligned}$$

□

Theorem 3.1. *Let L be a bounded linear operator from $W_3[0, 1]$ to $W_1[0, 1]$, $f(x) \in W_1[0, 1]$, then u(x) is a solution of (1.2) $\Leftrightarrow$ The following (3.8) holds.*

$$\begin{cases} (u(x), \psi_0(x)) = (f, f) \\ (u(x), \psi_n(x)) = 0, n = 1, 2, \ldots . \end{cases} \tag{3.8}$$

Proof. $\Leftarrow$) Since $\overline{\varphi}_0 = f(x), \psi_i(x) = L^*\overline{\varphi}_i(x), i = 0, 1, 2, \ldots,$ by (3.8), one obtains

$$(f(x), \overline{\varphi}_0(x)) = (f(x), f(x)) = (u(x), \psi_0(x)) = (u(x), L^*\overline{\varphi}_0(x)) = (Lu(x), \overline{\varphi}_0(x)),$$

i.e.,

$$(Lu(x), \overline{\varphi}_0) = (f(x), \overline{\varphi}_0(x)). \tag{3.9}$$

From

$$0 = (u(x), \psi_n(x)) = (u(x), L^*\overline{\varphi}_n(x)) = (Lu(x), \overline{\varphi}_n(x))$$

and $(f(x), \overline{\varphi}_n(x)) = (\overline{\varphi}_0(x), \overline{\varphi}_n(x)) = 0$, we can find that

$$(Lu(x), \overline{\varphi}_n) = (f(x), \overline{\varphi}_n(x)), n = 1, 2, \ldots . \tag{3.10}$$

By the (3.9), (3.10), we obtain $(Lu(x), \overline{\varphi}_n) = (f(x), \overline{\varphi}_n(x)), n = 0, 1, 2, \ldots .$ By means of the orthogonality of $\{\overline{\varphi}_i\}_{i=0}^{\infty}$, we get $Lu(x) = f(x)$.

$\Rightarrow$) Suppose $u(x)$ is solution of (1.2), then

$$(Lu(x), \overline{\varphi}_n(x)) = (f(x), \overline{\varphi}_n(x)), n = 0, 1, 2, \ldots .$$

If $n = 0$, then

$$(u(x), \psi_0(x)) = (u(x), L^*\overline{\varphi}_0(x)) = (Lu(x), \overline{\varphi}_0(x)) = (f(x), \overline{\varphi}_0(x)) = (f, f).$$

If $n = 1, 2, \ldots,$ by the orthogonality of $f(x)$ and $\{\overline{\varphi}_i\}_{i=1}^{\infty}$, we have

$$(u(x), \psi_n(x)) = (u(x), L^*\overline{\varphi}_n) = (Lu(x), \overline{\varphi}_n) = (f(x), \overline{\varphi}_n) = (\overline{\varphi}_0, \overline{\varphi}_n) = 0.$$

□

We introduce the following notation :

$$\Psi = \{u|u = \sum_{i=1}^{\infty} \lambda_i \psi_i, \{\lambda_i\} \in l^2\}$$

$$\Psi_n = \{u|u = \sum_{i=1}^{n} \lambda_i \psi_i, \{\lambda_i\} \in R\}$$

and $\Psi^{\perp}, \Psi_n^{\perp}$ denote the orthogonal complement of Ψ, Ψ_n in $W_3[0,1]$. In order to emphasize the first function of $\{\overline{\varphi}_i\}_{i=0}^{\infty}$ is $f(x)$, Ψ, Ψ_n are denoted as $\Psi(f), \Psi_n(f)$ respectively. $\{\psi_i(x)\}$ is written as $\{\psi_i(x,f)\}$.

Lemma 3.5. *The dimension of $\Psi(f)^{\perp}$ is one.*

Proof. We only need to prove that $\{\psi_i(x)\}_{i=0}^{\infty}$ is complete system of $W_3[0,1]$. Let $(u, \psi_i)_{W_3} = 0, i = 0, 1, 2, \cdots, u \in W_3$, with $i = 0$,

$$(u, \psi_0)_{W_3} = (u, L^*\overline{\varphi}_0)_{W_3} = (Lu, f)_{W_1} = 0. \tag{3.11}$$

Put $(u, \psi_1) = 0$, then

$$(u, \psi_1)_{W_3} = (u, L^*\overline{\varphi}_1)_{W_3} = (Lu, \beta_{10} f + \beta_{11}\varphi_1)_{W_1} = 0.$$

By (3.11),

$$(u, \psi_1)_{W_3} = (Lu, \beta_{11}\varphi_1)_{W_1} = \beta_{11}((Lu)(\cdot), R_{x_1}(\cdot))_{W_1} = \beta_{11}(Lu)(x_1) = 0.$$

In view of $\beta_{11} > 0$, then

$$Lu(x_1) = 0. \tag{3.12}$$

Let $(u, \psi_2)_{W_3} = 0$, then

$$\begin{aligned} (u, \psi_2)_{W_3} &= (u, L^*\overline{\varphi}_2)_{W_3} = (Lu, \overline{\varphi}_2)_{W_1} \\ &= (Lu, \beta_{20} f + \beta_{21}\varphi_1 + \beta_{22}\varphi_2)_{W_1} \\ &= (Lu, \beta_{20} f)_{W_1} + \beta_{21}(Lu, \varphi_1)_{w_1} + \beta_{22}(Lu, \varphi_2)_{w_1} \\ &= \beta_{20}(Lu, f) + \beta_{21}(Lu)(x_1) + \beta_{22}(Lu)(x_2) = 0 \end{aligned} \tag{3.13}$$

From (3.11), (3.12) and (3.13), one can obtain that

$$(Lu)(x_2) = 0. \tag{3.14}$$

We can deduce by induction

$$(Lu)(x_n) = 0, n = 1, 2, \cdots. \tag{3.15}$$

Hence $Lu = 0$ by the density of $\{x_n\}_{n=1}^{\infty}$ on $[0,1]$ and the continuity of Lu. It follows that $u = 0$ from the existence of L^{-1}.

□

Theorem 3.2. *The solution of Eq.*(1.2) *can be represented as*

$$u = \frac{(f,f)g(x,f)}{(g(x,f), L^*f)_{W_3}} \tag{3.16}$$

where $g(x,f) \in \Psi^{\perp}$.

Proof. Clearly,

$$(\frac{(f,f)g(x,f)}{(g(x,f), L^*f)_{W_3}}, L^*f)_{W_3} = (f,f),$$

Since $g(x,f) \in \Psi^{\perp}$, we must have

$$(\frac{(f,f)g(x,f)}{(g(x,f), L^*f)_{W_3}}, \psi_i)_{W_3} = 0, i = 1, 2, \cdots .$$

□

It is easy to obtain the following lemma by $g_n(x,f) \in \Psi_n^{\perp}(f)$.

Lemma 3.6. *If* $P\psi_n$ *is orthogonal projector from* W_3 *to* $\Psi_n(f)$, $\forall h \notin \Psi(f)$, *let* $g_n(x,f) = \frac{(h - P\psi_n h)}{\|h - P\psi_n h\|}$, *then*

$$u_n = \frac{(f,f)g_n(x,f)}{(g_n(x,f), L^*f)_{W_3}} \tag{3.17}$$

satisfies

$$\begin{array}{lcl} (u_n, L^*f)_{W_3} & = & (f,f), \\ (u_n, \psi_i)_{W_3} & = & 0, i = 1, 2, \cdots, n. \end{array} \tag{3.18}$$

Lemma 3.7. *Let* $\theta_n = (g_n(x,f), L^*f)$, *if* $\|u_n\|_{W_3} \le M$, *then* $\theta_n \ge \frac{\|f\|^2}{M} > 0$.

Proof. Note that

$$\begin{array}{lcl} \|u_n\|^2_{W_3} & = & (\frac{(f,f)g_n(x,f)}{(g_n(x,f),L^*f)_{W_3}}, \frac{(f,f)g_n(x,f)}{(g_n(x,f),L^*f)_{W_3}})_{W_3} \\ & = & \frac{\|f\|^4 (g_n(x,f), g_n(x,f))}{\theta_n^2} = \frac{\|f\|^4}{\theta_n^2} \le M^2. \end{array}$$

Hence

$$\theta_n \ge \frac{\|f\|^2}{M} > 0. \tag{3.19}$$

□

Theorem 3.3. *If* $\|u_n\|_{W_3} \le M$, *then Eq.*(1.2) *have a solution and the solution is* $\overline{u} = \frac{(f,f)\overline{g}(x,f)}{(\overline{g}(x,f), L^*f)_{W_3}}$.

Proof. From lemma 3.3, we know that $\{u_n\}$ is a compact set on $C[0,1]$. Thus there exists a subsequence $u_{n_k} \xrightarrow{C[0,1]} \overline{u}$. By (3.19), it is easy to obtain $\theta_{n_k} \to \overline{\theta} > 0$. Taking subsequence from (3.17)

$$u_{n_k} = \frac{(f,f)g_{n_k}(x,f)}{(g_{n_k}(x,f), L^*f)_{W_3}}$$

and taking limit, we get

$$\overline{u} = \frac{(f,f)\overline{g}(x,f)}{(\overline{g}(x,f), L^*f)_{W_3}}. \tag{3.20}$$

From lemma 3.6, it is easy to see that $\overline{u}$ satisfies the conditions of theorem 3.1. Thus $\overline{u}$ is the solution of $Eq.$(1.2). □

4 Example

In this section, an example will be tested by using the method discussed above. All experiments were performed in MATHEMATICA 5.0. Results obtained by the method are compared with the exact solution.

Example 1

Consider the equation

$$xu''(x) + u'(x) + xu(x) = f(x)$$

where

$$f(x) = xe^{x^2(x-1)^2}(5 - 18x + 20x^2 - 24x^3 + 52x^4 - 48x^5 + 16x^6),$$

$u(x) \in W_3[0,1]$ subject to boundary conditions $u(0) = u(1), u'(0) = u'(1)$. The exact solution is $u(x) = e^{x^2(x-1)^2}$. The exact and approximate solutions and the absolute error are displayed in Table 1 with $N = 50$.

5 Appendix

A The coefficients of the reproducing kernel $R_x(y)$

$$\Delta_1 = 96\,(-1+e)\,e^{3x}\,(239 + 244\,e - 246\,e^2 + 244\,e^3 + 239\,e^4)\,;$$

$$\Delta_2 = 60\,(-1+e)\,e^{3x}\,(239 + 483\,e - 2\,e^2 - 2\,e^3 + 483\,e^4 + 239\,e^5)\,;$$

$$\Delta_3 = 480\,(-1+e)\,e^{3x}\,(239 + 483\,e + 237\,e^2 + 242\,e^3 + 237\,e^4 + 483\,e^5 + 239\,e^6)\,;$$

$$c1 = \frac{1}{\Delta_1}(-27\,e^3 - 27\,e^4 + 5\,e^{4x} - 32\,e^{5x} + 27\,e^{6x} + 32\,e^{2+x} + 32\,e^{3+x} + 32\,e^{4+x} + 473\,e^{1+2x} + 478\,e^{2+2x} - 502\,e^{3+2x} + 483\,e^{4+2x} + 478\,e^{5+2x} + 10\,e^{1+4x} + 10\,e^{2+4x} + 5\,e^{3+4x} - 32\,e^{1+5x} - 32\,e^{2+5x} + 27\,e^{1+6x});$$

Table 1:

Node	True solution $u(x)$	Approximate solution $u_{50}(x)$	Absolute error
0.08	1.00543	1.0054	0.0000296116
0.16	1.01823	1.01843	0.000206946
0.24	1.03383	1.0342	0.000369521
0.32	1.04849	1.04899	0.000500041
0.40	1.05929	1.05988	0.000586402
0.48	1.06428	1.0649	0.000615768
0.56	1.06259	1.06318	0.000589435
0.64	1.05452	1.05503	0.000510008
0.72	1.04148	1.04187	0.000385639
0.80	1.02593	1.02616	0.000227555
0.88	1.01121	1.01127	0.0000515199
0.96	1.00148	1.00137	0.00010512

$$
\begin{aligned}
c2 = {} & \tfrac{1}{\Delta_1}(27\,e^{4} + 27\,e^{5} + 478\,e^{4\,x} - 32\,e^{3+x} - 32\,e^{4+x} - 32\,e^{5+x} + 5\,e^{2+2\,x} + 10\,e^{3+2\,x} + 10\,e^{4+2\,x} + 5 \\
& e^{5+2\,x} + 483\,e^{1+4\,x} - 502\,e^{2+4\,x} + 478\,e^{3+4\,x} + 473\,e^{4+4\,x} + 32\,e^{1+5\,x} + 32\,e^{2+5\,x} + 32\,e^{3+5\,x} - \\
& 27\,e^{1+6\,x} - 27\,e^{2+6\,x});\\
c3 = {} & -\tfrac{1}{\Delta_2}(108\,e^{3} + 108\,e^{4} - 20\,e^{4\,x} + 128\,e^{5\,x} - 108\,e^{6\,x} - 367\,e^{2+x} - 372\,e^{3+x} + 118\,e^{4+x} - 244\,e^{5+x} \\
& -239\,e^{6+x} + 20\,e^{1+2\,x} + 40\,e^{2+2\,x} + 40\,e^{3+2\,x} + 20\,e^{4+2\,x} - 40\,e^{1+4\,x} - 40\,e^{2+4\,x} - 20\,e^{3+4\,x} + \\
& 128\,e^{1+5\,x} + 128\,e^{2+5\,x} - 108\,e^{1+6\,x});\\
c4 = {} & -\tfrac{1}{\Delta_2}(-108\,e^{5} - 108\,e^{6} - 239\,e^{5\,x} + 128\,e^{4+x} + 128\,e^{5+x} + 128\,e^{6+x} - 20\,e^{3+2\,x} - 40\,e^{4+2\,x} - \\
& 40\,e^{5+2\,x} - 20\,e^{6+2\,x} + 20\,e^{2+4\,x} + 40\,e^{3+4\,x} + 40\,e^{4+4\,x} + 20\,e^{5+4\,x} - 244\,e^{1+5\,x} + 118\,e^{2+5\,x} - \\
& 372\,e^{3+5\,x} - 367\,e^{4+5\,x} + 108\,e^{2+6\,x} + 108\,e^{3+6\,x});\\
c5 = {} & \tfrac{1}{\Delta_3}(-251\,e^{3} - 241\,e^{4} - 492\,e^{5} + 488\,e^{6} + 478\,e^{7} + 135\,e^{4\,x} - 864\,e^{5\,x} + 729\,e^{6\,x} + 864\,e^{2+x} + 864 \\
& e^{3+x} + 864\,e^{4+x} - 135\,e^{1+2\,x} - 270\,e^{2+2\,x} - 270\,e^{3+2\,x} - 135\,e^{4+2\,x} + 270\,e^{1+4\,x} + 270\,e^{2+4\,x} + 135 \\
& e^{3+4\,x} - 864\,e^{1+5\,x} - 864\,e^{2+5\,x} + 729\,e^{1+6\,x});\\
c6 = {} & \tfrac{1}{\Delta_3}(729\,e^{6} + 729\,e^{7} + 478\,e^{6\,x} - 864\,e^{5+x} - 864\,e^{6+x} - 864\,e^{7+x} + 135\,e^{4+2\,x} + 270\,e^{5+2\,x} + 270 \\
& e^{6+2\,x} + 135\,e^{7+2\,x} - 135\,e^{3+4\,x} - 270\,e^{4+4\,x} - 270e^{5+4\,x} - 135\,e^{6+4\,x} + 864\,e^{3+5\,x} + 864\,e^{4+5\,x} + \\
& 864\,e^{5+5\,x} + 488\,e^{1+6\,x} - 492\,e^{2+6\,x} - 241\,e^{3+6\,x} - 251\,e^{4+6\,x});\\
d1 = {} & \tfrac{1}{\Delta_1}(-27\,e^{3} - 27\,e^{4} + 478\,e^{2\,x} + 5\,e^{4\,x} - 32\,e^{5\,x} + 27\,e^{6\,x} + 32\,e^{2+x} + 32\,e^{3+x} + 32\,e^{4+x} + 483\,e^{1+2\,x} \\
& -502\,e^{2+2\,x} + 478\,e^{3+2\,x} + 473\,e^{4+2\,x} + 10\,e^{1+4\,x} + 10\,e^{2+4\,x} + 5\,e^{3+4\,x} - 32\,e^{1+5\,x} - 32\,e^{2+5\,x} + \\
& 27\,e^{1+6\,x});\\
d2 = {} & \tfrac{1}{\Delta_1}(e^{1-3\,x}\,27\,e^{3} + 27\,e^{4} + 473\,e^{4\,x} + 32\,e^{5\,x} - 27\,e^{6\,x} - 32\,e^{2+x} - 32\,e^{3+x} - 32\,e^{4+x} + 5\,e^{1+2\,x} + 10 \\
& e^{2+2\,x} + 10\,e^{3+2\,x} + 5\,e^{4+2\,x} + 478\,e^{1+4\,x} - 502\,e^{2+4\,x} + 483\,e^{3+4\,x} + 478\,e^{4+4\,x} + 32\,e^{1+5\,x} + 32 \\
& e^{2+5\,x} - 27\,e^{1+6\,x});\\
d3 = {} & -\tfrac{1}{\Delta_2}(108\,e^{3} + 108\,e^{4} - 239\,e^{x} - 20\,e^{4\,x} + 128\,e^{5\,x} - 108\,e^{6\,x} - 244\,e^{1+x} + 118\,e^{2+x} - 372\,e^{3+x} - \\
& 367\,e^{4+x} + 20\,e^{1+2\,x} + 40\,e^{2+2\,x} + 40\,e^{3+2\,x} + 20\,e^{4+2\,x} - 40\,e^{1+4\,x} - 40\,e^{2+4\,x} - 20\,e^{3+4\,x} + 128 \\
& e^{1+5\,x} + 128\,e^{2+5\,x} - 108\,e^{1+6\,x});\\
d4 = {} & -\tfrac{1}{\Delta_2}(e^{2-3\,x}\; - 108\,e^{3} - 108\,e^{4} + 20\,e^{4\,x} - 367\,e^{5\,x} + 108\,e^{6\,x} + 128\,e^{2+x} + 128\,e^{3+x} + 128\,e^{4+x} - \\
& 20\,e^{1+2\,x} - 40\,e^{2+2\,x} - 40\,e^{3+2\,x} - 20\,e^{4+2\,x} + 40\,e^{1+4\,x} + 40\,e^{2+4\,x} + 20\,e^{3+4\,x} - 372\,e^{1+5\,x} + 118 \\
& e^{2+5\,x} - 244\,e^{3+5\,x} - 239\,e^{4+5\,x} + 108\,e^{1+6\,x});\\
d5 = {} & \tfrac{1}{\Delta_3}(478 + 488\,e - 492\,e^{2} - 241\,e^{3} - 251\,e^{4} + 135\,e^{4\,x} - 864\,e^{5\,x} + 729\,e^{6\,x} + 864\,e^{2+x} + 864\,e^{3+x} \\
& +864\,e^{4+x} - 135\,e^{1+2\,x} - 270\,e^{2+2\,x} - 270\,e^{3+2\,x} - 135\,e^{4+2\,x} + 270\,e^{1+4\,x} + 270\,e^{2+4\,x} + 135 \\
& e^{3+4\,x} - 864\,e^{1+5\,x} - 864\,e^{2+5\,x} + 729\,e^{1+6\,x});\\
d6 = {} & \tfrac{1}{\Delta_3}(e^{3-3\,x}\,729\,e^{3} + 729\,e^{4} - 135\,e^{4\,x} + 864\,e^{5\,x} - 251\,e^{6\,x} - 864\,e^{2+x} - 864\,e^{3+x} - 864\,e^{4+x} + 135 \\
& e^{1+2\,x} + 270\,e^{2+2\,x} + 270\,e^{3+2\,x} + 135\,e^{4+2\,x} - 270\,e^{1+4\,x} - 270\,e^{2+4\,x} - 135\,e^{3+4\,x} + 864\,e^{1+5\,x} + \\
& 864\,e^{2+5\,x} - 241\,e^{1+6\,x} - 492\,e^{2+6\,x} + 488\,e^{3+6\,x} + 478\,e^{4+6\,x}).
\end{aligned}
$$

B The proof of Theorem(2.1)

Through several integrations by parts for (2.1), then

$$\begin{array}{ll}(u(y),R_x(y))_{W_3}= & \int_0^1 u(y)(36R_x(y)-49R_x^{(2)}(y)+14R_x^{(4)}(y)-R_x^{(6)}(y))dy+u(y)(49R_x'(y)\\ & -14R_x^{(3)}(y)+R_x^{(5)}(y))|_0^1+u'(y)(14R_x^{(2)}(y)-R_x^{(4)}(y))|_0^1+u''(y)R_x^{(3)}(y)|_0^1.\end{array} \quad \text{(B.1)}$$

Since $R_x(y)\in W_3[0,1]$, it follows that

$$R_x(0)=R_x(1),R_x'(0)=R_x'(1). \quad \text{(B.2)}$$

Since $u\in W_3[0,1]$, $u(0)=u(1),u'(0)=u'(1)$. If

$$49R_x'(1)-14R_x^{(3)}(1)+R_x^{(5)}(1)-(49R_x'(0)-14R_x^{(3)}(0)+R_x^{(5)}(0))=0 \quad \text{(B.3)}$$

and

$$14R_x^{(2)}(1)-R_x^{(4)}(1)-(14R_x^{(2)}(0)-R_x^{(4)}(0))=0,R_x^{(3)}(0)=0,R_x^{(3)}(1)=0, \quad \text{(B.4)}$$

then $(B.1)$ implies that

$$(u(y),R_x(y))_{W_3}=\int_0^1 u(y)(36R_x(y)-49R_x^{(2)}(y)+14R_x^{(4)}(y)-R_x^{(6)}(y))dy.$$

For $\forall x\in[0,1]$, if $R_x(y)$ also satisfies

$$36R_x(y)-49R_x^{(2)}(y)+14R_x^{(4)}(y)-R_x^{(6)}(y)=\delta(y-x), \quad \text{(B.5)}$$

then

$$(u(y),R_x(y))_{W_3}=u(x).$$

Characteristic equation of (B.5) is given by

$$\lambda^6-14\lambda^4+49\lambda^2-36=0,$$

then we can obtain characteristic values $\lambda_1=1,\lambda_2=-1,\lambda_3=2,\ \lambda_4=-2,\lambda_5=3$, and $\lambda_6=-3$. So, let

$$R_x(y)=\begin{cases} c_1e^y+c_2e^{-y}+c_3e^{2y}+c_4e^{-2y}+c_5e^{3y}+c_6e^{-3y}, & y\le x,\\ d_1e^y+d_2e^{-y}+d_3e^{2y}+d_4e^{-2y}+d_5e^{3y}+d_6e^{-3y}, & y>x.\end{cases}$$

On the other hand, for (B.5), let $R_x(y)$ satisfy

$$R_x^{(k)}(x+0)=R_x^{(k)}(x-0),k=0,1,2,3,4. \quad \text{(B.6)}$$

Integrating (B.5) from $x-\varepsilon$ to $x+\varepsilon$ with respect to y and let $\varepsilon\to0$, we have the jump degree of $R_x^{(5)}(y)$ at $y=x$

$$R_x^{(5)}(x-0)-R_x^{(5)}(x+0)=1. \quad \text{(B.7)}$$

From (B.2),(B.3),(B.4),(B.6), (B.7), the unknown coefficients of (2.2) can be obtained.

References

[1] Fuyi Li, Zhanping Liang, Existence of positive periodic solutions to nonlinear second order differential equations, Applied mathematics Letters, 18(2005), 1256-1264.

[2] F.Merdivenci Atici, G.sh.Guseinov, On the existence of positive solutions for nonlinear differential equations with periodic boundary conditions, Journal of Computational and Applied Mathematics, 132(2001), 341-356.

[3] S.R.Baslandze, I.T.Kiguradze, On the unique solvability of a periodic boundary value problems for third order linear differential equations, Differential Equations, 42(2)(2006), 165-171.

[4] J.J.Nieto, Nonlinear second order periodic boundary value problems, Journal of Mathematical Analysis and Applications, 130(1988), 22-29.

[5] V.Seda, J.J.Nieto, M.Gera, Periodic boundary value problem for nonlinear higher ordinary differential equations, Applied Mathematics and Computation, 48(1992), 71-82.

[6] J.Y.Wang, D.Q.Jiang, A singular nonlinear second-order periodic boundary value problems, J.Tohohu Math. 50(1998), 203-210.

[7] Irena Rachunková, Existence of two positive solutions of a singular nonlinear periodic boundary value problem, Journal of Computational and Applied Mathematics, 113(2000), 27-34.

[8] Daqing Jiang, Jifeng Chu, Donal o'Regan, Ravi P.Agarwal, Multiple positive solutions to superlinear periodic boundary value problems with repulsive singular forces, J.Math.Anal.appl. 286(2003), 563-576.

[9] Zhongxin Zhang, Junyu Wang, On existence and multipleicity of positive solutions to periodic boundary value problems for singualr nonlinear second order differential equations, J.Math.Anal.appl. 281(2003), 99-107.

[10] Chunli Li,Minggen Cui. The exact solution for solving a class nonlinear operator equations in the reproducing kernel space. Appl.Math.Compu. 143(2-3)(2003), 393-399.

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