Applied Mathematical Sciences, Vol. 1, 2007, no. 2, 89 - 95

Fault Tolerant Routings in Complete Multipartite Graph[1]

Meirun Chen and Jianguo Qian[2]

School of Mathematical Sciences, Xiamen University
Fujian, Xiamen 361005, P.R. China

Abstract

The f-tolerant arc-forwarding index for a class of complete multipartite graphs are determined by constructing relevant fault tolerant routings. Furthermore, these routings are leveled for Cocktail-party graph and the complete 3-partite graph.

Mathematics Subject Classification: 05C38, 05C90, 90B10

Keywords: leveled fault tolerant routing, multipartite graph

1 Introduction

Routing communication demands is one of the fundamental problems in the area of networking. One of the most recognized recent applications of the problem is in the area of optical networking[1]. Most of the research concentrates on determining two invariants of a given optical network-the *arc-forwarding* and *optical indexes*[3]. In [4], fault-tolerant issues of optical networks were considered and the two invariants were generalized into so called *f-tolerant arc-forwarding* and *f-tolerant optical indexes*. The parameter f represents the number of faults that are tolerated in the optical networks. To determine the arc-forwarding index of a network, one has to design a delicate path system which uses every link in the network evenly.

An all-optical network can be modelled as a *symmetric* directed graph G with vertex set $V(G)$ and arc set $A(G)$, i.e. , if $(u, v) \in A(G)$ then $(v, u) \in A(G)$. Let $P(u, v)$ denote a directed path from u to v in G. An *f-fault tolerant routing* is defined by

$$R_f(G) = \{P_i(u, v) : (u, v) \in V(G) \times V(G), u \neq v, i = 0, \cdots, f\},$$

[1]supported by NSFC(10371102) and the Program of 985 Innovation Engineering on Information in Xiamen University (2004-2007)

[2]corresponding author, email: jgqian@xmu.edu.cn

where for each ordered pair $(u, v) \in V(G) \times V(G)$ of distinct nodes, the paths $P_0(u, v)$, $P_1(u, v)$, $\cdots$, $P_f(u, v)$ are internally node disjoint. As noted in [2], when links and/or nodes may fail in a network, it is important to establish connections for a required communication demands, that guarantee fault-free transmission. Assuming that at most f links/nodes may fail in the network, the set $\mathcal{R}_f(G)$ will obviously provide such a routing.

In practical applications the number of different signals a link can carry is limited. Therefore, the goal is to design a routing which minimizes the maximum load on arcs. Let $\pi(\mathcal{R}_f(G))$ denote the maximum load on arcs, that is, the maximum number of times an arc of G appears in directed paths of $\mathcal{R}_f(G)$. Then

$$\pi_f(G) = \min_{\mathcal{R}_f(G)} \pi(\mathcal{R}_f(G))$$

is called the *f-tolerant arc-forwarding index* of G.

Consider an f-fault tolerant routing $\mathcal{R}_f(G)$. We say that a routing $\mathcal{R}_f$ is *optimal* if its congestion achieves the index $\pi_f(G)$; and it is *balanced* if the difference between congestions of two arcs is at most one. For any $i = 0, \cdots, f$, the *level i* of the routing $\mathcal{R}_f$ is the set of paths $P_i(u, v) \in \mathcal{R}_f$ for all $u \neq v$. It follows that for any $f' < f$, the subrouting $\mathcal{R}_{f'}(G)$ consisting of levels $0, \cdots, f'$, is f'-fault tolerant. We say that an optimal balanced routing $\mathcal{R}_f(G)$ is *leveled* if every one of its subroutings is also optimal and balanced.

In [4], the optimal and balanced routing $\mathcal{R}_f$ was constructed for all hypercube. Using design theory approach, A. Gupta *et al.* found the optimal f-tolerant arc-forwarding indexes for complete graph of a prime power order [6]. Following this approach, in this paper we determine the f-tolerant arc-forwarding indexes for the complete multipartite graphs $K_{\{n,m\}}$ (here n is a prime power) by constructing a fault tolerant routing. Furthermore, for some f and P (*e.g,* the Cocktail-party graph), these routings are leveled.

2 Main results

The k-distance $d_k(u, v)$ of two vertices u and v is the minimum sums of lengths of k internally node disjoint paths connecting u and v. In particular, if $k = 1$ then $d_1(u, v)$ is the usual distance $d(u, v)$.

The following lower bound for $\pi_f(G)$ was observed by Gupta et. al.

Lemma 2.1 (A.Gupta, [6]) *For any graph $G = (V, E)$ and $f \leq |V| - 2$*

$$\pi_f(G) \geq \left\lceil \frac{1}{|E|} \sum_{u,v \in V} d_{f+1}(u, v) \right\rceil$$

□

For two natural numbers n and m, we denote by $K_{\{n,m\}}$ the complete n-partite graph with each partition part containing exactly m vertices. In particular, if $m = 2$ then $K_{\{n,m\}}$ is known as the Cocktail-party graph $CP(n)$. For convenience, in the following the n partition parts of $K_{\{n,m\}}$ will be labeled by the 0-th, 1-th, $\cdots$, and the $(n-1)$-th part, respectively. Similarly, the m vertices in a partition part is labeled by 0, 1, $\cdots, m-1$. Furthermore, we write the vertex j in the i-th partition part as $\langle i, j\rangle$, $i \in \{0, 1, \cdots, n-1\}$, $j \in \{0, 1, \cdots, m-1\}$.

By the definition of $K_{\{n,m\}}$, for two vertices $\langle i, s\rangle$ and $\langle i, t\rangle$ with $s \neq t$, one can see that every path from $\langle i, s\rangle$ to $\langle i, t\rangle$ has length at least 2, and thus we have $d_{f+1}(\langle i, s\rangle, \langle i, t\rangle) \geq 2(f + 1)$. On the other hand, for two vertices $\langle i, s\rangle$ and $\langle j, t\rangle$ with $i \neq j$, it can also be observed that there is only one path of length 1 and exactly $m(n-2)$ internally node disjoint paths of length 2 which connect $\langle i, s\rangle$ and $\langle j, t\rangle$, respectively. And all other paths have length at least 3. This implies that, if $f \leq m(n-2)$ then

$$d_{f+1}(\langle i, s\rangle, \langle j, t\rangle) \geq 2f + 1;$$

and if $f > m(n-2)$ then

$$d_{f+1}(\langle i, s\rangle, \langle j, t\rangle) \geq 1 + 3f - m(n-2).$$

Therefore, by Lemma 2.1 we have

a) If $f \leq m(n-2)$, then

$$\pi_f(K_{\{n,m\}}) \geq \left\lceil \frac{1}{m^2n(n-1)}[m(m-1)n \times 2(f+1) + m^2n(n-1)(2f+1)] \right\rceil$$

$$= \left\lceil \frac{2(m-1)(f+1)}{m(n-1)} \right\rceil + 2f + 1. \tag{1}$$

b) If $f > m(n-2)$, then

$$\pi_f(K_{\{n,m\}}) \geq$$

$$\left\lceil \frac{1}{m^2n(n-1)}[m(m-1)n \times 2(f+1) + m^2n(n-1)(1 + 3f - m(n-2))] \right\rceil$$

$$= \left\lceil \frac{2(m-1)(f+1)}{m(n-1)} \right\rceil + 3f - m(n-2) + 1. \tag{2}$$

In particular, if $m = 2$, then we have

$$\pi_f(K_{\{n,m\}}) = \pi_f(CP(n)) \geq \begin{cases} 2f+2, & f \leq n-2; \\ 2f+3, & n-2 < f \leq 2n-4; \\ 2f+4, & f = 2n-3. \end{cases} \tag{3}$$

Similarly, if $n = 3$, then

$$\pi_f(K_{\{3,m\}}) \geq \begin{cases} 3f+2, & f \leq m-2; \\ 3f+1, & f = m-1; \\ 4f+1-m, & m \leq f \leq 2m-2; \\ 4f-m, & f = 2m-1. \end{cases} \tag{4}$$

Theorem 2.2 *Let m be a positive integer and n be a prime power.*
(i) For every $f = \alpha(n-1)-1$, $\alpha = 1, 2, \cdots, m$, there is an f-fault tolerant routing for $K_{\{n,m\}}$, which implies that

$$\pi_f(K_{\{n,m\}}) = \begin{cases} 1+2f+\frac{2(f+1)}{n-1}, & if\ f \leq m(n-2); \\ 1+3f-m(n-2)+\frac{2(f+1)}{n-1}, & if\ m(n-2) < f \leq m(n-1)-1. \end{cases}$$

Furthermore, if $f \leq \lceil \frac{n}{2} \rceil - 2$ then there is a leveled f-fault tolerant routing, which implies that $\pi_f(K_{\{n,m\}}) = 2f+2$ for each $0 \leq f \leq \lceil \frac{n}{2} \rceil - 2$.
(ii) There is a leveled f-fault tolerant routing for the Cocktail-party graph $K_{\{n,2\}}$ and complete 3-partite graph $K_{\{3,m\}}$, which implies that, for any $f \leq 2n-3$,

$$\pi_f(K_{\{n,2\}}) = \begin{cases} 2f+2, & f \leq n-2; \\ 2f+3, & n-2 < f \leq 2n-4; \\ 2f+4, & f = 2n-3. \end{cases}$$

and for any $f \leq 2m-1$,

$$\pi_f(K_{\{3,m\}}) = \begin{cases} 3f+2 & f \leq m-2; \\ 3f+1 & f = m-1; \\ 4f+1-m & m \leq f \leq 2m-2; \\ 4f-m & f = 2m-1. \end{cases}$$

Before proving Theorem 2.2, let us introduce some properties related to Latin square which will play an important role in constructing a leveled routing for $K_{\{n,m\}}$.

A Latin square of order n is a pair (S, F) where F is a function $F : S \times S \longrightarrow S$ such that for any $u, w \in S$, the equation

$$F(u, v) = w \ (resp.\ F(v, u) = w)$$

has a unique solution $v \in S$. A Latin square is normally written as an $n \times n$ array for which the cell in row u and column v contains the symbol $F(u, v)$. We say that a Latin square (S, F) is idempotent if for every $u \in S$, $F(u, u) = u$. We say that two Latin squares (S, F_1) and (S, F_2) are independent if for all $u \neq v$, $F_1(u, v) \neq F_2(u, v)$.

Lemma 2.3 [5] *Let $n = p^r$ be a prime power. Then there exist $n-2$ mutually independent idempotent Latin squares.* □

Proof of Theorem 2.2 (i). By the existence of $n-2$ mutually independent idempotent Latin squares $F_h(i,j), h = 1,2,\cdots,n-2; i,j \in \{1,2,\cdots,n\}$, the $f+1$ internally node disjoint paths connecting any two nodes of different partition parts can be constructed as below:

$$\mathcal{A}_0 = \{P_0(\langle i,s\rangle,\langle j,t\rangle):\ \langle i,s\rangle \to \langle j,t\rangle,$$
$$i,j \in \{0,1,\cdots,n-1\}, i \neq j; s,t \in \{0,1,\cdots,m-1\}\}.$$

If $k \leq m(n-2)$, then define

$$\mathcal{A}_k = \{P_k(\langle i,s\rangle,\langle j,t\rangle):\ \langle i,s\rangle \to \left\langle F_{\lceil \frac{k}{m}\rceil}(i,j),(s+t+k) \bmod(m)\right\rangle \to \langle j,t\rangle,$$
$$i,j \in \{0,1,\cdots,n-1\}, i \neq j; s,t \in \{0,1,\cdots,m-1\}\}.$$

If $m(n-2)+1 \leq k \leq m(n-1)-1$, then

$$\mathcal{A}_k = \{P_k(\langle i,s\rangle,\langle j,t\rangle):\ \langle i,s\rangle \to \langle j,(t-k) \bmod(m)\rangle \to \langle i,(s+k) \bmod(m)\rangle \to \langle j,t\rangle,$$
$$i,j \in \{0,1,\cdots,n-1\}, i \neq j; s,t \in \{0,1,\cdots,m-1\}\}.$$

On the other hand, for any k, $0 \leq k \leq f$, let

$$\mathcal{B}_k = \{P_k(\langle i,s\rangle,\langle i,t\rangle):\ \langle i,s\rangle \to$$
$$\left\langle (i+k+1-\left\lfloor\frac{k}{n-1}\right\rfloor\cdot(n-1)) \bmod(n), (F(s,t)+\left\lfloor\frac{k}{n-1}\right\rfloor) \bmod(m)\right\rangle$$
$$\to \langle i,t\rangle, i \in \{0,1,\cdots,n-1\}; s,t \in \{0,1,\cdots,m-1\}; s \neq t\ \},$$

where $F(s,t)$ is the function of an arbitrary idempotent Latin square $(\{0,1,\cdots,m-1\},F)$.

One can verify that every arc is used exactly once by $\mathcal{A}_0$. Furthermore, if $1 \leq k \leq m(n-2)$ then by the property of mutually independent idempotent Latin squares, an arc $(\langle i,s\rangle,\langle j,t\rangle)$ is used by a path $P_k(\langle\alpha,\beta\rangle,\langle\alpha',\beta'\rangle)$ if and only if

1) $\alpha = i, \beta = s$, $\beta' = t-k-s(\mathrm{mod} m)$ and α' is the unique solution of $F_{\lceil\frac{k}{m}\rceil}(i,\alpha') = j$; or

2) $\alpha' = j, \beta' = t$, $\beta = s-k-t(\mathrm{mod} m)$ and α is the unique solution of $F_{\lceil\frac{k}{m}\rceil}(\alpha,j) = i$.

The above argument means that each arc is used exactly twice by $\mathcal{A}_k$ for $k \in \{1,2,\cdots,m(n-2)\}$. Similarly, if $m(n-2)+1 \leq k \leq m(n-1)-1$ then the arc $(\langle i,s\rangle,\langle j,t\rangle)$ is used by the paths $P_k(\langle i,s\rangle,\langle j,k+t(\mathrm{mod} m)\rangle)$, $P_k(\langle i,s-k(\mathrm{mod} m)\rangle,\langle j,t\rangle)$ and $P_k(\langle j,t-k(\mathrm{mod} m)\rangle,\langle i,s+k(\mathrm{mod} m)\rangle)$ in $\mathcal{A}_k$

or equivalent to say, each arc is used exactly three times by $\mathcal{A}_k$ for $k \in \{m(n-2)+1, m(n-2)+2, \cdots, m(n-1)-1\}$.

On the other hand, for every $k = \alpha(n-1)-1$, $\alpha \in \{1, 2, \cdots, m\}$, the union of $\mathcal{B}_0$, $\mathcal{B}_1$, $\cdots$, $\mathcal{B}_k$ uses every arc exactly 2α or $2\alpha - 2$ times. More precisely, an arc $(\langle i, s\rangle, \langle j, t\rangle)$ is used $2\alpha - 2$ times if and only if $s \leq t \leq \alpha + s - 1$ or $t \leq s \leq \alpha + t - 1$.

Let $\mathcal{P}_k = \mathcal{A}_k \cup \mathcal{B}_k, k = 0, \cdots, f$ and $\mathcal{R}_f = \{\mathcal{P}_k : k = 0, 1, \cdots, f\}$. Then from the above discussion, for any $f = \alpha(n-1) - 1$ and $f \leq m(n-2)$, we have

$$\pi(\mathcal{R}_f) = 1 + 2f + \frac{2(f+1)}{n-1} = 1 + 2f + \left\lceil \frac{2(m-1)(f+1)}{m(n-1)} \right\rceil,$$

where the last equality holds because $\frac{f+1}{n-1}$ is an integer less than m. Thus, by (1), $\mathcal{R}_f$ is optimal.

Similarly, for any $f = \alpha(n-1) - 1$ and $m(n-2)+1 \leq f \leq m(n-1)-1$, we have

$$\begin{aligned}\pi(\mathcal{R}_f) &= 1 + 2m(n-2) + \frac{2(f+1)}{n-1} + 3(f - m(n-2)) \\ &= 1 + 3f - m(n-2) + \left\lceil \frac{2(m-1)(f+1)}{m(n-1)} \right\rceil.\end{aligned}$$

So by (2), $\mathcal{R}_f$ is optimal.

Finally, if $f \leq \lceil \frac{n}{2} \rceil - 2$ then it is easy to verify that $\mathcal{R}_k$ is optimal and balanced for each $k \in \{0, 1, \cdots, f\}$, which implies that $\mathcal{R}_{\lceil \frac{n}{2} \rceil - 2}$ is leveled.

(ii). Consider firstly the case $m = 2$. For any k, $0 \leq k \leq f \leq 2n-3$, let $\mathcal{A}_k$ be defined as in (i) and let

$$\begin{aligned}\mathcal{B}_k = \{ & P_k(\langle i, s\rangle, \langle i, t\rangle) : \langle i, s\rangle \\ & \to \left\langle (i + k + 1 + \left\lfloor \frac{k}{n-1} \right\rfloor) \bmod(n), (s + \left\lfloor \frac{k}{n-1} \right\rfloor) \bmod(2) \right\rangle \\ & \to \langle i, t\rangle, i \in \{0, 1, \cdots, n-1\}; s, t \in \{0, 1\} \ \}.\end{aligned}$$

By the definition of $\mathcal{B}_k$ we can verify that each arc is used at most once by the union $\bigcup_{i=0}^{k} \mathcal{B}_i$ if $k \leq n-2$, or at most twice if $n-1 \leq k \leq 2n-3$. Consequently, if $k \leq n-2$ then

$$\pi(\mathcal{R}_k) = \pi(\{\mathcal{A}_i \cup \mathcal{B}_i : i = 0, 1, \cdots, k\}) = 1 + 2f + 1$$

and if $n-1 \leq k \leq 2n-3$ then

$$\pi(\mathcal{R}_k) = \pi(\{\mathcal{A}_i \cup \mathcal{B}_i : i = 0, 1, \cdots, k\}) = 1 + 2f + 2.$$

Combining with (3), the routing $\mathcal{R}_f = \{\mathcal{A}_i \cup \mathcal{B}_i : i = 0, 1, \cdots, f\}$ is a leveled f-fault tolerant routing for $K_{\{n,2\}}$.

We now consider the case $n = 3$. For $0 \leq k \leq 2m - 1$, let

$$\mathcal{B}_k = \{P_k(\langle i, s\rangle, \langle i, t\rangle) : \langle i, s\rangle \to \left\langle (i + 1 + \left\lfloor \frac{k}{m} \right\rfloor) \bmod(3), (s + t + k) \bmod(m) \right\rangle \to \langle i, t\rangle \quad i \in \{0, 1, 2\}; s, t \in \{0, 1, \cdots, m - 1\}\ \}.$$

Then for every i, $0 \leq i \leq 2m - 1$, each arc is used i or $i + 1$ times by the union of $\mathcal{B}_0$, $\mathcal{B}_1$, $\cdots$, $\mathcal{B}_i$. Similar to the argument as in the case $m = 2$, by a direct calculating and combining with (4), we can see that the routing $\mathcal{R}_f = \{\mathcal{A}_i \cup \mathcal{B}_i : i = 0, 1, \cdots, f\}, f \leq 2m - 1$, is a leveled f-fault tolerant routing for $K_{\{3,m\}}$. This completes our proof. □

References

[1] A. Aggarwal, A. Bar-Noy, D. Coppersmith, R. Ramaswami,B. Schieber and M. Sudan, Efficient routing in optical networks, J. of the ACM 46(1996), 973-1001.

[2] B. Beauquier, All-to-all communication for some wavelength-routed all-optical networks, Networks, 33(1999), 179-187.

[3] B. Beauquier, C. J. Bermond, L. Gargano, P. Hell, S. Perennes, and U. Vaccaro, Graph Problems arising from wavelength-routing in all-optical networks, in Proc. of the 2nd Workshop on Optics and Computer Science, part of IPPS'97, 1997.

[4] J. Manuch and L. Stacho, On f-wise arc forwarding index and wavelength allocations in faulty all-optical hypercubes, Theor. Informatics and Appl., 37(2003), 255-270.

[5] N.S. Mendelsohn, On maximal sets of mutuallly orthogonal idempotent Latin squares, Canad. Math. Bull., 14(1971), 449.

[6] Arvind Gupta, Jan Manuch, and Ladislav Stacho, Fault Toletant Forwarding and Optical Indexes: A Design Theory Approach, Springer-Verlag Berlin Heidelberg, 197-208, 2004.

[7] Norman Biggs, Algebraic Graph Theory (2nd ed.), Cambridge University Press, London, New York(1993).

Received: June 23, 2006