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Classification of Decision Making Units with Interval Data Using SBM Model

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Abstract

In original data envelopment analysis (DEA) models, the data for all inputs and outputs are known exactly. When some inputs and outputs are unknown decision variables, such as interval data, ordinal data, and ratio bounded data, the DEA model is called imprecise DEA (IDEA). In this paper, We develop an alternative approach based upon slacks-based measure of efficiency (SBM) for dealing with interval data in DEA. Upper and lower bounds for the SBM-efficiency scores of the decision making units (DMUs) are then defined, and DMUs are classified in terms of the variability of their SBM-efficiency scores.

Keywords: DEA, Slacks, SBM-efficiency, Interval data

1 Introduction

Data envelopment analysis (DEA) is a nonparametric technique for measuring and evaluating the relative efficiency of peer decision making units (DMUs) with multiple inputs and multiple outputs (Charnes et al., 1978). The original DEA models [4] assume that data on the outputs and inputs are known exactly. However, this assumption may not be true. For example, some outputs and inputs may be only known in the form of interval data, ordinal data, and ratio bounded data. Cooper et al. (1999) addressed the problem of imprecise data in DEA in its general form (see also Entali et al., 2002; Zhu 2003; Zhu 2004, for further information). Also Despotis and Smirlis (2002) calculated upper and lower bounds for the radial efficiency scores of DMUs with interval data. In this paper we consider SBM model with interval data. First we introduce SBM model (Tone, 2001), then according to Despotis and Smirlis (2002) we follow SBM-efficiency analysis. The rest of this paper is as follows: Section 2 introduces SBM model. Section 3 is the main part of this paper where we

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introduce SBM model with interval data. In Section 4 We get upper and lower bounds of SBM-efficiency. In Section 5 we will classify DMUs based upon their SBM-efficiency scores. Section 6 contains a numerical example. And conclusions are given in Section 7.

2 SBM model

Suppose we have n DMUs. Each $DMU_j,\ (j = 1, 2, \dots, n)$, produces s different outputs $y_{rj}, (r = 1, 2, \dots, s)$, using m different inputs $x_{ij}, (i = 1, 2, \dots, m)$. When a DMU_o is under evaluation by the SBM model, we have:

$$
\begin{aligned}
(SBM) \quad \rho_o^* = \quad & \min \frac{1-\frac{1}{m}\sum_{i=1}^{m}\frac{s_i^-}{x_{io}}}{1+\frac{1}{s}\sum_{r=1}^{s}\frac{s_i^+}{y_{ro}}} && (1)\\
& s.t. \quad \sum_{j=1}^{n}\lambda_j x_{ij} + s_i^- = x_{io}, && i = 1, 2, \dots, m\\
& \sum_{j=1}^{n}\lambda_j y_{rj} - s_r^+ = y_{ro}, && r = 1, 2, \dots, s\\
& \lambda_j, s_i^-, s_r^+ \geq 0, \forall j,\ i \ and \ r,
\end{aligned}
$$

where x_{io} and y_{ro} are the ith input and rth output of DMU_o $(o \in \{1, 2, \dots, n\})$, respectively.

Definition 1.(SBM-efficiency) A DMU_o is SBM-efficient iff $\rho_o^* = 1$.

This condition is equivalent to $\mathbf{S}^{-*} = (s_1^{-*}, s_2^{-*}, \dots, s_m^{-*}) = \mathbf{0}$ and $\mathbf{S}^{+*} = (s_1^{+*}, s_2^{+*}, \dots, s_s^{+*}) = \mathbf{0}$, i.e., no input excesses and no output shortfalls in any optimal solution. Otherwise it is called SBM-inefficient.

Theorem 1. If $DMU_{\mathtt{A}}$ dominates $DMU_{\mathtt{B}}$, then $\rho_{\mathtt{A}}^* \geq \rho_{\mathtt{B}}^*$.

Proof. See Cooper et al., (2001).□

Model (1) can be transformed into an equivalent linear program using the Charnes-Cooper transformation as follows (See Charnes and Cooper, 1962; Tone, 2001).

$$(LSBM) \quad \min \ \tau = t - \tfrac{1}{m}\sum_{i=1}^{m}\frac{S_i^-}{x_{io}} \tag{2}$$
$$s.t. \ \ 1 = t + \tfrac{1}{s}\sum_{r=1}^{s}\frac{S_r^+}{y_{ro}},$$
$$\sum_{j=1}^{n}\Lambda_j x_{ij} + S_i^- = t x_{io}, \qquad i = 1,2,\dots,m$$
$$\sum_{j=1}^{n}\Lambda_j x_{rj} - S_r^+ = t y_{ro}, \qquad r = 1,2,\dots,s$$
$$t > 0, \Lambda_j, S_i^-, S_r^+ \geq 0, \forall j, \ i \ and \ r,$$

where t=$\frac{1}{1+\frac{1}{s}\sum_{r=1}^{s}\frac{s_r^+}{y_{ro}}}$, $\Lambda_j = t\lambda_j$, $S_i^- = ts_i^-$, $S_r^+ = ts_r^+$, $\forall j, \ i \ and \ r.$
The dual program of Model (2) is as follows:

$$(DLSBM) \quad \max \ 1 + \sum_{r=1}^{s} u_r y_{ro} - \sum_{i=1}^{m} v_i x_{io} \tag{3}$$
$$s.t. \ \ \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \qquad j = 1,2,\dots,n$$
$$v_i x_{io} \geq \tfrac{1}{m}, \qquad i = 1,2,\dots,m$$
$$u_r y_{ro} \geq \tfrac{1}{s}(1 + \sum_{r=1}^{s} u_r y_{ro} - \sum_{i=1}^{m} v_i x_{io}), \quad r = 1,2,\dots,s.$$

3 SBM Model with interval data

Unlike the original DEA models, we assume further that the levels of inputs and outputs are not known exactly, the true input and output data known to lie within bounded intervals, i.e., $x_{ij} \in [\underline{x}_{ij}, \overline{x}_{ij}]$ and $y_{rj} \in [\underline{y}_{rj}, \overline{y}_{rj}]$ with upper and lower bounds of the intervals given as constants and assumed strictly positive. In this case, the SBM-efficiency can be an interval. Now Consider the following SBM-DEA model with imprecise data:

$$
\begin{aligned}
\max\ & 1+\sum_{r=1}^{s} u_r y_{ro} - \sum_{i=1}^{m} v_i x_{io} && (4)\\
s.t.\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, && j=1,2,\dots,n\\
& v_i x_{io} \ge \tfrac{1}{m}, && i=1,2,\dots,m\\
& u_r y_{ro} \ge \tfrac{1}{s}(1+\sum_{r=1}^{s} u_r y_{ro} - \sum_{i=1}^{m} v_i x_{io}), && r=1,2,\dots,s.\\
& x_{ij} \in [\underline{x}_{ij}\quad \overline{x}_{ij}]\\
& y_{rj} \in [\underline{y}_{rj}\quad \overline{y}_{rj}]\\
& \lambda_j, s_i^-, s_r^+ \ge 0, \forall j,\ i \ and\ r.
\end{aligned}
$$

Obviously Model (4) is non-linear and non-convex, because in addition to variables $u_1, u_2, \dots, u_s$ and $v_1, v_2, \dots, v_m$ (weights for outputs and inputs, respectively), the levels of outputs y_{rj} and inputs x_{ij} are also variables whose exact values are to be determined. The SBM-efficiency score attained by DMU_o in Model (4) is not worse (less) than any other SBM-efficiency score that the DMU might attain, by adjusting the levels of the outputs and inputs within the limits of the bounded intervals. Now we convert Model (4) into an equivalent linear program. For this task, first we apply the following transformations to variables x_{ij} and y_{rj}:
$x_{ij} = \underline{x}_{ij} + s_{ij}(\overline{x}_{ij} - \underline{x}_{ij}), \quad 0 \le s_{ij} \le 1;\ i=1,2,\dots,m;\ j=1,2,\dots,n,$
$y_{rj} = \underline{y}_{rj} + t_{rj}(\overline{y}_{rj} - \underline{y}_{rj}), \quad 0 \le t_{rj} \le 1;\ r=1,2,\dots,s;\ j=1,2,\dots,n.$
With these transformations, Model (4) is as follows:

$$
\begin{aligned}
\max\quad & 1+\sum_{r=1}^{s} u_r(\underline{y}_{ro} + t_{ro}(\overline{y}_{ro} - \underline{y}_{ro})) - \sum_{i=1}^{m} v_i(\underline{x}_{io} + s_{io}(\overline{x}_{io} - \underline{x}_{io}))\\
s.t.\ & \sum_{r=1}^{s} u_r(\underline{y}_{rj} + t_{rj}(\overline{y}_{rj} - \underline{y}_{rj})) - \sum_{i=1}^{m} v_i(\underline{x}_{ij} + s_{ij}(\overline{x}_{ij} - \underline{x}_{ij})) \le 0, && \forall j\\
& v_i(\underline{x}_{io} + s_{io}(\overline{x}_{io} - \underline{x}_{io})) \ge \tfrac{1}{m}, && \forall i\\
& u_r(\underline{y}_{ro} + t_{ro}(\overline{y}_{ro} - \underline{y}_{ro})) \ge \tfrac{1}{s}[1+\sum_{r=1}^{s} u_r(\underline{y}_{ro} + t_{ro}(\overline{y}_{ro} - \underline{y}_{ro}))\\
& \qquad - \sum_{i=1}^{m} v_i(\underline{x}_{io} + s_{io}(\overline{x}_{io} - \underline{x}_{io}))], && \forall r\\
& 0 \le s_{ij} \le 1, \quad \forall i,\ j\\
& 0 \le t_{rj} \le 1, \quad \forall r,\ j.
\end{aligned}
$$

Set

$q_{ij} = s_{ij}v_i, \quad i = 1, 2, \ldots, m; j = 1, 2, \ldots, n,$
$p_{rj} = t_{rj}u_r, \quad r = 1, 2, \ldots, s; j = 1, 2, \ldots, n.$
So, inequalities $0 \leq s_{ij} \leq 1$ and $0 \leq t_{rj} \leq 1$ convert to $0 \leq q_{ij} \leq v_i$ and $0 \leq p_{rj} \leq u_r$, respectively. With the above substitutions, finally Model (5) is transformed into the following equivalent linear program:

$$\begin{aligned}
\rho^*_{\mathtt{SBM}} = \quad & \max \ 1 + \sum_{r=1}^{s} u_r \underline{y}_{ro} + p_{ro}(\overline{y}_{ro} - \underline{y}_{ro}) - \sum_{i=1}^{m} v_i \underline{x}_{io} + q_{io}(\overline{x}_{io} - \underline{x}_{io}) \qquad (5) \\
& s.t. \ \sum_{r=1}^{s} u_r \underline{y}_{rj} + p_{rj}(\overline{y}_{rj} - \underline{y}_{rj}) - \sum_{i=1}^{m} v_i \underline{x}_{ij} + q_{ij}(\overline{x}_{ij} - \underline{x}_{ij}) \leq 0, \qquad \forall j \\
& v_i \underline{x}_{io} + q_{io}(\overline{x}_{io} - \underline{x}_{io}) \geq \tfrac{1}{m}, \qquad \forall i \\
& u_r \underline{y}_{ro} + p_{ro}(\overline{y}_{ro} - \underline{y}_{ro}) \geq \tfrac{1}{s}[1 + \sum_{r=1}^{s} u_r \underline{y}_{ro} + p_{ro}(\overline{y}_{ro} - \underline{y}_{ro}) - \\
& \qquad \sum_{i=1}^{m} v_i \underline{x}_{io} + q_{io}(\overline{x}_{io} - \underline{x}_{io})], \qquad \forall r \\
& 0 \leq q_{ij} \leq v_i, \quad \forall i, \ j \\
& 0 \leq p_{rj} \leq u_r, \quad \forall r, \ j.
\end{aligned}$$

It is clear that Model (5) is linear. If we take the length of each interval equal to zero then Model (5) becomes Model (3). To find the SBM-efficiency value of DMU_o with interval data in SBM model, we must solve Model (5).

4 Upper and lower bounds of SBM-efficiency

The upper bound of interval SBM-efficiency is obtained from the optimistic viewpoint and the lower bound is obtained from the pessimistic viewpoint. The following model provides such an upper bound of interval SBM-efficiency for DMU_o:

$$\begin{aligned}
\overline{\rho}^*_o = \quad & \max \ 1 + \sum_{r=1}^{s} u_r \overline{y}_{ro} - \sum_{i=1}^{m} v_i \underline{x}_{io} \qquad (6) \\
& s.t. \ \sum_{r=1}^{s} u_r \underline{y}_{rj} - \sum_{i=1}^{m} v_i \overline{x}_{ij} \leq 0, \qquad j = 1, 2, \ldots, n, \ j \neq o \\
& \sum_{r=1}^{s} u_r \overline{y}_{ro} - \sum_{i=1}^{m} v_i \underline{x}_{io} \leq 0 \\
& v_i \underline{x}_{io} \geq \tfrac{1}{m}, \qquad i = 1, 2, \ldots, m \\
& u_r \overline{y}_{ro} \geq \tfrac{1}{s}(1 + \sum_{r=1}^{s} u_r \overline{y}_{ro} - \sum_{i=1}^{m} v_i \underline{x}_{io}), \qquad r = 1, 2, \ldots, s.
\end{aligned}$$

Model (6) is an SBM-DEA model with exact data, where the levels of inputs and outputs are adjusted in favour of DMU_o and aggressively against the other DMUs. For DMU_o, the inputs are adjusted at the lower bounds and the outputs at the upper bounds of the intervals. Unfavourably for the other DMUs, the inputs are contrarily adjusted at their upper bounds and the outputs at their lower bounds. The model below provides a lower bound of SBM-efficiency score for DMU_o:

$$\begin{aligned} \underline{\rho}_o^* = \quad & \max \ 1 + \sum_{r=1}^{s} u_r \underline{y}_{ro} - \sum_{i=1}^{m} v_i \overline{x}_{io} && (7) \\ & s.t. \ \sum_{r=1}^{s} u_r \overline{y}_{rj} - \sum_{i=1}^{m} v_i \underline{x}_{ij} \leq 0, && j = 1, 2, \ldots, n, \ j \neq o \\ & \sum_{r=1}^{s} u_r \underline{y}_{ro} - \sum_{i=1}^{m} v_i \overline{x}_{io} \leq 0 \\ & v_i \overline{x}_{io} \geq \tfrac{1}{m}, && i = 1, 2, \ldots, m \\ & u_r \underline{y}_{ro} \geq \tfrac{1}{s}\Big(1 + \sum_{r=1}^{s} u_r \underline{y}_{ro} - \sum_{i=1}^{m} v_i \overline{x}_{io}\Big), && r = 1, 2, \ldots, s. \end{aligned}$$

Model (7) is also an SBM-DEA model with exact data. For DMU_o, the inputs are adjusted at their upper bounds and the outputs at their lower bounds and for the other DMUs, the inputs are adjusted at their lower bounds and the outputs at their upper bounds.
Therefore, Models (6) and (7) provide for each DMU a bounded interval $[\underline{\rho}_o^*, \ \overline{\rho}_o^*]$ in which its possible SBM-efficiency scores lie, from the worst to the best case.

Theorem 2. If $\rho_{\texttt{SBM}}^*$ and $\overline{\rho}_o^*$ are the optimal values of (5) and (6), respectively, then $\rho_{\texttt{SBM}}^* = \overline{\rho}_o^*$.
Proof. Suppose $\mathbf{u}^* = (u_r^*, r = 1, 2, \ldots, s)$ and $\mathbf{v}^* = (v_i^*, i = 1, 2, \ldots, m)$ is an optimal solution of Model (6).
Set
$\overline{p}_{rj} = 0, \quad r = 1, 2, \ldots s; j = 1, 2, \ldots n, j \neq o,$
$\overline{p}_{ro} = u_r^*, \quad r = 1, 2, \ldots, s,$
$\overline{q}_{ij} = v_i^*, \quad i = 1, 2, \ldots m; j = 1, 2, \ldots n, j \neq o,$
$\overline{q}_{io} = 0, \quad i = 1, 2, \ldots m.$
Then the augmented solution $(\mathbf{u}, \ \mathbf{v}, \ \mathbf{P}, \ \mathbf{Q})$=$(\mathbf{u}^*, \ \mathbf{v}^*, \ \overline{\mathbf{P}}, \ \overline{\mathbf{Q}})$ is a feasible solution for Model (5). Hence $\overline{\rho}_o^* \leq \rho_{\texttt{SBM}}^*$.
On the basis of the strategy of Model (5), which determines the best inputs and outputs level of DMU_o in its interval, $\overline{\rho}_o^*$ is the highest possible SBM-efficiency score that DMU_o can obtain. Then we have $\rho_{\texttt{SBM}}^* \leq \overline{\rho}_o^*$. So $\rho_{\texttt{SBM}}^* = \overline{\rho}_o^*$.□

5 Classification of the DMUs

On the basis of the above SBM-efficiency score intervals, the DMUs can be classified in three subsets as follows:

- $\mathbf{E}^{++}_{\texttt{SBM}} = \{j \in \{1, 2, \ldots, n\} | \ \underline{\rho}^*_o = 1\}$,
- $\mathbf{E}^{+}_{\texttt{SBM}} = \{j \in \{1, 2, \ldots, n\} | \ \underline{\rho}^*_o < 1, \ and \ \overline{\rho}^*_o = 1\}$,
- $\mathbf{E}^{-}_{\texttt{SBM}} = \{j \in \{1, 2, \ldots, n\} | \ \overline{\rho}^*_o < 1\}$.

The set $\mathbf{E}^{++}_{\texttt{SBM}}$ consists of the DMUs that are SBM-efficient in any case (any combination of input/output levels). The set $\mathbf{E}^{+}_{\texttt{SBM}}$ consists of the DMUs that are SBM-efficient in a maximal sense, but there are input/output adjustments under which they cannot maintain their SBM-efficiency; and according to the differences of upper and lower bounds of SBM-efficiency values $(\overline{\rho}^*_o - \underline{\rho}^*_o)$, the SBM-efficient DMUs in $\mathbf{E}^{+}_{\texttt{SBM}}$ can be ranked. Finally, the set $\mathbf{E}^{-}_{\texttt{SBM}}$ consists of the definitely SBM-inefficient DMUs.

6 An application

To illustrate our approach, consider the interval data setting of Table **1** (5 DMUs with 2 inputs and 2 outputs). The SBM-efficiency scores obtained by applying Models (6) and (7) are given in Table **2**.

Table 1: Interval data

	Input				Output			
	$\underline{x}_{1j}$	$\overline{x}_{1j}$	$\underline{x}_{2j}$	$\overline{x}_{2j}$	$\underline{y}_{1j}$	$\overline{y}_{1j}$	$\underline{y}_{2j}$	$\overline{y}_{2j}$
DMU_1	7	7.5	0.6	0.9	15.7	16.1	14	20
DMU_2	6	7.5	2.1	4.8	13.8	14.4	10.5	11
DMU_3	5	8.5	1	7	14.3	15.9	14	17.5
DMU_4	2	6	1.6	0.5	15.7	19.8	10.5	14.5
DMU_5	9.5	11	1.2	1.9	15.8	18.1	10.5	12.5

Table 2: SBM-efficiency scores and classification

	$\underline{\rho}^*_o$	$\overline{\rho}^*_o$	Classification
DMU_1	1.00000	1.00000	$\mathbf{E}^{++}_{\texttt{SBM}}$
DMU_2	0.21308	1.00000	$\mathbf{E}^{+}_{\texttt{SBM}}$
DMU_3	0.31757	1.00000	$\mathbf{E}^{+}_{\texttt{SBM}}$
DMU_4	0.41365	1.00000	$\mathbf{E}^{+}_{\texttt{SBM}}$
DMU_5	0.32566	0.77462	$\mathbf{E}^{-}_{\texttt{SBM}}$

As can be observed from Table 2, for DMU_1, for instance, $\underline{\rho}_o^*$=1, and then DMU_1 is classified in $\mathbf{E}^{++}_{\mathtt{SBM}}$ as it is SBM-efficient in any case. According to $(\overline{\rho}_j^* - \underline{\rho}_j^*)$ we rank SBM-efficient DMUs in $\mathbf{E}^{+}_{\mathtt{SBM}}$ as follows: DMU_4, DMU_3, and DMU_2.

7 Conclusion

After introducing IDEA by Cooper et al., in 1999, this subject has been investigated in some perspectives. The current paper discusses and develops an IDEA procedure based upon the SBM model. Also the discussion in the current study is based upon multiplier SBM model. We define upper and lower bounds for the possible SBM-efficiency scores that a unit might attain in an imprecise data setting. We then use these bounds to classify the units as follows:
(i) SBM-efficient in any case
(ii) SBM-efficient in maximal sense
(iii) always SBM-inefficient.

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