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Selection of an Eligible Benchmark for Interval Decision Making Units with Input Contraction

F. Hosseinzadeh Lotfi[1]

Department of Mathematics, Science and Research Branch
Islamic Azad University
P.O. Box 14155/775 and 14155/4933, Post code: 1477893855, Tehran, Iran

F. Rezai Balf, M. Adabitabar Firozja and S. A. Hosseiny Matikolai

Department of Mathematics
Islamic Azad University, Qaemshahr, Iran

Abstract

This paper offers an extension of obtaining an eligible benchmark using the concept of input contraction on interval data. Here, it is worthwhile obtaining most proportionality benchmark, which it will be most similarity to the evaluated Interval Decision Making Unit (IDMU). Concept of similarity means that the evaluated IDMU and its benchmark have inputs and outputs closest to each other. In continue, we give concept of contract (input contract) such that it gets possible to obtain shortest path to efficient subsets.

Keywords: Data envelopment Analysis, Input Contraction, Interval Data

1. Introduction

Data Envelopment Analysis (DEA) is a non-parametric method for evaluating the relative efficiency of Decision Making Units (DMUs) of multiple inputs and outputs [5]. The original DEA models (Charnes et al (1978)[1], Banker et. al (1984)[2], assume that inputs and outputs are measured by exact values and the value of efficiency be assessed on base relationship between the evaluated unit and its projection point on efficient frontier. Using DEA models for evaluating relative efficiency of DMUs, usually give score 1 for a set of efficient

[1]Corresponding author. e-mail: hosseinzadeh_lotfi@yahoo.com

DMUs and less than 1 for inefficient DMUs. We know, technical inefficient relate a failure of DMUs in order to obtain maximal outputs using of inputs. In order to improve technical efficient, obtaining drawback actions and recognize eligible ways for use the better than inputs have special important.

2. Background

2.1. Efficiency improvement

Some question posed by inefficient DMUs in management programs are "How can I become efficient?" or "What am I doing wrong? ". A reasonable strategy would be that, after the DMU is informed that it is inefficient, its manager visits some of the efficient DMUs to observe how they do things. This benchmarking procedure is common in DMU management programs.
A non-trivial question here is how to choose which of the efficient DMUs it should visit. It seems that an inefficient DMU will prefer to visit the efficient DMU that is most similar to it [3].

2.2 Definition 1

Koopmans (1951)[8] defines a DMU as technically efficient if and only if, increasing any output or decreasing any input is possible only by decreasing some other output or increasing some other input.

2.3 Definition 2

Here we present a definition of input isoquant as

$$Isoq\ L(Y) = \{X \in R^m_+ :\ X \in L(Y), \lambda X \notin L(Y), \lambda \in [0, 1)\}. \qquad (1)$$

Where $L : R^s_+ \to R^m_+$ is a mapping from the output vector $Y \in R^s_+$ into the set of input vectors $X \in R^m_+$ that allow to produce Y. It is mentioned that in order to two inputs and one constant output with constant return to scale, $L(Y)$ is Farrell frontier (see Fig. 1).

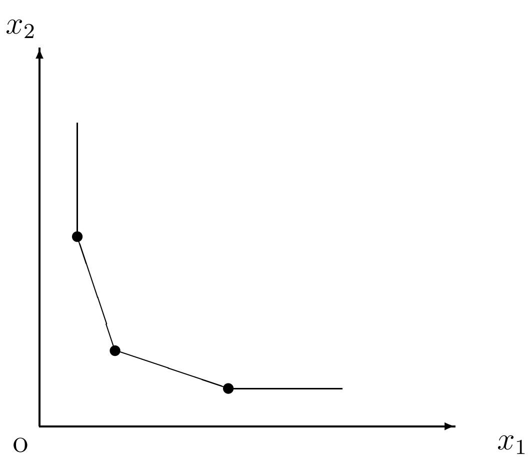

Fig. 1. Farrell frontier

2.4 Definition 3
An efficient subset of the isoquant is defined as
$Eff\ L(Y) = \{X \in R_{+}^{m}:\ X \in L(Y), \hat{X} \leq X, \hat{X} \notin L(Y)\}.$ (2)

2.5 Definition 4
Farrell (1957)[6] presented a definition of efficiency that can be achieved holding constant the output vector as
$F(X,Y) = Min\{\theta : \theta X \in L(Y), \theta \in R_{+}\}.$ (3)
This measure shrinks the input vector along a ray until a point in the isoquant is reached. Note that Farrell efficiency is done with respect to the isoquant, such as Koopmans definition of efficiency is base on the concept of efficient subset.

2.6 Definition 5
The Russell measure [4], introduced by Fare and Lovell (1978)[9] is defined as
$R(X,Y) = Min\{\frac{\sum_{i=1}^{m}\theta_i}{m} : (\theta_1 X_1, \theta_2 X_2, ..., \theta_m X_m) \in L(Y), \theta_i \in [0,1] \forall i\}.$ (4)
The Russell measure shrinks the input vector all coordinate directions until a point in the subset of the isoquant is reached.
2.7 Definition 6
Kopp[7] introduces the notion of single-factor efficiency measures as an attempt to understand the individual contribution of each input to inefficiency. The single-factor efficiency measure of $k-th$ input is given by
$K_k(X,Y) = Min\{\theta_k : (X_1, ..., \theta_k X_k, ..., X_m) \in L(Y), \theta_k \in R_{+}\}.$ (5)
Expression (5) gives the contraction in input needed to reach the isoquant.

3. Input-specific contraction model

In this section after some definition, we present input-specific contraction model which introduced by Gonzalez et al. (2001)[3].

3.1 Definition 7
The smallest contraction to the efficient subset is defined as
$C(X,Y) = Min\{\sum_{i=1}^{m}(1-\theta_i) : (\theta_1 x_1, \theta_2 x_2, ..., \theta_m x_m) \in Eff\ L(Y), \theta_i \leq 1\ \forall i\}.$ (6)
Expression (6) determinates the shortest path to the efficient subset.
3.2 Definition 8
The $k-th$ input-specific contraction measures the contraction needed to reach the isoquant along the $k-th$ axis is defined as

$$C(X,Y)_k = Max\left\{\sum_{i=1}^{m}(1-\theta_i) : \begin{array}{cc} (x_1,...,\theta_k x_k,...,x_m) \in IsoqL(Y) & \theta_i \leq 1 \\ (\theta_1 x_1,...,\theta_m x_m) \in EffL(Y) & \forall i \end{array}\right\}. \quad (7)$$

3.3 Theorem 1

The smallest contraction to the efficient subset is the smallest input-specific contraction [3]:

$$C(X,Y) = Min\{C(X,Y)_k \mid \ k = 1,...,m\ \}. \quad (8)$$

3.4 programming model

Gonzalez et al (2001) introduced the below formulation that computes all feasible reductions in order to reach the efficient subset.

$$\begin{array}{lll}
 & Min\ \ M\theta_k + \sum_{i=1, i\neq k}^{m} \theta_i & \\
s.t & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta_i x_{io} & i = 1,...,m \\
 & \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro} & r = 1,...,s \qquad (9) \\
 & \sum_{j=1}^{n} \lambda_j = 1 & \\
 & \theta_i \leq 1 & i = 1,...,m, i \neq k \\
 & \lambda_j \geq 0 & j = 1,...,n
\end{array}$$

Where M is a large enough scalar to force the model (9) to identity $k-th$ input toward the isoquant. After computing θ in Eq. (9), the input-specific contraction defined in Eq. (8) can be derived as

$$C(X_i, Y_i) = \sum_{i=1}^{m}(1-\theta_i) = (1-\theta_k) + \sum_{i=1, i\neq k}^{m}(1-\theta_i). \quad (10)$$

4. Contraction input-special on subset efficient using interval inputs

In this section we will extend the work Gonzalez et al. (2001) on interval data (here, we note to interval input). According to model (9), we have model (11) for interval inputs:

$$\begin{array}{lll}
 & Min\ \ M\theta_k + \sum_{i=1, i\neq k}^{m} \theta_i & \\
s.t & \sum_{j=1}^{n} \lambda_j [x_{ij}^L, x_{ij}^U] \leq \theta_i [x_{io}^L, x_{io}^U] & i = 1,...,m \\
 & \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro} & r = 1,...,s \qquad (11) \\
 & \sum_{j=1}^{n} \lambda_j = 1 & \\
 & \theta_i \leq 1 & i = 1,...,m, i \neq k \\
 & \lambda_j \geq 0 & j = 1,...,n
\end{array}$$

Where M is a large number, such that it force that θ_k get possible minimum value. Model (11) first decrease $k-th$ input and then the other inputs so that we reach on isoquant. Suppose that $\theta^* = (\theta_1^*,...,\theta_k^*,...,\theta_m^*)$ is optimal solution in model (11), then we get contraction input-special with interval input by $C([X_i^L, X_i^U], Y_i)_k$:

$$C([X_i^L, X_i^U], Y_i)_k = \sum_{i=1}^{m}(1-\theta_i^*) \qquad (12)$$

The relation (12) also can be written as follows:

$$C([X_i^L, X_i^U], Y_i)_k = (1-\theta_k^*) + \sum_{i=1, i\neq k}^{m}(1-\theta_i^*). \qquad (13)$$

In formula (13), first factor is the value contraction for arrive to isoquant and second factor show the values of slack variables in order to reach the efficient subset.

4.1 Definition 9

The smallest contraction to the efficient subset with interval input is defined as

$$C([X^L, X^U], Y) = Min\{\sum_{i=1}^{m}(1-\theta_i) : (\theta_1[X_1^L, X_1^U], \theta_2[X_2^L, X_2^U], ..., \theta_m[X_m^L, X_m^U]) \in Eff\ L(Y), \theta_i \leq 1\ \forall i\}. \qquad (14)$$

4.2 Definition 10

The smallest contraction to the efficient subset is the smallest input-specific contraction

$$C([X^L, X^U], Y) = [C([X^L, X^U], Y)^L, C([X^L, X^U], Y)^U]$$
$$C([X^L, X^U], Y)^L = Min\ C([X^L, X^U], Y)_k^L, \qquad k = 1, ..., m \qquad (15)$$
$$C([X^L, X^U], Y)^U = Max\ C([X^L, X^U], Y)_k^U, \qquad k = 1, ..., m$$

Regarding model (11), with interval input, we obtain in below two models

$$\begin{aligned} Z_k^L = Min\ \ & M\theta_k + \sum_{i=1, i\neq k}^{m}\theta_i \\ s.t\ \ & \sum_{j=1, j\neq o}^{n}\lambda_j x_{ij}^L + \lambda_o x_{io}^U \leq \theta_i x_{io}^U \qquad i = 1, ..., m \\ & \sum_{j=1}^{n}\lambda_j y_{rj} \geq y_{ro} \qquad r = 1, ..., s \qquad (16) \\ & \sum_{j=1}^{n}\lambda_j = 1 \\ & \theta_i \leq 1 \qquad i = 1, ..., m, i \neq k \\ & \lambda_j \geq 0 \qquad j = 1, ..., n \end{aligned}$$

$$\begin{aligned} Z_k^U = Min\ \ & M\theta_k + \sum_{i=1, i\neq k}^{m}\theta_i \\ s.t\ \ & \sum_{j=1, j\neq o}^{n}\lambda_j x_{ij}^U + \lambda_o x_{io}^L \leq \theta_i x_{io}^L \qquad i = 1, ..., m \\ & \sum_{j=1}^{n}\lambda_j y_{rj} \geq y_{ro} \qquad r = 1, ..., s \qquad (17) \end{aligned}$$

$$\sum_{j=1}^{n}\lambda_j = 1$$
$$\theta_i \leq 1 \qquad i = 1, ..., m, i \neq k$$
$$\lambda_j \geq 0 \qquad j = 1, ..., n$$

Then $C([X^L, X^U], Y)_k^L = \sum_{i=1}^{m}(1-\theta_i)$ with θ_i in model (17) and $C([X^L, X^U], Y)_k^U = \sum_{i=1}^{m}(1-\theta_i)$ with θ_i in model (16) are values of contraction input-special for models (16) and (17), respectively. In order to further explain, consider Farrell model (two inputs and one output) accorded to Figures 2 and 3, for data and A, B, C and D which inputs are intervals. Let DMU_c be as a unit under evaluate.

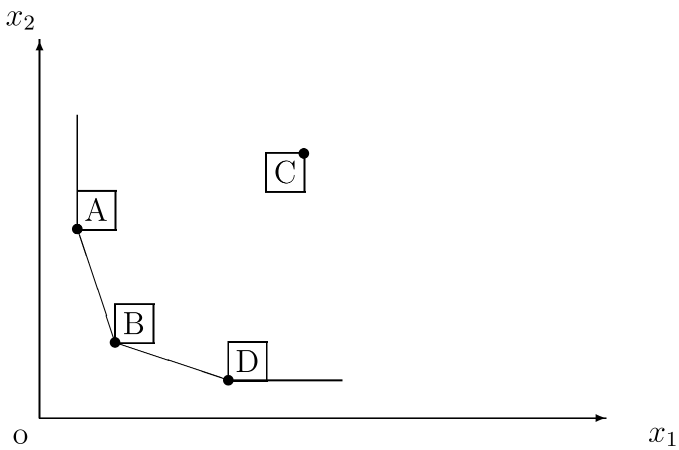

Fig. 2. Input of evaluated DMU is best condition and other DMUs in worst condition

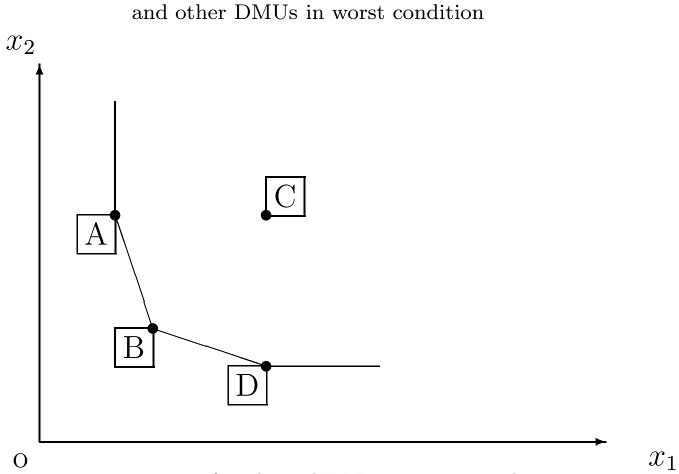

Fig. 3. Input of evaluated DMU is worst condition and other DMUs in best condition

4.3 Theorem 2

Let, $(\lambda^*, \theta^*) \in R^{m+n}$ is a optimal solution (16)then

$$\sum_{j=1, j\neq o}^{n} \lambda_j^* x_{ij}^L + \lambda_o^* x_{io}^U = \theta_i^* x_{io}^U \quad i = 1, ..., m \qquad (18)$$

Proof: Suppose that λ^*, θ^* be optimal solution of model (16), such that

$\sum_{j=1, j\neq o}^{n} \lambda_j^* x_{ij}^L + \lambda_o^* x_{io}^U < \theta_i^* x_{io}^U \quad i = 1, ..., m$. Then, we get

$\sum_{j=1, j\neq o}^{n} \lambda_j^* x_{ij}^L + \lambda_o^* x_{io}^U \leq \theta_i^* x_{io}^U - s_{io} \quad i = 1, ..., m$. Where $s_{io} > 0$. We set $s_{io} = l_i x_{io}^U$, such that $l_i > 0$, because x and s are input quantities. Therefore, we have

$\sum_{j=1, j\neq o}^{n} \lambda_j^* x_{ij}^L + \lambda_o^* x_{io}^U \leq \theta_i^* x_{io}^U - l_i x_{io}^U \quad i = 1, ..., m$. Thereby

$\sum_{j=1, j\neq o}^{n} \lambda_j^* x_{ij}^L + \lambda_o^* x_{io}^U \leq \theta_i^{\sim} x_{io}^U \quad i = 1, ..., m$. Where $\theta_i^{\sim} = \theta_i^* - l_i, \quad i = 1, ..., m$.

Hence, $(\lambda^{\sim}, \theta^*)$ is a solution for model (16), and we will have objective value

$Z_k^{\sim L} = M\theta_k^{\sim} + \sum_{i=1, i\neq k}^{m} \theta_i^{\sim} = M(\theta_k^* - l_k) + \sum_{i=1, i\neq k}^{m} (\theta_i^* - l_i) = Z_k^{*L} - (Ml_k + \sum_{i=1, i\neq k}^{m} l_i) < Z_k^{*L}$

This is a contradict.

4.4 Theorem 3

Let, $(\lambda^*, \theta^*) \in R^{m+n}$ is a optimal solution (17)then

$$\sum_{j=1, j\neq o}^{n} \lambda_j^* x_{ij}^U + \lambda_o^* x_{io}^L = \theta_i^* x_{io}^L \quad i = 1, ..., m \qquad (18)$$

Proof: The proof is also as Theorem 2.

4.5 Theorem 4

Assume that $x_{ij} \in [x_{ij}^L, x_{ij}^U]$, then according to models (11), (16) and (17) we have: $Z_k^{*L} \leq Z_k^* \leq Z_k^{*U}$.

Proof: Suppose that λ^*, θ^* be optimal solution of model (11), then

$$\sum_{j=1, j\neq o}^{n} \lambda_j^* x_{ij}^L \leq \sum_{j=1, j\neq o}^{n} \lambda_j^* x_{ij} \leq (\theta_i^* - \lambda_o^*) x_{io} \leq (\theta_i^* - \lambda_o^*) x_{io}^U, \quad i = 1, ..., m$$

It implied that λ^*, θ^* is a feasible solution for model (16). Therefore we have:

$Z_k^{*L} \leq Z_k^* \qquad (20)$

Also suppose that λ^*, θ^* be optimal solution of model (17), then

$$\sum_{j=1, j\neq o}^{n} \lambda_j^* x_{ij} \leq \sum_{j=1, j\neq o}^{n} \lambda_j^* x_{ij}^U \leq (\theta_i^* - \lambda_o^*) x_{io}^L \leq (\theta_i^* - \lambda_o^*) x_{io}, \quad i = 1, ..., m$$

It implied that λ^*, θ^* is a feasible solution for model (11). Therefore we have:

$Z_k^* \leq Z_k^{*U} \qquad (21)$

Hence from relations (20) and (21) we have: $Z_k^{*L} \leq Z_k^* \leq Z_k^{*U}$.

The proof is complete.

4.6 Theorem 5
Assume that $x_{ij} \in [x_{ij}^L, x_{ij}^U]$, then according to models (11), (16) and (17) we have:
$C([X_i^L, X_i^U], Y)_k^L \leq C([X_i^L, X_i^U], Y)_k \leq\leq C([X_i^L, X_i^U], Y)_k^U$

Proof: The assertion is confirm with respect to models (11), (16), (17) and the definition (15) and also inequality $Z_k^{*L} \leq Z_k^* \leq Z_k^{*U}$, because we have: first $\theta_k^{*L} \leq \theta_k^* \leq \theta_k^{*U}$, where $\theta_k^{*L}, \theta_k^*$ and θ_k^{*U} are the optimal values of $k-th$ input toward the isoquant for models (17), (11) and (16), respectively. Second $C([X_i^L, X_i^U], Y)_k^L = m - \sum_{i=1}^{m} \theta_i^{*U} \leq m + (M-1)\theta_k^{*U} - Z_k^{*U} \leq m + (M-1)\theta_k^* - Z_k^* = C([X_i^L, X_i^U], Y)_k \leq m + (M-1)\theta_k^{*L} - Z_k^{*L} = C([X_i^L, X_i^U], Y)_k^U$

5. Numerical Example

In this article, branches of Bank are chosen. There are 18 branches in this district. Each branch uses 9 inputs to produce 3 outputs. Table 1 shows a list of results regarding models (15), (16) and (17).

Table1. The results interval input contraction by models (15), (16) and (17)

DMU_j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
θ_j^L	1	1	0.0972	1	1	1	1	1	1	1	1	1	1	1	1	0.2308	0.1304	1
θ_j^U	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

In Table 1 we have three branches have the lower bounded of input contraction less than 1 and the upper bounded 1, meanwhile the other branches have lower and upper bounded of input contraction is equal 1. Note that, the lower bounded of input contraction in order to branch 3 arise in $7-th$ input index and branches 16 and 17 in $1-th$ input index.

6. Conclusion

In this paper we present eligible benchmarks for inefficient Interval Decision Making Units (IDMU). In this approach we first compute the $k-th$ contraction interval data (Here, interval input) with models (16) and (17). In continue we give minimum contraction of inputs with relation of (15). In this study also we performed an application of data envelopment analysis to the Iran commercial banking system, by using interval data and determination of input contraction as an interval. As it was predictable, these results show that in order to some of bank branches the least input contraction is not membership to an especial input.

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