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Further Results on Global Stability Criterion of Neural Networks with Continuously Distributed Delays

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Abstract

Based on the recent result given in Park [19], a further result for global asymptotic stability of the equilibrium point for a class of uncertain neural networks with discrete and distributed delays is investigated. A stability condition is derived in terms of linear matrix inequalities (LMIs), which can be solved easily by various convex optimization algorithms.

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1 Introduction

During the last decade, neural networks have been applied to various signal processing problems such as optimization image processing, content-addressable memory, associative memory design, fixed point computation, pattern classification, and so on. Such applications heavily rely on the dynamic behavior of the networks. Hence the analysis of these dynamics is necessary for practical design of neural networks. So far numerous works on global asymptotic stability of equilibrium of the networks have been extensively studied [1-4]. On the other hand, it has been well recognized that time delays are often encountered in various neural networks, and the delays are often the sources of oscillations, instability and poor performance of the networks [5-12]. It is noticed that most works about delayed neural networks have focused on discrete delays. However, neural networks usually have a spatial extent due to the presence of a multitude of parallel pathways with a variety of axon sizes and lengths. Therefore, there will be a distribution of conduction velocities along these pathways and a distribution of propagation delays [13]. In recent years, the problem of stability analysis for various neural networks with distributed delays is investigated by the researchers [14-17].

In this paper, the recent result in [19] will be extended to uncertain neural networks with continuously distributed delays. The derived condition is expressed in terms of two linear matrix inequalities (LMIs) which can be solved numerically very efficiently by resorting to recently developed standard algorithms such as interior-point methods, and no tuning of parameters will be involved [18].

Notation: Throughout this paper, $\mathcal{R}^n$ denotes the n dimensional Euclidean space, and $\mathcal{R}^{n\times m}$ is the set of all $n \times m$ real matrices. I denotes the identity matrix with appropriate dimensions. $\|\cdot\|$ denotes the Euclidean norm of given vector. $\star$ denotes the elements below the main diagonal of a symmetric block matrix. $diag\{\cdots\}$ denotes the diagonal matrix. For symmetric matrices X and Y, the notation $X > Y$(respectively, $X \geq Y$) means that the matrix $X - Y$ is positive definite, (respectively, nonnegative).

2 Main Results

Consider a continuous time-delayed neural network which is described by the following nonlinear retarded functional differential equations:

$$\begin{aligned} \dot{x}(t) &= -(C+\Delta C)x(t) + (W+\Delta W)f(x(t)) + (A+\Delta A)f(x(t-h)) \\ &\quad +(B+\Delta B)\int_{-\infty}^{t} K(t-s)f(x(s))ds, \end{aligned} \tag{1}$$

where $x(t) = [x_1, x_2, \cdots, x_n]^T \in \mathcal{R}^n$ is the state vector at time t, $C = diag\{c_1, c_2, \cdots, c_n\} \in \mathcal{R}^{n\times n} > 0$ denotes the passive decay rate, $W = diag\{w_1, w_2, \cdots, w_n\} \in \mathcal{R}^{n\times n}$ is the feedback term, $A = [a_{ij}] \in \mathcal{R}^{n\times n}, B = [b_{ij}] \in \mathcal{R}^{n\times n}$ are the synaptic connection strengths, $f = [f_1(x_1), f_2(x_2), \cdots, f_n(x_n)]^T$ denotes the neuron activations, $h > 0$ is the discrete transmission delay from one neuron to another, and the delay kernel $K(t-s) = diag\{k_1(t-s), k_2(t-s), \cdots, k_n(t-s)\}$ is a real value non-negative continuous function defined on $[0, \infty)$ satisfying

$$\int_0^{\infty} k_j(s)ds = 1, \quad \forall\, i. \tag{2}$$

Also, $\Delta A, \Delta W_0, \Delta W_1, \Delta W_2$ are the uncertainties of system matrices of the form:

$$\begin{bmatrix} \Delta C & \Delta W & \Delta A & \Delta B \end{bmatrix} = HF(t) \begin{bmatrix} E & E_0 & E_1 & E_2 \end{bmatrix}, \tag{3}$$

where H, E, E_0, E_1, E_2 are known constant matrices of appropriate dimensions and the time-varying nonlinear function $F(t)$ satisfy

$$F^T(t)F(t) \leq I \text{ for } \forall\, t \in \mathcal{R}. \tag{4}$$

In this paper, it is assumed that the neuron activation functions are bounded and satisfies the following property:

$$|f_j(x_j(t))| \leq l_j|x_j(t))|, \quad j = 1, 2, \cdots, n. \tag{5}$$

By employing the well-known Brouwer's fixed point theorem, note that one can easily prove that there exists an equilibrium point for system (1).

Before presenting the main results, we give the the following facts and lemma.

Fact 1. (Schur complement) Given constant symmetric matrices $\Sigma_1, \Sigma_2, \Sigma_3$ where $\Sigma_1 = \Sigma_1^T$ and $0 < \Sigma_2 = \Sigma_2^T$, then $\Sigma_1 + \Sigma_3^T\Sigma_2^{-1}\Sigma_3 < 0$ if and only if

$$\begin{bmatrix} \Sigma_1 & \Sigma_3^T \\ \Sigma_3 & -\Sigma_2 \end{bmatrix} < 0, \quad \text{or} \quad \begin{bmatrix} -\Sigma_2 & \Sigma_3 \\ \Sigma_3^T & \Sigma_1 \end{bmatrix} < 0.$$

Fact 2. For any $z, y \in \mathbb{R}^{n\times m}$ and a positive scalar ϵ, the following inequality

$$2z^Ty \leq \epsilon z^Tz + \epsilon^{-1}y^Ty$$

holds.

Lemma 1. [19] For given $L = diag\{l_1, l_2, \cdots, l_n\}$, the equilibrium point **0** of (1) without uncertainties, i.e., $\Delta C = \Delta W = \Delta A = \Delta B = 0$, is globally asymptotically stable if there exist positive definite matrices P, Q, Z, X, positive diagonal matrices $E = diag\{e_1, e_2, \cdots, e_n\}$ $D = diag\{d_1, d_2, \cdots, d_n\}$ and $Y \geq 0$, satisfying the following two LMIs:

$$\Pi = \begin{bmatrix} \Pi_1 & PW & PA+Y^T & PB & -hC^TZ \\ \star & \Pi_2 & DA & DB & hW^TZ \\ \star & \star & \Pi_3 & 0 & hA^TZ \\ \star & \star & \star & -E & hB^TZ \\ \star & \star & \star & \star & -hZ \end{bmatrix} < 0, \tag{6}$$

$$\Gamma = \begin{bmatrix} X & Y \\ \star & Z \end{bmatrix} \geq 0, \tag{7}$$

where

$$\begin{aligned} \Pi_1 &= -PC - CP \\ \Pi_2 &= -DCL^{-1} - L^{-1}CD + DW + W^TD + Q + E \\ \Pi_3 &= -Q + hX - YL^{-1} - L^{-1}Y^T. \end{aligned}$$

Now, we give a sufficient condition for the global asymptotic stability of the equilibrium point for the neural network (1) based on Lemma 1.
Then, we have the following theorem.

Theorem 1. The equilibrium point **0** of system (1) is globally asymptotically stable if there exist positive definite matrices P, Q, Z, X, two positive diagonal matrices D, E, three positive scalars $\epsilon_i (i = 1, 2, 3)$, and $Y \geq 0$, satisfying the following two LMIs:

$$\begin{bmatrix} \Sigma_1 & \begin{pmatrix} PW - (\epsilon_1 \\ +\epsilon_3) E^T E_0 \end{pmatrix} & \Sigma_5 & \Sigma_9 & -hA^T Z & PH & 0 & 0 \\ \star & \Sigma_2 & \Sigma_4 & \Sigma_8 & hW_0^T Z & 0 & DH & 0 \\ \star & \star & \Sigma_3 & \Sigma_7 & hW_1^T Z & 0 & 0 & 0 \\ \star & \star & \star & \Sigma_6 & hW_2^T Z & 0 & 0 & 0 \\ \star & \star & \star & \star & -hZ & 0 & 0 & ZH \\ \star & \star & \star & \star & \star & -\epsilon_1 I & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -\epsilon_2 I & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\epsilon_3 I \end{bmatrix} < 0 \quad (8)$$

$$\begin{bmatrix} X & Y \\ \star & Z \end{bmatrix} \geq 0, \quad (9)$$

where

$$\begin{aligned} \Sigma_1 &= -PC - CP + (\epsilon_1 + \epsilon_3) E^T E, \\ \Sigma_2 &= -DCL^{-1} - L^{-1}CD + DW + W^T D + Q + E + (\epsilon_1 + \epsilon_3) E_0^T E_0 \\ &\quad +\epsilon_2 (-L^{-1} E^T + E_0^T)(-EL^{-1} + E_0), \\ \Sigma_3 &= -Q + hX - YL^{-1} - L^{-1} Y^T + (\epsilon_1 + \epsilon_2 + \epsilon_3) E_1^T E_1, \\ \Sigma_4 &= DA + (\epsilon_1 + \epsilon_3) E_0^T E_1 + \epsilon_2 (-L^{-1} E^T + E_0^T) E_1, \\ \Sigma_5 &= PA + Y^T - (\epsilon_1 + \epsilon_3) E^T E_1, \\ \Sigma_6 &= -E + (\epsilon_1 + \epsilon_2 + \epsilon_3) E_2^T E_2, \\ \Sigma_7 &= (\epsilon_1 + \epsilon_2 + \epsilon_3) E_1^T E_2, \\ \Sigma_8 &= DB + (\epsilon_1 + \epsilon_3) E_0^T E_2 + \epsilon_2 (-L^{-1} E^T + E_0^T) E_2, \\ \Sigma_9 &= PB - (\epsilon_1 + \epsilon_3) E^T E_2. \end{aligned} \quad (10)$$

Proof. By Lemma 1, the system is globally asymptotically stable if the inequality (9) and the following inequality hold

$$\Pi + 2\Omega_1 F(t) \Omega_2^T + 2\Omega_3 F(t) \Omega_4^T + 2\Omega_5 F(t) \Omega_6^T < 0, \quad (11)$$

where

$$\Omega_1 = \begin{bmatrix} PH \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \Omega_2 = \begin{bmatrix} -E^T \\ E_0^T \\ E_1^T \\ E_2^T \\ 0 \end{bmatrix}, \Omega_3 = \begin{bmatrix} 0 \\ DH \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\Omega_4 = \begin{bmatrix} 0 \\ -L^{-1}E^T + E_0^T \\ E_1^T \\ E_2^T \\ 0 \end{bmatrix}, \quad \Omega_5 = h\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ ZH \end{bmatrix}, \quad \Omega_6 = \begin{bmatrix} -E^T \\ E_0^T \\ E_1^T \\ E_2^T \\ 0 \end{bmatrix}.$$

By Fact 2, Eq. (11) holds if the following inequality satisfies

$$\begin{aligned} &\Pi + \epsilon_1^{-1}\Omega_1\Omega_1^T + \epsilon_1\Omega_2\Omega_2^T + \epsilon_2^{-1}\Omega_3\Omega_3^T + \epsilon_2\Omega_4\Omega_4^T + \epsilon_3^{-1}\Omega_5\Omega_5^T + \epsilon_3\Omega_6\Omega_6^T \\ &\equiv \Pi + \Omega < 0, \end{aligned} \tag{12}$$

where $\epsilon_i > 0, i = 1, 2, 3$, and

$$\Omega = \begin{bmatrix} (1,1) & -(\epsilon_1+\epsilon_3)E^TE_0 & (1,3) & -(\epsilon_1+\epsilon_3)E^TE_2 & 0 \\ \star & (2,2) & (2,3) & (2,4) & 0 \\ \star & \star & (3,3) & (3,4) & 0 \\ \star & \star & \star & (4,4) & 0 \\ \star & \star & \star & \star & (5,5) \end{bmatrix},$$

$$\begin{aligned}
(1,1) &= \epsilon_1^{-1}PHH^TP + (\epsilon_1+\epsilon_3)E^TE, \\
(1,3) &= -(\epsilon_1+\epsilon_3)E^TE_1, \\
(2,2) &= \epsilon_2^{-1}DHH^TD + (\epsilon_1+\epsilon_3)E_0^TE_0 + \epsilon_2(-L^{-1}E^T + E_0^T)(-EL^{-1} + E_0), \\
(2,3) &= (\epsilon_1+\epsilon_3)E_0^TE_1 + \epsilon_2(-L^{-1}E^T + E_0^T)E_1, \\
(2,4) &= (\epsilon_1+\epsilon_3)E_0^TE_2 + \epsilon_2(-L^{-1}E^T + E_0^T)E_2, \\
(3,3) &= (\epsilon_1+\epsilon_2+\epsilon_3)E_1^TE_1, \\
(3,4) &= (\epsilon_1+\epsilon_2+\epsilon_3)E_1^TE_2, \\
(4,4) &= (\epsilon_1+\epsilon_2+\epsilon_3)E_2^TE_2, \\
(5,5) &= \epsilon_3^{-1}ZHH^TZ.
\end{aligned}$$

Using the relationship (10), the inequality $\Pi + \Omega < 0$ is equivalent to

$$\begin{bmatrix} \begin{pmatrix} \Sigma_1 + \epsilon_1^{-1}PH \\ \times H^TP \end{pmatrix} & \begin{pmatrix} PW - (\epsilon_1 \\ +\epsilon_3)E^TE_0 \end{pmatrix} & \Sigma_5 & \Sigma_9 & -hA^TZ \\ \star & \Sigma_2 + \epsilon_2^{-1}DHH^TD & \Sigma_4 & \Sigma_8 & hW_0^TZ \\ \star & \star & \Sigma_3 & \Sigma_7 & hW_1^TZ \\ \star & \star & \star & \Sigma_6 & hW_2^TZ \\ \star & \star & \star & \star & -hZ + (5,5) \end{bmatrix} < 0. \tag{13}$$

Then, by Fact 1, the inequality given (13) is equivalent to the LMI (8). Thus, if the LMIs given in (8) and (9) hold, the equilibrium point of system (1) is globally asymptotically stable in the sense of Lemma 1. This completes our proof. ■

Remark 1. The criterion given in Theorem 1 is delay-dependent with respect to h. It is well known that the delay-dependent criteria are less conservative than delay-independent criteria when the delay is small. The solutions of Theorem 1 can be obtained by solving the eigenvalue problem with respect to solution variables, which is a convex optimization problem [18].

3 Concluding remarks

We have further studied the stability property of a class of uncertain neural networks with continuously distributed delays based on the recent work [19]. A new delay-dependent criterion on global asymptotic stability of the neural networks. The condition is formulated in terms of LMIs, which is solved by various convex optimization algorithms.

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