

**Variable Viscosity Effects on Hydromagnetic Boundary  
Layer Flow along a Continuously Moving Vertical Plate  
In the Presence of Radiation**

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**Abstract.** This work presents a study of the flow and heat transfer of an incompressible viscous electrically conducting fluid over a continuously moving vertical infinite plate with uniform suction and heat flux in the presence of radiation taking into account the effects of variable viscosity. It is found that the velocity increases as the viscosity of air or the magnetic parameter decreases and the thermal boundary layer thickness increases as the radiation parameter increases. The skin-friction coefficient is computed and discussed for various values of the parameters.

**Keywords:** MHD, variable viscosity, radiation.

Mathematics Subject Classification: 76W05

## **1- Introduction**

Flow and heat transfer in the boundary layer induced by a moving surface in a quiescent fluid is important in many engineering applications. For examples, in the extrusion of polymer sheet from a dye, the cooling of an infinite metallic plate in a cooling path, glass blowing continuous casting and spinning of fibers.

Sakiadis [5] studied the boundary layer flow over a continuous solid surface moving with constant velocity in an ambient fluid. The flow is quite different from the boundary layer flow over a semi-infinite flat plate due to the entrainment of the ambient fluid. Tsou et al. [7] presented a combined analytical and experimental study of the flow and temperature fields in the boundary layer on a continuous moving surface. Erickson et al. [9] extended Sakiadis problem to include blowing or suction at the moving surface. Crane [10] studied the boundary layer flow caused by a stretching sheet whose velocity varies linearly with the distance from a fixed point on the surface. Gupta and Gupta [15] studied the momentum, heat and mass transfer in the boundary layer over a stretching sheet with suction or blowing. Soundalgekar and Ramana [17] investigated the constant surface case with a power law temperature variation.

The magnetohydrodynamics of an electrically conducting fluid is encountered in many problems in geophysics, astrophysics, engineering applications and other industrial areas. Hydromagnetic free convection flow have a great significance for the applications in the fields of stellar and planetary magnetospheres, aeronautics. Engineers employ magnetohydrodynamics principles in the design of heat exchangers, pumps, in space vehicle propulsion, thermal protection, control and re-entry and in creating novel power generating systems. However, hydromagnetic flow and heat transfer problems have become more important industrially. In many metallurgical processes involve the cooling of many continuous strips or filaments by drawing them through an electrically conducting fluid subject to a magnetic field, the rate of cooling can be controlled

and final product of desired characteristics can be achieved. Another important application of hydromagnetics to metallurgy lies in the purification of molten metals from non-metallic inclusions by the application of a magnetic field. Chakrabartia and Gupta [1] investigated hydromagnetic flow, heat and mass transfer over a stretching sheet. Kumar et al. [6] studied hydromagnetic flow and heat transfer on a continuously moving vertical plate. Vajravelu and Hadjinicolaou [8] studied the flow and heat transfer characteristic in an electrically conducting fluid near an isothermal stretching sheet. Sharma and Mathur [14] investigated steady laminar free convection flow of an electrically conducting fluid along a porous hot vertical infinite plate in the presence of heat source or sink.

On the other hand, at high temperature the effects of radiation in space technology, solar power technology, space vehicle re-entry, nuclear engineering applications are very significant. Many processes in industrial areas occur at high temperature and the knowledge of radiation heat transfer in the system can perhaps lead to a desired product with a desired characteristic. Raptis and Massalas [4] studied the radiation effect on the unsteady magnetohydrodynamic flow of an electrically conducting viscous fluid past a plate. Chamkha [2] investigated thermal radiation and buoyancy effects on hydromagnetic flow over an accelerating permeable surface with heat source or sink. Raptis et al.[3] discussed the effect of thermal radiation on MHD asymmetric flow of an electrically conducting fluid past a semi-infinite plate.

All the above studies were confined to a fluid with constant viscosity. However, it is known that this physical property may change significantly with temperature. Hossain and Munir [12] analyzed a two-dimensional mixed convection flow of a viscous incompressible fluid of temperature dependent viscosity past a vertical plate. Fang [16] studied the influence of fluid property

variation on the boundary layers of a stretching surface. Hossain et al.[11] discussed the effect of radiation on free convection flow of a fluid with variable viscosity from a porous vertical plate.

The purpose of the present work is to study the effects of radiation and variable viscosity on hydromagnetic boundary layer flow along a continuously moving vertical plate with uniform suction and heat flux.

## 2- Physical model and governing equations

consider a steady two- dimensional laminar boundary layer flow of an incompressible electrically conducting viscous fluid on an infinite plate, issuing from a slot and moving vertically with uniform velocity in a fluid and heat is supplied from the plate to the fluid at a uniform rate. The x-axis is taken along the plate in the upwards direction and y- axis is normal to it. A transverse constant magnetic field  $B_0$  is applied, i.e, in the direction of y-axis. The physical model of the problem is shown in figure 1.

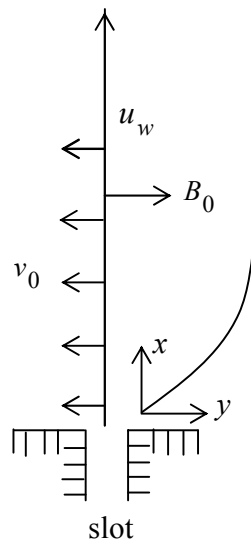


Figure 1. Physical model of the problem

The induced magnetic field is assumed to be negligible. It is also assumed that the external electric field is zero and the electric field due to polarization of charges is negligible. The viscous dissipation and Joule heating are also neglected. We further assume that property variation with temperature are limited to viscosity and with the density taken into account only in the buoyancy term in the momentum equation. Since the motion is two – dimensional and the length of the plate is large, therefore, all the physical variables are independent of  $x$ . Under the above assumptions and Boussinesq approximation the hydromagnetic flow relevant to the problem is governed by the following equations

$$\frac{dv}{dy} = 0, \tag{1}$$

$$v \frac{du}{dy} = \frac{1}{\rho_\infty} \frac{d}{dy} \left( \mu \frac{du}{dy} \right) + g\beta(T - T_\infty) - \frac{\sigma B_0^2}{\rho_\infty} u, \tag{2}$$

$$v \frac{dT}{dy} = \frac{\kappa}{\rho c_p} \frac{d^2T}{dy^2} - \frac{1}{\rho c_p} \frac{dq_r}{dy}. \tag{3}$$

The boundary conditions are

$$y=0 : u = u_w, \quad v = -v_o, \quad \frac{dT}{dy} = \frac{-q}{\kappa}, \tag{4}$$

$$y \rightarrow \infty : u \rightarrow 0, \quad T \rightarrow T_\infty.$$

From equation (1) we take

$$v = -v_o, \tag{5}$$

where  $u, v$  are the velocities along  $x, y$  coordinates, respectively.  $g$  is the acceleration due to gravity,  $\beta$  is the coefficient of thermal expansion,  $T$  is the fluid temperature,  $T_\infty$  is the ambient temperature,  $\rho$  is the density of the fluid,  $\sigma$  is the electrical conductivity,  $\rho_\infty$  is the ambient density,  $\mu$  is the fluid viscosity,  $q$

is the heat flux,  $\kappa$  is the thermal conductivity,  $c_p$  is the specific heat at constant pressure,  $q_r$  is the radiative heat flux and  $v_0$  is the normal velocity at the plate .

By using Rosseland approximation  $q_r$  takes the form [3]

$$q_r = \frac{4\sigma^*}{3k^*} \frac{dT^4}{dy}, \quad (6)$$

where  $k^*$  is the mean absorption coefficient and  $\sigma^*$  is the Stefan-Boltzmann constant. The temperature differences within the fluid assumed sufficiently small such that  $T^4$  may be expressed as a linear function of the temperature. Expanding  $T^4$  in a Taylor series about  $T_\infty$  and neglecting higher order terms, we get

$$T^4 = 4T_\infty^3 T - 3T_\infty^4. \quad (7)$$

By using equations (5), (6) and (7) then equation (3) gives

$$-v_0 \frac{dT}{dy} = \frac{\kappa}{\rho_\infty c_p} \frac{d^2 T}{dy^2} + \frac{16\sigma^* T_\infty^3}{3\rho_\infty c_p k^*} \frac{d^2 T}{dy^2}. \quad (8)$$

Introducing the following non-dimensional quantities

$$\eta = \frac{\rho_\infty v_0}{\mu_\infty} y, \quad f = \frac{u}{u_w}, \quad \theta = \frac{T - T_\infty}{\left(\frac{q\mu_\infty}{\kappa\rho_\infty v_0}\right)}, \quad (9)$$

into equations (2) and (8), one gets the following non-dimensional equations governing the flow and the energy distribution:

$$\mu f'' + \mu' f' + f' + G_r \theta - Mf = 0, \quad (10)$$

$$(1+R) \theta'' + P_r \theta' = 0. \quad (11)$$

The appropriate boundary conditions are

$$\eta=0: \quad f=1, \quad \theta' = -1,$$

(12)

$$\eta \rightarrow \infty : f \rightarrow 0, \quad \theta \rightarrow 0,$$

where the prime denotes the differentiation with respect to  $\eta$  and

$G_r = \frac{\mu_\infty g \beta}{\rho_\infty u_w v_o^2} \frac{q \mu_\infty}{\rho_\infty v_w \kappa}$  is the Grashof number,  $P_r = \frac{\mu_\infty c_p}{\kappa}$  is the Prandtl number,  $M = \frac{\sigma B_o^2 \mu_\infty}{\rho_\infty^2 v_o^2}$  is the magnetic number and  $R = \frac{16 a \sigma_* T_\infty^3}{3 \kappa k^*}$  is the radiation parameter.

The fluid viscosity  $\mu(\theta)$  was assumed to obey the Reynolds model [13]

$$\frac{\mu}{\mu_\infty} = e^{-\alpha \theta}, \tag{13}$$

where  $\alpha$ , is parameter depending on the nature of the fluid.

Using equation (13) in equation (10) we obtain

$$f'' - \alpha f' \theta' + e^{\alpha \theta} (f' - Mf + G_r \theta) = 0, \tag{14}$$

### 3. Method of solution

(i) Case of constant viscosity :

For  $\alpha = 0$ , from equation (14) we have

$$f'' + f' - Mf + G_r \theta = 0. \tag{15}$$

Solving equation (11) and (14) with the boundary conditions (12), we get

$$\theta = \frac{(1+R)}{P_r} e^{-\left(\frac{P_r}{1+R}\right)\eta}, \tag{16}$$

$$f = 1 + \frac{G_r (1+R)^3}{P_r (P_r^2 - (1+R)P_r - (1+R)^2 M)} e^{-(1+\sqrt{1+4M})\eta/2}$$

$$-\frac{G_r (1+R)^3}{P_r (P_r^2 - (1+R)P_r - (1+R)^2 M)} e^{-\left(\frac{P_r}{1+R}\right)\eta} \quad (17)$$

(ii) Variable viscosity case:

On taking into account the solution for temperature, we solved numerically the equation (14) under the boundary conditions (12) using the fourth order Runge-Kutta method algorithm with systematic guessing  $f'(0)$  by the shooting technique until the boundary condition  $f(\eta)$  at infinity decay exponentially to zero. If the boundary condition at infinity is not satisfied then the numerical routine uses the Newton-Raphson method to calculate corrections to the estimate value of  $f'(0)$ . This process is repeated iteratively until convergence is achieved to a specified accuracy,  $10^{-5}$ .

The physical quantity of most interest in such problem is the skin- friction coefficient which is defined by

$$C_f = \frac{2\left(\mu \frac{\partial u}{\partial y}\right)_{y=0}}{\rho_\infty v_0 u_w} = 2 e^{-\alpha\theta(0)} f'(0) \quad (18)$$

#### 4. Results and discussion

In order to validate our results, we have compared our numerical results with  $\alpha=0$  for  $f'(0)$  with those of analytical results. The results are found to be in a good agreement as given in table 1. For constant viscosity, i.e.  $\alpha=0$ , the numerical solutions for the velocity distribution  $f(\eta)$  are in agreement with the analytical solutions as shown in figures 2-4 for various values of  $M$ ,  $G_r$  and  $R$ , respectively. For the purpose of discussing the effect of various parameters on the flow profiles and the temperature distributions within the boundary layer. Numerical calculations have been carried out for various values of  $\alpha, M, G_r$  and



$R$  with fixed values of  $P_r$ . The value of  $P_r$  is taken to be 0.733 for air. The effect of  $\alpha$  on the dimensionless velocity  $f$  illustrated in figure 5 with  $M = 2$ ,  $R = 0.2$  and  $G_r = 5$ .

Table.1. comparison of analytical and numerical values of  $f'(0)$  with  $P_r = 0.733$

$M$	$R$	$G_r$	Analytical	Numerical
0	0.5	5	19.9385	19.9371
1	0.5	5	7.6274	7.6273
2	0.5	5	4.8732	4.8723
1	0	5	3.4309	3.4309
1	0.1	5	4.2239	4.2239
1	0.5	5	7.6274	7.6264
1	0.5	5	7.6274	7.6264
1	0.5	8	13.1747	13.1731
1	0.5	10	16.8728	16.8709

From this figure, one sees that the velocity  $f$  increases as the viscosity of air decreases. The fluid velocity increased and reached its maximum value at very short distance from the plate and then decreased to zero. The velocity  $f$  at any vertical plane near the plate decreases as the magnetic parameter  $M$  increases as shown in figure 6. It is observed that the velocity increased to its maximum value near the plate and then decreased to zero. Figures 7 and 8 show that the velocity  $f$  and the temperature  $\theta$  increases as the radiation parameter  $R$  increases. As the radiation parameter  $R$  increase the maximum value of the velocity increases. Also it is noticed that a decreases in the fluid temperature with maximum value at the plate and minimum at a distance away from the plate. The Grashof number effect on the velocity  $f$  is shown in figure 9. It is shown that the velocity  $f$

increases as the Grashof number increases. From this figure, we see that the velocity increased to its maximum value near the plate and then decreased to zero. The maximum value increased with the increasing  $G_r$ . The variation of the skin-friction coefficient  $f'(0)$  for various values of  $\alpha$ ,  $M$ ,  $R$  and  $G_r$  with  $P_r = 0.733$  is shown in table 2. It can be seen from this table that the skin-friction coefficient increases as the viscosity parameter, the radiation parameter or the Grashof number increases. Increasing of the magnetic parameter leads to a decrease in the skin-friction.

Table.2. The values of  $f'(0)$  for different with  $P_r = 0.733$

$\alpha$	$M$	$R$	$G_r$	$f'(0)$
0	1	0.5	5	7.6264
-0.2	1	0.5	5	5.3720
-0.4	1	0.5	5	3.7820
-0.2	0	0.5	5	13.2331
-0.2	1	0.5	5	5.3720
-0.2	2	0.5	5	3.4921
-0.2	1	0	5	2.6836
-0.2	1	0.1	5	3.2356
-0.2	1	0.5	5	5.3720
-0.2	1	0.5	5	5.3720
-0.2	1	0.5	8	9.3035
-0.2	1	0.5	10	11.9246

## 5-Conclusion

In this work, the problem of boundary layer flow of a steady viscous, incompressible electrically conducting fluid with variable viscosity over a continuously moving vertical porous plate in the presence of magnetic field and radiation has been investigated. The major results from this study can be summarized :

1. the velocity increases as the viscosity parameter increases, while it decreases as the magnetic parameter increases.
2. the maximum value of the velocity increases as the radiation parameter or the Grashof number increases.
3. the thermal boundary layer thickness decreases as the radiation parameter increases.
4. the skin-friction coefficient increases as the viscosity parameter, the radiation parameter or the Grashof number increases, while it decreases as the magnetic parameter increases.

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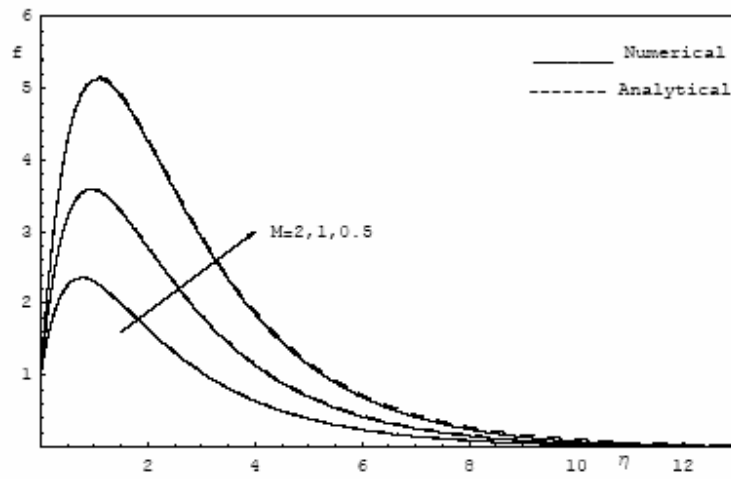


Fig .2. Velocity distribution for various values of  $M$  with  $R = 0.5$  and  $G_x = 5$

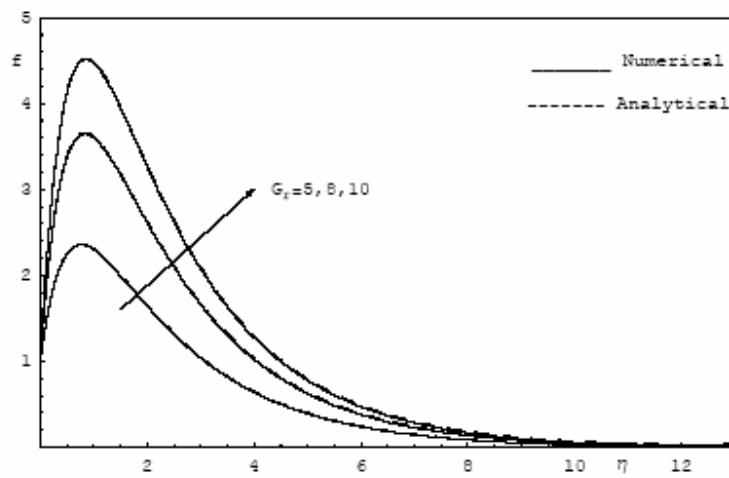


Fig .3. Velocity distribution for various values of  $G_x$  with  $R = 0.5$  and  $M = 2$

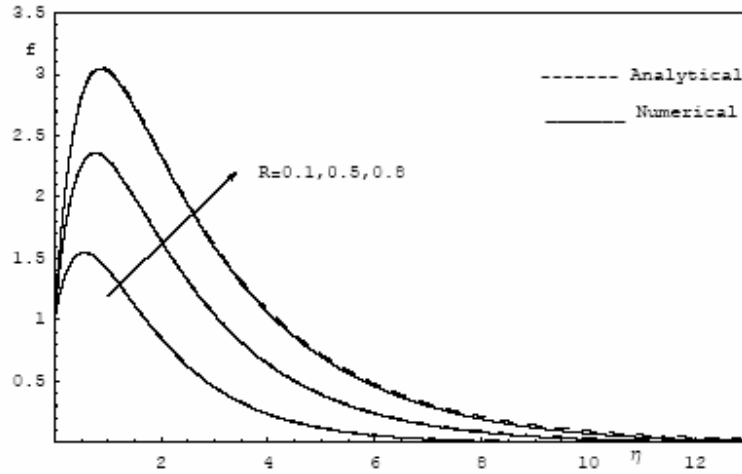


Fig .4. Velocity distribution for various values of  $R$  with  $M = 2$  and  $G_r = 5$

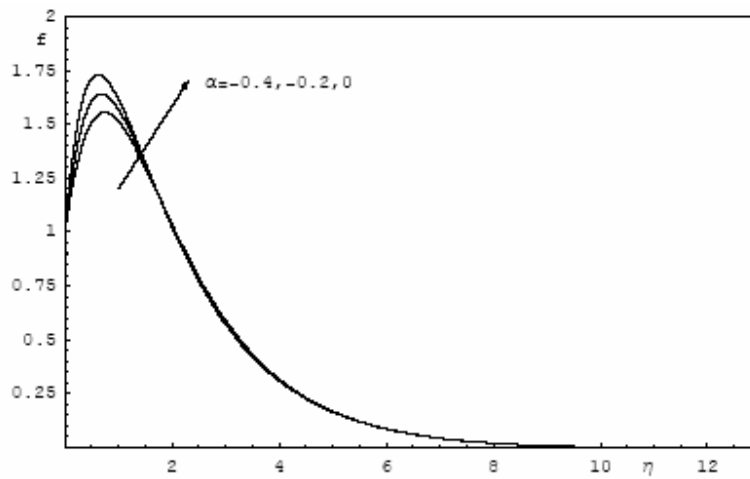


Fig .5. Velocity distribution for various values of  $\alpha$  with  $G_r = 5$ ,  $R = 0.2$  and  $M = 2$

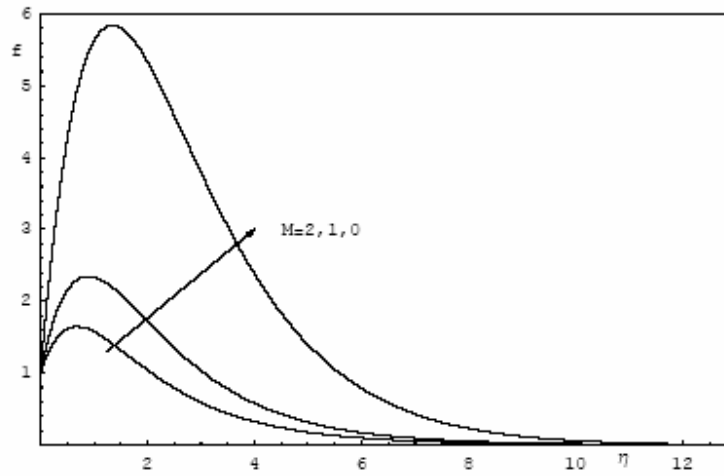


Fig .6. Velocity distribution for various values of  $M$  with  $\alpha = -0.2$ ,  $R = 0.2$  and  $G_x = 5$

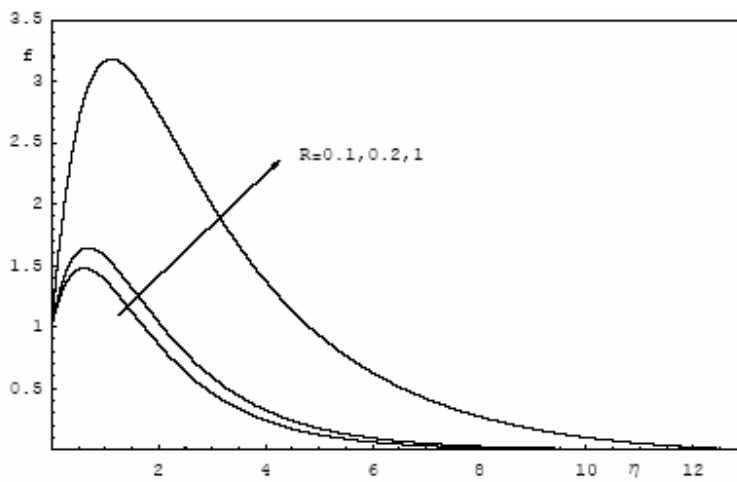


Fig .7. Velocity distribution for various values of  $R$  with  $\alpha = -0.2$ ,  $G_x = 5$  and  $M = 2$

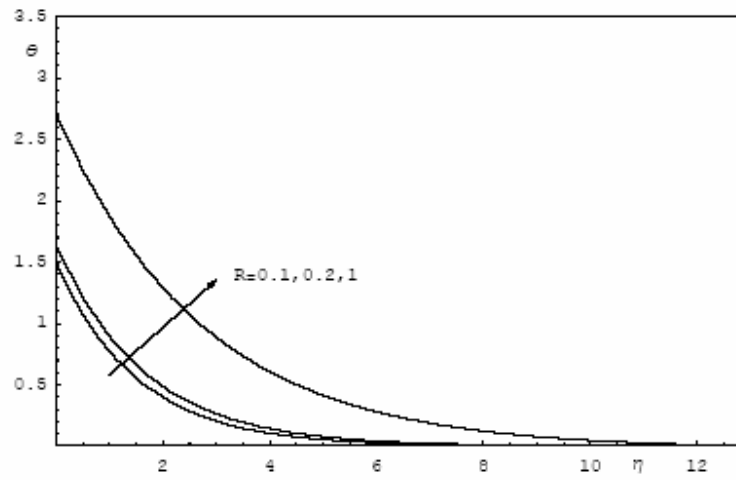


Fig .8. Temperature distribution for various values of  $R$  with  $\alpha = -0.2$  and  $G_x = 5$

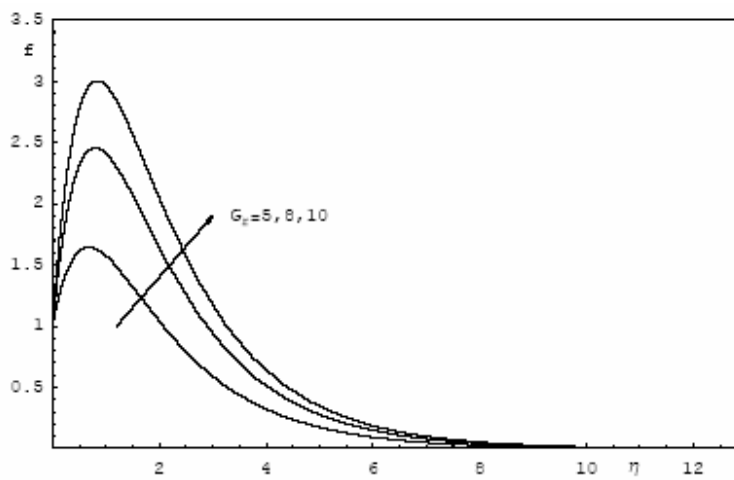


Fig .9. Velocity distribution for various values of  $G_x$  with  $\alpha = -0.2$  and  $R = 0.2$