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Iterative Approximation to Common Fixed Points of Two Nonexpansive Mappings in Banach Spaces

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Abstract. Let E be a real reflexive Banach space which admits a weakly sequentially continuous duality mapping from E to E^*, and K be a nonempty closed convex subset of E. Suppose that $T, S : K \to K$ are two nonexpansive mappings such that $F := F(ST) = F(T) \cap F(S) \neq \emptyset$. For arbitrary initial value $x_0 \in K$ and fixed anchor $u \in K$, define iteratively a sequence $\{x_n\}$ as follows:

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n)Tx_n \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)Sy_n, \quad n \geq 0, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ satisfies proper conditions. We prove that $\{x_n\}$ converges strongly to $P_F u$ as $n \to \infty$, where P_F is a unique sunny nonexpansive retraction of K onto F. Also we prove that the same conclusions still hold in a uniformly convex Banach space with uniformly Géteaux differentiable norm or uniformly smooth Banach spaces. Our results extend and improve the corresponding ones by Tae-Hwa Kim and Hong-Kun Xu [Strong convergence of modified Mann iterations, Nonlinear Anal. 61(2005) 51-60].

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1. Introduction

Let K be a nonempty closed convex subset of a Banach space E and let T, S be two nonexpansive mappings from K into itself (recall that a mapping $T : K \to K$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\| \quad \forall x, y \in K$), $\{\alpha_n\}, \{\beta_n\}$ is two sequences in $[0, 1]$. In 1953, Mann [6] introduced the following iterative procedure in Hibert space: for $x_0 \in K$,

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0. \tag{1.1}$$

Later Reich [10] studied this iterative procedure (1.1) in a uniformly convex Banach space whose norm is Fréchet differentiable and obtained that if T has a fixed point and $\sum\limits_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ converges weakly to a fixed point of T. A generalization of Mann iterative schemes was given by Takahashi and Tamura [5]. This scheme dealt with two nonexpansive mappings:

$$\begin{cases} x_{n+1} = \alpha_n Sy_n + (1 - \alpha_n)x_n \\ y_n = \beta_n Tx_n + (1 - \beta_n)x_n, n \geq 1. \end{cases} \tag{1.2}$$

S.H. Khan and H. Fukhar-ud-din [3] further generalize the iteration scheme (1.2) to the one with errors.

In 1967, Halpern [12] firstly introduced the following iteration scheme: for nonexpansive mappings T and $y, x_0 \in K$,

$$x_{n+1} = \alpha_n y + (1 - \alpha_n)Tx_n, \quad n \geq 0. \tag{1.3}$$

He pointed out that the control conditions $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ are necessary for the convergence of the iteration scheme (1.3) to a fixed point of T. Later, many authors studied the iteration scheme (1.3), for instance, see [4, 13, 15, 19].

Recently, Y. Kimura, W. Takahashi and M. Toyoda [9] studied the following iterative scheme for two nonexpansive mappings T, S in a uniformly convex Banach space with uniformly Géteaux differentiable norm. For $x_0,\ u \in K$

$$\begin{cases} x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n \\ y_n = \beta_n Sx_n + (1 - \beta_n)Tx_n, \quad n \geq 0. \end{cases} \tag{1.4}$$

At the same time, Tae-Hwa Kim and Hong-Kun Xu [2] dealt with the following iterative scheme for one nonexpansive mapping T in a uniformly smooth Banach space. For$x_0,\ u \in K$,

$$\begin{cases} x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n \\ y_n = \beta_n x_n + (1 - \beta_n)Tx_n, \quad n \geq 0. \end{cases} \tag{1.5}$$

In this paper, motivated by Tae-Hwa Kim and Hong-Kun Xu[2] and S.H. Khan and H. Fukhar-ud-din[3] and Y. Kimura, W. Takahashi and M. Toyoda[9], we consider the following the following iterative scheme (1.6) for two nonexpansive mappings S, T. For $x_0,\ u \in K$,

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n) S y_n \\ y_n = \beta_n x_n + (1-\beta_n) T x_n, \quad n \geq 0. \end{cases} \tag{1.6}$$

We will prove several strong convergence results in a reflexive Banach space E which admits a weakly sequentially continuous duality mapping J from E to E^* or in a uniformly convex Banach space E with uniformly Géteaux differentiable norm or in a uniformly smooth Banach space E.

2. **Preliminaries**

Throughout this paper, it is assumed that E is a real Banach space with norm $\|\cdot\|$, and let J denote the normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \{f \in E^*, \langle x, f\rangle = \|x\| \, \|f\|\,, \|x\| = \|f\|\}, \forall\, x \in E,$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. In the sequel, we shall denote the single-valued duality mapping by j, and denote $F(T) = \{x \in E; Tx = x\}$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$(respectively $x_n \rightharpoonup x$, $x_n \overset{*}{\rightharpoonup} x$)will denote strong (respectively weak, weak*) convergence of the sequence $\{x_n\}$ to x. In Banach space E, the following result (*the Subdifferential Inequality*) is well known (Theorem 4.2.1 of [16]). $\forall x, y \in E$, $\forall j(x+y) \in J(x+y)$,

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle. \tag{2.1}$$

Let E be a real Banach space and T a mapping with domain $D(T)$ and range $R(T)$ in E. T is called (respectively, *contractive*) *nonexpansive* if for any $x, y \in D(T)$, such that

$$\|Tx - Ty\| \leq \|x - y\|\,,$$

$$\text{(respectively, } \|Tx - Ty\| \leq \beta \|x - y\| \quad \text{for some } 0 < \beta < 1.)$$

Recall that a Banach space E is said to be *smooth*, if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.2}$$

exists for each x, y on the unit sphere $S(E)$ of E. In this case, the duality mapping J is strong-weak* continuous (Lemma 4.3.3 of [16]). Moreover, if for each y in $S(E)$ the limit defined by (2.2) is uniformly attained for x in $S(E)$, we say that the norm of E is *uniformly Gâteaux differentiable*. A Banach space E is said to *uniformly smooth*, if the limit (2.2) is attained uniformly for $(x, y) \in S(E) \times S(E)$. A Banach space E is said to *uniformly convex* if its dual space E^* is uniformly smooth.

It is well known that the (normalized) duality mapping $J(x)$ $(\forall x \in E)$ is single-valued if and only if E is smooth (Theorem 4.3.1 and theorem 4.3.2 of [16]). If E has a uniformly Gâteaux differentiable norm, then the (normalized) duality mapping $J : E \longrightarrow E^*$ is norm to weak* uniformly continuous on bounded sets of E (Theorem 4.3.6 of [16]). Every uniformly smooth Banach space E is a reflexive Banach space with uniformly Gâteaux differentiable norm, and the (normalized) duality mapping $J : E \longrightarrow E^*$ is single-valued and norm to norm uniformly continuous on bounded sets of E (Theorem 4.3.4 and Theorem 4.3.7 of [16]).

If C are nonempty convex subsets of a Banach space E and D is nonempty subset of C, then a mapping $P : C \to D$ is called a *retraction* if P is continuous with $F(P) = D$. A mapping $P : C \to D$ is called *sunny* if

$$P(Px + t(x - Px)) = Px, \ \forall x \in C$$

whenever $Px+t(x-Px) \in C$ and $t > 0$. A subset D of C!!is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction of C onto D. For more details, see [7, 8, 16]. The following Lemma is well known [7, 16].

Lemma 2.1. *Let C be nonempty convex subset of a smooth Banach space E, $\emptyset \neq D \subset C$, $J : E \to E^*$ the normalized duality mapping of E, and $P : C \to D$ a retraction. Then the following are equivalent:*

(i) $\langle x - Px, j(y - Px)\rangle \leq 0$ *for all $x \in C$ and $y \in D$;*

(ii) *P is both sunny and nonexpansive.*

Hence, there exists at most a sunny nonexpansive retraction P from C onto D.

In 1980, Reich [13] showed that if E is uniformly smooth and $F(T)$ is the fixed point set of a nonexpansive mapping T from K into itself, then there is the unique sunny nonexpansive retraction from K onto $F(T)$.

Lemma 2.2. *(Reich*[13]*) Let K be nonempty closed convex subset of a uniformly smooth Banach space E and let $T : K \to K$ be a nonexpansive mapping with a fixed point. Then $F(T)$ is a sunny nonexpansive retract of K. Further, For each fixed $u \in K$ and every $t \in (0, 1)$, and let $z_t \in K$ be a point satisfying*

$$z_t = tu + (1 - t)Tz_t.$$

Then $\{z_t\}$ converges strongly to Pu as $t \to 0$, where P is the unique sunny nonexpansive retraction from K onto $F(T)$.

In 1984, Takahashi and Ueda [14] also proved the existence of sunny nonexpansive retractions in uniformly convex Banach space with a uniformly Géteaux differentiable norm.

Lemma 2.3. *(Takahashi -Ueda* [14]*) Let K be nonempty closed convex subset of a uniformly convex Banach space E with uniformly Géteaux differentiable norm. Supose $T : K \to K$ is a nonexpansive mapping with a fixed point. Then $F(T)$ is a sunny nonexpansive retract of K. Further, For each fixed $u \in K$ and every $t \in (0, 1)$, and let $z_t \in K$ be a point satisfying*

$$z_t = tu + (1 - t)Tz_t.$$

Then $\{z_t\}$ converges strongly to Pu as $t \to 0$, where P is the unique sunny nonexpansive retraction from K onto $F(T)$.

If Banach space E admits sequentially continuous duality mapping J from weak topology to weak star topology, then by Lemma 1 of reference [11], we get that duality mapping J is single-valued. In this case, duality mapping J is also said to be *weakly sequentially continuous*, i.e. for each $\{x_n\} \subset E$ with $x_n \rightharpoonup x$, then $J(x_n) \overset{*}{\rightharpoonup} J(x)$ ([7, 11]).

A Banach space E is said to be satisfy *Opial's condition* if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ $(n \to \infty)$ implies

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \forall y \in E \text{ with } x \neq y.$$

By Theorem 1 of reference [11], we know that if E admits a weakly sequentially continuous duality mapping, then E satisfies Opial's condition, and E is smooth, for more details, see reference [11].We also know that in l^p spaces with $p > 1$ and $p \neq 2$, the normalized duality mapping J is not weakly sequentially continuous, but there exists a duality mapping that is weakly sequentially continuous.

In the sequel, we also need the following lemma which can be found in the existing literature [7, Lemma 2].

Lemma 2.4. *Let C be a nonempty closed convex subset of a reflexive Banach space E which satisfies Opial's condition, and suppose $T : C \to E$ is nonexpansive. If as $n \to \infty$, $x_n \rightharpoonup x, x_n - Tx_n \to 0$, Then $x = Tx$.*

The following lemma modified Xu [17, Lemma 2.1], also see L.S.liu[18] and Y.Song and R. Chen[20, 1].

Lemma 2.5. *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\lambda_n + \beta_n, \quad n \geq 0,$$

where $\{\gamma_n\} \subset (0, 1)$, $\{\beta_n\} \subset \mathbb{R}$ and $\{\lambda_n\} \subset \mathbb{R}$ such that

(i) $\sum\limits_{n=0}^{\infty} \gamma_n = \infty$ or equivalently $\prod\limits_{n=0}^{\infty} (1 - \gamma_n) = 0$;

(ii) $\limsup\limits_{n\to\infty} \lambda_n \leq 0$;

(iii) $\sum\limits_{n=0}^{\infty} |\beta_n| < +\infty$.

Then $\{a_n\}$ converges to zero, as $n \to \infty$.

3. Main Results

At first, we will show the fixed point set $F(T)$ of a nonexpansive mapping T is sunny nonexpansive retraction of K in a reflexive Banach space E which admits a weakly sequentially continuous duality mapping J from E to E^*.

Lemma 3.1. *Let E be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J from E to E^*. Suppose K is a nonempty closed convex subset of E. Suppose that $T : K \to K$ is a nonexpansive mapping with $F(T) \neq \emptyset$. Then the fixed point set $F(T)$ of T is sunny nonexpansive retract of K.*

Proof. For a given $u \in K$, we can define a contraction $T_t : K \to K$ by $T_t = tu + (1-t)T$. An application of the Banach contraction principle yields a unique fixed point x_t of T_t for each $t \in (0,1)$. Notice that the assumption that $F(T) \neq \emptyset$ guarantees the boundedness of the net $\{x_t\}$. At this point, following the proof lines of Theorem 2.2 in Yisheng Song and Rudong Chen [1]($f(x_t) \equiv u$ and $P = I$), we can get that

$$x_t \to p \in F(T), \quad t \to 0,$$

and p satisfies the following variational inequality:

$$\langle u - p, j(y-p)\rangle \leq 0, \ \forall y \in F(T).$$

Let $P_{F(T)}u = \lim_{t\to 0} x_t = p$, we obtain that

$$\langle u - P_{F(T)}u, j(y - P_{F(T)}u)\rangle \leq 0, \ \forall y \in F(T).$$

By Lemma 2.1, we obtain that $P_{F(T)}$ is sunny nonexpansive retraction from K to $F(T)$. The proof is completed. □

Theorem 3.2. *Let E be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J from E to E^*. Suppose K is a nonempty closed convex subset of E, and S, T is nonexpansive mappings from K into itself such that $F := F(ST) = F(T) \cap F(S) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be a sequence of positive numbers in $[0,1]$ satisfying the following conditions:*

(A1) $\lim_{n\to\infty} \alpha_n = 0$;

(A2) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < +\infty$ *or* $(A2)'$ $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$;

(A3) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(B1) $\lim_{n\to\infty} \beta_n = 0$;

(B2) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < +\infty$.

For arbitrary initial value $x_0 \in K$ and fixed anchor $u \in K$, define iteratively a sequence $\{x_n\}$ as follows:

$$\begin{cases} y_n = T_n x_n = \beta_n x_n + (1-\beta_n)Tx_n \\ x_{n+1} = \alpha_n u + (1-\alpha_n)Sy_n, \quad n \geq 0, \end{cases} \tag{3.6}$$

Then $\{x_n\}$ converges strongly to $P_F u$, where P_F is the unique sunny nonexpansive retraction from K onto F.

Proof. First, we show we use an inductive argument to prove $\|x_n - p\| \le M$, $\forall p \in F$ and $n \ge 0$, where $M = \max\{\|x_0 - p\|, \|u - p\|\}$. The result is clearly true for $n = 0$. Suppose $\|x_k - p\| \le M$, then by the nonexpansivity of T, S, clearly $T_n = \beta_n I + (1 - \beta_n)T$ is nonexpansive and $F(T) = F(T_n)$, so we have

$$\begin{aligned}\|x_{k+1} - p\| &= \|\alpha_k(u - p) + (1 - \alpha_k)(Sy_k - p)\| \\ &\le \alpha_k \|u - p\| + (1 - \alpha_k)\|y_k - p\| \\ &\le \alpha_k \|u - p\| + (1 - \alpha_k)\|T_k x_k - p\| \\ &\le \alpha_k \|u - p\| + (1 - \alpha_k)\|x_k - p\| \le M.\end{aligned}$$

Therefore, $\|x_n - p\| \le M, \forall n \ge 0$. So the sets $\{x_n\}$ is bounded, and the set $\{Tx_n\}$, $\{Sx_n\}$ are also bounded. Putting $C = \sup_{n\in\mathbb{N}}\{\|x_n\|, \|Tx_n\|, \|Sx_n\|\}$. Now we show

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Using (3.6), we have

$$\begin{aligned}\|y_n - y_{n-1}\| =&\|T_n x_n - T_{n-1}x_{n-1}\| \\ \le&\|T_n x_n - T_n x_{n-1}\| + \|T_n x_{n-1} - T_{n-1}x_{n-1}\| \\ \le&\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1} + Tx_{n-1}\| \\ \le&\|x_n - x_{n-1}\| + 2C|\beta_n - \beta_{n-1}|.\end{aligned}$$

Thus,

$$\begin{aligned}\|x_{n+1} - x_n\| =&\|\alpha_n u + (1 - \alpha_n)Sy_n - (\alpha_{n-1}u + (1 - \alpha_{n-1})Sy_{n-1})\| \\ \le&|\alpha_n - \alpha_{n-1}|\|u\| + (1 - \alpha_n)\|Sy_n - Sy_{n-1}\| \\ &+ |\alpha_n - \alpha_{n-1}|\|Sy_{n-1}\| \\ \le&(1 - \alpha_n)\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|(\|u\| + C) \\ \le&(1 - \alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|(\|u\| + C) \\ &+ 2C|\beta_n - \beta_{n-1}| \\ =&(1 - \alpha_n)\|x_n - x_{n-1}\| + \alpha_n|1 - \frac{\alpha_{n-1}}{\alpha_n}|(\|u\| + C) \\ &+ 2C|\beta_n - \beta_{n-1}|.\end{aligned}$$

By (A2) and (B2) we have

$$\sum_{n=1}^{\infty}(|\alpha_n - \alpha_{n-1}|(\|u\| + C) + 2C|\beta_n - \beta_{n-1}|) < +\infty.$$

So adding to (A3), which satisfies Lemma 2.5 ($\lambda_n \equiv 0$). By $(A2)'$ we have

$$\lim_{n\to\infty} |1 - \frac{\alpha_{n-1}}{\alpha_n}|(\|u\| + C) = 0,$$

So adding to (A3) and (B2), which also satisfies Lemma 2.5. Hence, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.7}$$

Setting $A = ST$, obviously A is nonexpansive mapping from K to itself. Next we prove

$$\lim_{n\to\infty} \|x_{n+1} - Ax_{n+1}\| = 0. \tag{3.8}$$

By (3.6), condition $(A1)$ and $\{Tx_n\}$, $\{Sx_n\}$ boundary, we get

$$\|x_{n+1} - Sy_n\| = \alpha_n \|u - Sy_n\| \to 0. \tag{3.9}$$

and

$$\begin{aligned}\|x_{n+1} - STx_{n+1}\| \leq & \|x_{n+1} - Sy_n\| + \|Sy_n - STx_n\| + \|STx_n - STx_{n+1}\| \\ \leq & \|x_n - x_{n+1}\| + \|x_{n+1} - Sy_n\| + \|y_n - Tx_n\| \\ \leq & \|x_n - x_{n+1}\| + \|x_{n+1} - Sy_n\| + \beta_n \|x_n - Tx_n\|\end{aligned}$$

Using (3.7), (3.9) and condition $(B1)$, we obtain (3.8) holds.

On the other hand, by Lemma 3.1, we know $F = F(A)$ is the sunny nonexpansive retract of K, and denote P_F a sunny nonexpansive retraction of K onto F. We next show that

$$\limsup_{n\to\infty} \langle u - P_F u, j(x_{n+1} - P_F u)\rangle \leq 0. \tag{3.10}$$

Indeed, we can take a subsequence $\{x_{n_k+1}\}$ of $\{x_{n+1}\}$ such that

$$\limsup_{n\to\infty} \langle u - P_F u, j(x_{n+1} - P_F u)\rangle = \lim_{k\to\infty} \langle u - P_F u, j(x_{n_k+1} - P_F u)\rangle.$$

We may assume that $x_{n_k+1} \rightharpoonup x^*$ by E reflexive and $\{x_{n+1}\}$ bounded. It follow from Lemma 2.4 and (3.8) that $x^* \in F(A) = F$. Hence by Lemma 2.1 and the duality mapping J is weakly sequentially continuous from E to E^*, we obtain

$$\limsup_{n\to\infty} \langle u - P_F u, j(x_{n+1} - P_F u)\rangle = \langle u - P_F u, j(x^* - P_F u)\rangle \leq 0.$$

Finally we show that $x_n \to P_F u$. As a matter of fact, Using (2.1) and $(1 - \alpha_n)^2 \leq (1 - \alpha_n)$, we get

$$\begin{aligned}\|x_{n+1} - P_F u\|^2 = & \|(1 - \alpha_n)(A_n x_n - P_F u) + \alpha_n(u - P_F u)\|^2 \\ \leq & (1 - \alpha_n) \|x_n - P_F u\|^2 \\ & + 2\alpha_n \langle u - P_F u, j(x_{n+1} - P_F u)\rangle).\end{aligned} \tag{3.11}$$

Using Lemma 2.5, (A3), (3.10) and (3.11), we conclude that $x_n \to P_F u$. □

Remark 1. Since the condition $(A2), (A2)'$ and $(B2)$ are mainly used to show $x_{n+1} - x_n \to 0$, if condition $(B2)$ is replaced by condition

$(B2)'$ $\lim_{n\to\infty} \dfrac{|\beta_n - \beta_{n-1}|}{\alpha_n} = 0$, then the result still holds.

Using Lemma 2.3, we know that in a uniformly convex Banach space E with uniformly Géteaux differentiable norm, the fixed point set $F(T)$ of nonexpansive self-mapping T on nonempty closed convex subset K of E is sunny nonexpansive retract of K. So we can also get the following theorem:

Theorem 3.3. *Let E be a uniformly convex Banach space with uniformly Géteaux differentiable norm. Suppose K is a nonempty closed convex subset of E, and S, T are nonexpansive mapping from K to K such that $F := F(ST) = F(S) \cap F(T) \neq \emptyset$. The sequence $\{x_n\}$ is defined by (3.6), and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1]$ satisfying the conditions (A1), (A2) or (A2)$'$, (A3), (B1), (B2) or (B2)$'$. Then $x_n \to P_F u$, where P_F is a sunny nonexpansive retraction from K into F.*

Proof. As in the proof of Theorem 3.2, we can reach the following objectives:
(1) $\{x_n\}$, $\{Sx_n\}$ and $\{Tx_n\}$is bounded;
(2) $x_{n+1} - x_n \to 0$ as $n \to \infty$;
(3) Putting $A = ST$, then

$$F = F(A) = F(ST) = F(S) \cap F(T), \quad x_n - Ax_n \to 0 \ (n \to \infty).$$

We next show that $x_n \to P_F(u),\ (n \to \infty)$.

Indeed, putting $z_t = tu + (1-t)Az_t$, by Lemma 2.3, we obtain $P_F u = \lim_{t\to 0} z_t$, where P_F is a sunny nonexpansive retract from K to F. Using (2.1) and equality $z_t - x_n = (1-t)(Az_t - x_n) + t(u - x_n)$, we get

$$\begin{aligned} \|z_t - x_n\|^2 \le &(1-t)^2 \|Az_t - x_n\|^2 + 2t\langle u - x_n, j(z_t - x_n)\rangle \\ \le &(1 - 2t + t^2) \|z_t - x_n\|^2 + a_n(t) \\ &+ 2t\langle u - z_t, j(z_t - x_n)\rangle + 2t \|z_t - x_n\|^2 \end{aligned}$$

where

$$a_n(t) = \|x_n - Ax_n\| (2 \|z_t - x_n\| + \|x_n - Ax_n\|).$$

It follows that

$$\langle u - z_t, j(x_n - z_t)\rangle \le \frac{t}{2} \|z_t - x_n\|^2 + \frac{1}{2t} a_n(t). \tag{3.12}$$

We claim that $a_n(t) \to 0$ *as* $n \to \infty$. Indeed, since $\{x_n\}$, $\{z_t\}$ is bounded, $\{Ax_n\}$ is bounded. So that $\{x_n - Ax_n\}$ and $\{z_t - x_n\}$ is bounded. Using $\lim_{n\to\infty} \|x_n - Ax_n\| = 0$, we obtain

$$a_n(t) = \|x_n - Ax_n\| (2 \|z_t - x_n\| + \|x_n - Ax_n\|) \to 0 \ \ as \ \ n \to \infty.$$

Taking $n \to \infty$ in(3.12),

$$\limsup_{n\to\infty} \langle u - z_t, j(x_n - z_t)\rangle \le \frac{t}{2} M, \tag{3.13}$$

where $M > 0$ is a constant such that $M \ge \max \|z_t - x_n\|^2$ for $t \in (0,1)$ and for all $n \ge 1$. By letting $t \to 0$ in (3.13) we have

$$\limsup_{t\to 0} \limsup_{n\to\infty} \langle u - z_t, j(x_n - z_t)\rangle \le 0. \tag{3.14}$$

Next we show that

$$\limsup_{n\to\infty} \langle u - P_F u, j(x_n - P_F u)\rangle \le 0.$$

In fact, noting that z_t converges strongly to $P_F u$, as $t \to 0$, and that the set $\{z_t - x_n\}$ is bounded, together with the fact that the duality map J is single-valued and norm-weak* uniformly continuous on bounded sets of a Banach space E with a uniformly Géteaux differentiable norm, we get

$$\begin{aligned}
&|\langle u - P_F u, j(x_n - P_F u)\rangle - \langle u - z_t, j(x_n - z_t)\rangle| \\
&= |\langle u - P_F u, j(x_n - P_F u) - j(x_n - z_t)\rangle + \langle z_t - P_F u, j(x_n - z_t)\rangle| \\
&\leq |\langle u - P_F u, j(x_n - P_F u) - j(x_n - z_t)\rangle| \\
&\quad + \|z_t - P_F u\| \, \|x_n - z_t\| \to 0, \ as \ \ t \to 0.
\end{aligned}$$

Hence, $\forall \varepsilon > 0, \ \ \exists \delta > 0$, such that $\forall t \in (0, \delta)$, for all n, we have

$$\langle u - P_F u, j(x_n - P_F u)\rangle \leq \langle u - z_t, j(x_n - z_t)\rangle + \varepsilon.$$

Therefore,

$$\limsup_{n\to\infty} \langle u - P_F u, j(x_n - P_F u)\rangle \leq \limsup_{n\to\infty} (\langle u - z_t, j(x_n - z_t)\rangle + \varepsilon.$$

Taking $t \to 0$ and noting (3.14), we have

$$\begin{aligned}
&\limsup_{n\to\infty} \langle u - P_F u, j(x_n - P_F u)\rangle \\
&\leq \limsup_{t\to 0} \limsup_{n\to\infty} (\langle u - z_t, j(x_n - z_t)\rangle + \varepsilon \\
&\leq \varepsilon.
\end{aligned}$$

Since ε is arbitrary, we get

$$\limsup_{n\to\infty} \langle u - p, j(x_{n+1} - P_F u)\rangle \leq 0. \tag{3.15}$$

Finally we show that $x_n \to P_F u$. In fact, Using (2.1), we get

$$\begin{aligned}
\|x_{n+1} - P_F u\|^2 &= \|(1-\alpha_n)(A_n x_n - P_F u) + \alpha_n (u - P_F u)\|^2 \\
&\leq (1-\alpha_n) \|x_n - P_F u\|^2 \\
&\quad + 2\alpha_n \langle u - P_F u, j(x_{n+1} - P_F u)\rangle).
\end{aligned} \tag{3.16}$$

Using Lemma 2.5, (A3), (3.15) and (3.16), we can easily obtain that x_n strongly converge to $P_F(u)$. The proof is complete. □

Since every uniformly smooth Banach space must be reflexive Banach space with uniformly Géteaux differentiable norm, so using Lemma 2.2, we also obtain the following theorem. As the proof is similar to theorem 3.3, we omit it.

Theorem 3.4. *Let E be a uniformly smooth Banach space. Suppose K is a nonempty closed convex subset of E, and S, T are nonexpansive mapping from K to K such that $F := F(ST) = F(S) \cap F(T) \neq \emptyset$. The sequence $\{x_n\}$ is defined by (3.6), and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1]$ satisfying the conditions (A1), (A2) or $(A2)'$, (A3), (B1), (B2) or $(B2)'$. Then $x_n \to P_F u$, where P_F is a sunny nonexpansive retraction from K into F.*

In theorem 3.4, taking $S = I$, we can easily obtain the following corollary which is Theorem 1 of Tae-Hwa Kim and Hong-Kun Xu[2].

Corollary 3.5. *(T.H. Kim-H.K.Xu[2] Theorem 1) Let E be a uniformly smooth Banach space. Suppose K is a nonempty closed convex subset of E, and T is nonexpansive mappings from K into itself such that $F := F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be a sequence of positive numbers in $[0,1]$ satisfying the following conditions:*

(i) $\alpha_n \to 0, \beta_n \to 0$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < +\infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < +\infty$.

For arbitrary initial value $x_0 \in K$ and fixed anchor $u \in K$, define iteratively a sequence $\{x_n\}$ as follows:

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n) T x_n \\ x_{n+1} = \alpha_n u + (1-\alpha_n) y_n, \quad n \geq 0, \end{cases} \tag{3.17}$$

Then $\{x_n\}$ converges strongly to Pu, where P is the unique sunny nonexpansive retraction from K onto $F(T)$.

Corollary 3.6. *Let E be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J from E to E^* or a uniformly smooth Banach space or uniformly convex Banach space with uniformly Géteaux differentiable norm. Suppose K is a nonempty closed convex subset of E, and T are nonexpansive mapping from K to K such that $F := F(T) \neq \emptyset$. The sequence $\{x_n\}$ is defined by (3.17), and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1]$ satisfying the conditions (A1), (A2) or $(A2)'$, (A3), (B1), (B2) or $(B2)'$. Then $x_n \to P_F u$, where P_F is a sunny nonexpansive retraction from K into F.*

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