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A New Convergence Theorem for Newton's Method in Banach Space Using Assumptions on the First Fréchet-derivative

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Abstract

In this study, we provide a new Kantorovich-type convergence theorem for Newton's method in Banach space. Its condition is different from earlier ones, and therefore it has theoretical and practical value. A simple numerical example is given to show that our results apply, but earlier ones fail.

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1 Introduction

Let X,Y be Banach spaces, $F : D \subseteq X \longrightarrow Y$ be Fréchet-differentiable. We are concerned with the problem of approximating a solution x^* of the equation

$$F(x) = 0.$$

Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n)(n \geq 0) \tag{1}$$

has been applied extensively to generate a sequence $\{x_n\}(n \geq 0)$ converging to x^*. In particular, the following conditions have been used.

Condition A(Kantorovich[6]) Let $F : D \subseteq X \longrightarrow Y$ be Fréchet-differentiable, and $F'(x_0)^{-1} \in L(Y, X)$ exists for some $x_0 \in D$, where $L(Y, X)$ is the space of bounded linear operator from Y into X. Assume

$$\|F'(x_0)^{-1}[F'(x) - F'(y)]\| \leq K\|x - y\|(\forall x, y \in D), \tag{2}$$

$$\|F'(x_0)^{-1}F(x_0)\| \leq \eta \tag{3}$$

and

$$2K\eta \leq 1. \tag{4}$$

Condition B(Huang[5], Gutierrez[3,4]) Let $F : D \subseteq X \longrightarrow Y$ be twice Fréchet-differentiable, $F'(x) \in L(X, Y), F''(x) \in L(X, L(X, Y))(x \in D), F'(x_0)^{-1}$ exists at some $x_0 \in D$. Assume

$$\|F'(x_0)^{-1}[F'(x) - F'(y)]\| \leq a_0\|x - y\|(\forall x, y \in D),$$

$$\|F'(x_0)^{-1}F(x_0)\| \leq \eta, \|F'(x_0)^{-1}F(x_0)\| \leq b_0,$$

and

$$3\eta a_0^2 + 3a_0b_0 + b_0^3 \leq (b_0^2 + 2a_0)^{\frac{3}{2}}.$$

Condition C(Argyros[1]) Let $F : D \subseteq X \longrightarrow Y$ be twice Fréchet-differentiable. Assume

(a) there exists $x_0 \in X$ and non-negative numbers a,b,c such that

$$\|F''(x) - F''(x_0)\| \leq a\|x - x_0\|, \|F''(x_0)\| \leq b$$

and

$$\|F'(x)^{-1}\| \leq c(\forall x \in X).$$

(b) the following conditions hold:

$$\alpha = \frac{c}{2}[\frac{a}{3}\|F'(x_0)^{-1}F(x_0)\| + b]\|F'(x_0)^{-1}F(x_0)\| \in [0, 1)$$

and

$$d = \frac{c}{2}[(\frac{1}{1-\alpha} + \frac{\alpha}{3})a\|F'(x_0)^{-1}F(x_0)\| + b]\|F'(x_0)^{-1}F(x_0) \in [0, 1).$$

Under condition A,B or C, one can obtain many results. But sometimes conditions A,B and C fail.

Example Let $D = X = Y = R, x_0 = 0$ and consider the function F on D given by

$$F(x) = \int_{-1}^{x} (1 + |u|)du.$$

Using (2) and (3) ,we get K=1, $\eta = \frac{3}{2}$. Kantorovich assumption (4) is violated, since $2K\eta = 3 > 1$. Therefore condition A fails. Furthermore, conditions B and C fail because $F''(0)$ doesn't exist.

In this paper, we put forth a new condition, under which the Newton method starting from $x_0 = 0$ in above example converges.

2 The main result

In this section, we present our convergence result concerning Newton's method using hypotheses on the first Fréchet-derivative. It is assumed that a solution of a nonlinear equation exists.

Theorem *Let $F : X \longrightarrow Y$ be a Fréchet-differentiable operator. Assume (a) there exists $x_0 \in X$ and non-negative numbers a,b such that*

$$\|F'(x) - F'(y)\| \leq a\|x - y\| \tag{5}$$

and

$$\|F'(x)^{-1}\| \leq b(\forall x \in X). \tag{6}$$

(b) Define parameters α by

$$\alpha = \frac{ab}{2}\|F'(x_0)^{-1}F(x_0)\|$$

and the following condition holds:

$$\alpha \in [0, 1). \tag{7}$$

Then the following hold:

$$\|x_{n+2} - x_{n+1}\| \leq \frac{ab}{2}\|x_{n+1} - x_n\|^2 (n \geq 0), \tag{8}$$

$$\|x_{n+1} - x_n\| \leq \alpha^n \|F'(x_0)^{-1}F(x_0)\|, \tag{9}$$

$$\|x_n - x^*\| \leq \|F'(x_0)^{-1}F(x_0)\| \sum_{j=n}^{+\infty} \alpha^j (n \geq 0) \tag{10}$$

and

$$\lim_{n\longrightarrow\infty} x_n = x^* \text{ with } F(x^*) = 0.$$

Proof We first note that Newton iterates $\{x_n\}(n \geq 0)$ generated by (1) are well defined for all $n \geq 0$ since $F'(x)^{-1}$ exists for all $x \in X$. Using (1) we obtain the approximation

$$\begin{aligned} F(x_{n+1}) &= F(x_{n+1}) - F(x_n) - F'(x_n)(x_{n+1} - x_n) \\ &= \int_0^1 [F'(x_n + t((x_{n+1} - x_n))) - F'(x_n)]dt(x_{n+1} - x_n) \end{aligned} \tag{11}$$

Then, by (5),(6) and (11) we obtain

$$\|x_{n+2} - x_{n+1}\| \leq \frac{ab}{2}\|x_{n+1} - x_n\|^2 (n \geq 0), \tag{12}$$

which shows (8). For n=0, (12) gives

$$\|x_2 - x_1\| \leq \frac{ab}{2}\|F'(x_0)^{-1}F(x_0)\|\|x_1 - x_0\| = \alpha\|x_1 - x_0\|.$$

Let us assume

$$\|x_{k+2} - x_{k+1}\| \leq \alpha\|x_{k+1} - x_k\| \tag{13}$$

for $k = 0, 1, 2, \cdots, n-1$. Then, by (7) and (13) we get

$$\begin{aligned} \|x_{n+2} - x_{n+1}\| &\leq \frac{ab}{2}\|x_{n+1} - x_n\|^2 \\ &\leq \cdots \leq \frac{ab}{2}\alpha^n\|x_1 - x_0\|\|x_{n+1} - x_n\| \\ &= \alpha^{n+1}\|x_{n+1} - x_n\| \leq \alpha\|x_{n+1} - x_n\|, \end{aligned}$$

which shows (13) for k=n.

Furthermore, it follows that

$$\|x_{n+1} - x_n\| \leq \alpha\|x_n - x_{n-1}\| \leq \cdots \leq \alpha^n\|x_1 - x_0\|,$$

which shows (9). For $p \geq 0$, estimate (9) implies

$$\|x_{n+p} - x_n\| \leq \|F'(x_0)^{-1}F(x_0)\| \sum_{j=n}^{n+p-1} \alpha^j (n \geq 1). \tag{14}$$

It follows from (7) and (14) that $\{x_n\}(n \geq 0)$ is a Cauchy sequence in a Banach space X and it converges to some $x^* \in X$. So estimate (10) holds. Finally, by letting $n \longrightarrow \infty$ in (1) we get $F(x^*) = 0$.

That completes the proof.

3 Numerical example

Returning back to above example, we first note that there exists a zero x^* of F on R since $F(0)F(-2) < 0$. Moreover, we get a=1, b=1 by (5) and (6). Then, we obtain $\alpha = \frac{ab}{2}\|F'(x_0)^{-1}F(x_0)\| = \frac{3}{4} < 1$. Hence, all hypotheses of our theorem are satisfied. That is, our convergence theorem guarantees that the Newton method generated by (1) and starting from $x_0 = 0$ converges to a zero x^* of function F.

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