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Fourth-Order Method for Non-Homogeneous Heat Equation with Nonlocal Boundary Conditions

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Abstract

In this paper a fourth-order numerical scheme is developed and implemented for the solution of non-homogeneous heat equation $u_t = u_{xx} + q(x,t)$ with integral boundary conditions. The results obtained show that the numerical method based on the proposed technique is fourth-order accurate as well as L-acceptable. Also the efficiency and the accuracy of the new scheme is in good agreement with the exact ones as compared to the alternative techniques existing in the literature.

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1 Introduction

In this paper we have considered the non-homogeneous heat equation in one-dimension with the non-local boundary conditions. Much attention has been paid in the literature for the development, analysis and implementation of accurate methods for the numerical solution of this typical problem.
Consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + q(x,t), \quad 0 < x < 1, \quad 0 < t \leq T \tag{1}$$

subject to the given initial condition

$$u(x,0) = f(x), \quad 0 \le x \le 1 \tag{2}$$

and the non-local boundary conditions

$$u(0,t) = \int_0^1 \phi(x,t)u(x,t)dx + g_1(t), \quad 0 < t \le T \tag{3}$$

$$u(1,t) = \int_0^1 \psi(x,t)u(x,t)dx + g_2(t), \quad 0 < t \le T \tag{4}$$

where f, g_1, g_2, ϕ, ψ and q are known functions and are assumed to be sufficiently smooth to produce a smooth classical solution of u. T is given positive constant. A number of numerical procedures are suggested in the literature to address the problem: see, for instant [2, 3, 5, 6, 9, 12].

Inspiring from great accuracy achieved in [11] the authors aim to attempt this problem. In this paper the method of lines, semi discretization approach, will be used to transform the model partial differential equation (PDE) into a system of first-order, linear, ordinary differential equations (ODEs), the solution of which satisfies a recurrence relation involving matrix exponential terms. A fourth-order rational approximation will be used to approximate exponential functions which will lead to an L-acceptable algorithm which may be parallelized through the partial fraction splitting technique.

2 DISCRETIZATION AND TREATMENT OF THE NON-LOCAL BOUNDARY CONDITIONS

Choosing a positive odd integer $N > 6$ and dividing the interval $[0,1]$ into $N+1$ subintervals each of width h, so that $h = 1/(N+1)$, and the time variable t into time steps each of length l, gives a rectangular mesh of points with coordinates $(x_m, t_n) = (mh, nl)$ where $(m = 0, 1, 2, ..., N+1$ and $n = 0, 1, 2, ...)$ covering the region $R = [0 < x < 1] \times [t > 0]$ and its boundary ∂R consisting of lines $x = 0, \quad x = 1$ and $t = 0$.

Assuming that $u(x,t)$ is six times continuously differentiable with respect to variable x and that these derivatives are uniformly bounded, the space derivative in (1) may be approximated to the fourth-order accuracy at some general point (x,t) of the mesh by using the five point central difference approximation

$$\begin{aligned} \frac{\partial^2 u(x,t)}{\partial x^2} &= \frac{1}{12h^2}\{-u(x-2h,t) + 16u(x-h,t) - 30u(x,t) + 16u(x+h,t) \\ &- u(x+2h,t)\} + \frac{h^4}{90}\frac{\partial^6 u(x,t)}{\partial x^6} + O(h^5) \quad as \quad h \to 0 \end{aligned} \tag{5}$$

also used by [1, 5, 11].

It is worth noting that the equation (5) is valid only for $(x,t) = (x_m, t_n)$ with $m = 2, 3, ..., N-1$. To attain the same accuracy at the points (x_i, t_n) for $i = 1$ and $i = N$, special formulae developed by [11] are used, which approximate $\frac{\partial^2 u(x,t)}{\partial x^2}$ not only to fourth-order but also with dominant error term $\frac{h^4}{90}\frac{\partial^6 u(x,t)}{\partial x^6}$ for $x = x_1, \quad x_N$ and $t = t_n$. Such approximations are

$$\begin{aligned} \frac{\partial^2 u(x,t)}{\partial x^2} &= \frac{1}{12h^2}\{9u(x-h,t) - 9u(x,t) - 19u(x+h,t) + 34u(x+2h,t) \\ &- 21u(x+3h,t) + 7u(x+4h,t) - u(x+5h,t)\} + \frac{h^4}{90}\frac{\partial^6 u(x,t)}{\partial x^6} \\ &+ O(h^5) \quad as \quad h \to 0 \end{aligned} \tag{6}$$

and

$$\begin{aligned} \frac{\partial^2 u(x,t)}{\partial x^2} &= \frac{1}{12h^2}\{-u(x-5h,t) + 7u(x-4h,t) - 21u(x-3h,t) + 34u(x-2h,t) \\ &- 19u(x-h,t) - 9u(x,t) + 9u(x+h,t)\} + \frac{h^4}{90}\frac{\partial^6 u(x,t)}{\partial x^6} \\ &+ O(h^5) \quad as \quad h \to 0 \end{aligned} \tag{7}$$

at the mesh points (x_1, t_n) and (x_N, t_n) respectively. Applying (1) with (5), (6) and (7) to all the interior mesh points of the grid at time level $t = t_n$ produces a system of N linear equations in $N+2$ unknowns $U_0, U_1, ..., U_{N+1}$. The integrals in (3) and (4) are approximated by using Simpson's $\frac{1}{3}$ rule as used by [6, 7, 8]. Here

$$\begin{aligned} u(0,t) &= \frac{h}{3}\{\phi(0,t)u(0,t) + 4\sum_{i=1}^{\frac{N+1}{2}} \phi((2i-1)h,t)u((2i-1)h,t) \\ &+ 2\sum_{i=1}^{\frac{N+1}{2}-1} \phi(2ih,t)u(2ih,t) + u((N+1)h,t)\} \\ &+ g_1(t) + O(h^4) \end{aligned} \tag{8}$$

and

$$\begin{aligned} u(1,t) &= \frac{h}{3}\{\psi(0,t)u(0,t) + 4\sum_{i=1}^{\frac{N+1}{2}} \psi((2i-1)h,t)u((2i-1)h,t) \\ &+ 2\sum_{i=1}^{\frac{N+1}{2}-1} \psi(2ih,t)u(2ih,t) + u((N+1)h,t)\} \\ &+ g_2(t) + O(h^4) \end{aligned} \tag{9}$$

Solving (8) and (9) for U_0 and U_{N+1} and substituting their values in the above system we have a system of N linear ordinary differential equations which can be written in vector matrix form as

$$\frac{d\mathbf{U}(t)}{dt} = A\mathbf{U}(t) + \mathbf{v(t)}, \quad t > 0 \tag{10}$$

with initial distribution

$$\mathbf{U}(0) = \mathbf{f} \tag{11}$$

in which $\mathbf{U}(t) = [U_1(t), U_2(t), ..., U_N(t)]^T$ and $\mathbf{f} = [f(x_1), f(x_2), ..., f(x_N)]^T$, where T denoting the transpose and matrix A of order $N \times N$ which is given by

$$A = \frac{1}{12h^2}\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \cdots & & \alpha_{N-1} & \alpha_N \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \cdots & & \beta_{N-1} & \beta_N \\ -1 & 16 & -30 & 16 & -1 & & & & & \\ & -1 & 16 & -30 & 16 & -1 & & & & \\ & & & -1 & 16 & -30 & 16 & -1 & & \\ & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & & & -1 & 16 & -30 & 16 & -1 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \cdots & & \gamma_{N-1} & \gamma_N \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 & \delta_6 & \cdots & & \delta_{N-1} & \delta_N \end{bmatrix}$$

where
$\alpha_1 = 9m_1 - 9,\ \alpha_2 = 9m_2 - 19,\ \alpha_3 = 9m_3 + 34,\ \alpha_4 = 9m_4 - 21,$
$\alpha_5 = 9m_5 + 7,\ \alpha_6 = 9m_6 - 1,$ and $\alpha_i = 9m_i$ for $i \geq 7$
$\beta_1 = -m_1 + 16,\ \beta_2 = -m_2 - 30,\ \beta_3 = -m_3 + 16,\ \beta_4 = -m_4 - 1,$ and $\beta_i = -m_i$ for $i \geq 5$
$\gamma_{N-3} = -n_{N-3} - 1,\ \gamma_{N-2} = -n_{N-2} + 16,\ \gamma_{N-1} = -n_{N-1} - 30,$
$\gamma_N = -n_N + 16,\ \gamma_i = -2n_i$ for $1 \leq i \leq N-4$
$\delta_{N-5} = 9n_{N-5} - 1,\ \delta_{N-4} = 9n_{N-4} + 7,\ \delta_{N-3} = 9n_{N-3} - 21,$
$\delta_{N-2} = 9n_{N-2} + 34,\ \delta_{N-1} = 9n_{N-1} - 19,\ \delta_N = 9n_N - 9,$
and $\delta_i = 9n_i\ \ for\ \ 1 \leq i \leq N-6$
in which

$$m_i = \begin{cases} \frac{4\frac{h}{3}(c_4\phi_i - c_2\psi_i)}{c_1c_4 - c_2c_3} & \text{for i=1,3,5,...N} \\ \frac{2\frac{h}{3}(c_4\phi_i - c_2\psi_i)}{c_1c_4 - c_2c_3} & \text{for i=2,4,6,...N-1} \end{cases}$$

and

$$n_i = \begin{cases} \frac{4\frac{h}{3}(c_3\phi_i - c_1\psi_i)}{c_2c_3 - c_1c_4} & \text{for i=1,3,5,...N} \\ \frac{2\frac{h}{3}(c_3\phi_i - c_1\psi_i)}{c_2c_3 - c_1c_4} & \text{for i=2,4,6,...N-1} \end{cases}$$

Here $c_1 = 1 - \frac{h}{3}\phi_0, c_2 = -\frac{h}{3}\phi_{N+1}, c_3 = -\frac{h}{3}\psi_0$ and $c_4 = 1 - \frac{h}{3}\psi_{N+1}$, also $\phi_i = \phi(ih, t)$ and $\psi_i = \psi(ih, t)$. The column vector $\mathbf{v}(t)$ contains the contribution from the functions $q(x, t), g_1(t)$ and $g_2(t)$ and is given as

$$\mathbf{v}(t) = [\frac{9l_1}{12h^2} + q_1, \frac{-l_1}{12h^2} + q_2, q_3, ..., q_{N-2}, \frac{-l_2}{12h^2} + q_{N-1}, \frac{9l_2}{12h^2} + q_N]^T \tag{12}$$

where $l_1 = \frac{c_4 g_1(t) - c_2 g_2(t)}{c_1 c_4 - c_2 c_3}$ and $l_2 = \frac{c_1 g_2(t) - c_3 g_1(t)}{c_1 c_4 - c_2 c_3}$
The solution of the system (10) subject to (11) is given by

$$\mathbf{U}(t) = exp(tA)\mathbf{f} + \int_0^t exp[(t-s)A]\mathbf{v(s)}ds \tag{13}$$

[11] which satisfies the recurrence relation

$$\mathbf{U}(t+l) = exp(lA)\mathbf{U}(t) + \int_t^{t+l} exp[(t+l-s)A]\mathbf{v(s)}ds, \quad t = 0, l, 2l, ... \tag{14}$$

. Eigenvalues of the matrix A are calculated using MATLAB 5.3 for $N = 9, 19, 39, 79$ and it is observed that they are distinct negative real ones or complex with negative real parts.

To approximate the matrix exponential function in (14) following [11] we use a rational approximation consisting of three parameters a_1, a_2, a_3 and a real scalar θ given by

$$E_4(\theta) = \frac{1 + (1-a_1)\theta + (\frac{1}{2} - a_1 + a_2)\theta^2 + (\frac{1}{6} - \frac{a_1}{2} + a_2 - a_3)\theta^3}{1 - a_1\theta + a_2\theta^2 - a_3\theta^3 + (-\frac{1}{24} + \frac{a_1}{6} - \frac{a_2}{2} + a_3)\theta^4} = \frac{p(\theta)}{q(\theta)} \tag{15}$$

with error constant $C = \frac{1}{30} - \frac{1}{8}a_1 + \frac{1}{3}a_2 - \frac{1}{2}a_3$. Stability of the method is guaranteed by [11].
So we have

$$exp(lA) = G^{-1}(I + (1-a_1)lA + (\frac{1}{2} - a_1 + a_2)l^2A^2 + (\frac{1}{6} - \frac{a_1}{a_2} + a_2 - a_3)l^3A^3) \tag{16}$$

where

$$G = I - a_1 lA + a_2 l^2A^2 - a_3 l^3 A^3 + (-\frac{1}{24} + \frac{a_1}{6} - \frac{a_2}{2} + a_3)l^4A^4 \tag{17}$$

The quadrature term in (14) is approximated by

$$\int_t^{t+l} exp[(t+l-s)A]\mathbf{v}(s)ds = W_1\mathbf{v}(s_1) + W_2\mathbf{v}(s_2) + W_3\mathbf{v}(s_3) + W_4\mathbf{v}(s_4) \tag{18}$$

where $s_1 = t, s_2 = t + l/3, s_3 = t + 2l/3, s_4 = t + l$ and W_1, W_2, W_3 and W_4 are matrices given by [11] and are mentioned here for convenience.

$$W_1 = \frac{l}{24}\{3I - (19 - 78a_1 + 216a_2 - 324a_3)lA + (3 - 8a_1 + 12a_2)l^2A^2\}G^{-1} \tag{19}$$

$$W_2 = \frac{3l}{16}\{2I + (16 - 56a_1 + 144a_2 - 216a_3)lA + (1 - 4a_1 + 12a_2 - 24a_3)l^2A^2\}G^{-1} \tag{20}$$

$$W_3 = \frac{3l}{8}\{I - (7 - 26a_1 + 72a_2 - 108a_3)lA - (1 - 4a_1 + 12a_2 - 24a_3)l^2A^2\}G^{-1} \tag{21}$$

$$
\begin{aligned}
W_4 &= \frac{l}{48}\{6I + (44 - 168a_1 + 432a_2 - 648a_3)lA + (11 - 44a_1 + 132a_2 \\
&- 216a_3)l^2A^2 + (2 - 8a_1 + 24a_2 - 48a_3)l^3A^3\}G^{-1} \qquad (22)
\end{aligned}
$$

Hence

$$
\mathbf{U}(t+l) = exp(lA)\mathbf{U}(t) + W_1\mathbf{v}(t) + W_2\mathbf{v}(t+l/3) + W_3\mathbf{v}(t+2l/3) + W_4\mathbf{v}(t+l) \qquad (23)
$$

3 NUMERICAL EXPERIMENTS

In this section the numerical method described in this paper will be applied to four problems from the literature and results obtained will be compared with exact solutions as well as with the results existing in the literature. Following [11] we have chosen here $a_1 = 64/25, a_2 = 7/3$ and $a_3 = 547/600$ which fulfil all the requirements.

EXAMPLE (1):- Consider the problem (1)-(4) with
$f(x) = x^2, \quad 0 < x < 1,$
$g_1(t) = \frac{-1}{4(t+1)^2}, \quad 0 < t < 1,$
$g_2(t) = \frac{3}{4(t+1)^2}, \quad 0 < t < 1,$
$\phi(x,t) = x, \quad 0 < x < 1,$
$\psi(x,t) = x, \quad 0 < x < 1,$
$q(x,t) = \frac{-2(x^2+t+1)}{(t+1)^3}, \quad 0 < t \leq 1, \quad 0 < x < 1$
which has theoretical solution $u(x,t) = (\frac{x}{t+1})^2$ [7].

The problem is solved using the scheme developed in this paper for $h = l = 0.05, 0.025, 0.01, 0.005, 0.0025, 0.001$ at $x = 0.6$ and $t = 1$. The relative errors obtained by the new scheme are given in Table 1 and the results are compared with different schemes, BTCS implicit scheme, Crandall method, FTCS scheme and Dufort-Frankel scheme given by [7]. From the table we can see that the results of the new scheme are far better than those of the schemes given in [7].

EXAMPLE (2):-Now consider the problem (1)-(4) with
$f(x) = exp(-x), \quad 0 < x < 1,$
$\phi(x,t) = ax, \quad 0 < x < 1,$
$\psi(x,t) = bcos(x), \quad 0 < x < 1,$
$g_1(t) = 0, \quad 0 < t < 1,$
$g_2(t) = 0, \quad 0 < t < 1,$
where $a = e/(e-2)$ and $b = 2/(sin(1) - cos(1) + e)$
and $q(x,t) = -exp[-(x + sint)](1 + cost), \quad 0 < t \leq 1, \quad 0 < x < 1$
which has theoretical solution $u(x,t) = exp(-(x + sint))$

For Example (2) results are given in Table 2 and Table 3. In Table 2 the results are computed for $h = l = 0.05, 0.025, 0.01, 0.005, 0.0025, 0.001$ at

$x = 0.6$ and $t = 0.1$. The relative errors developed in the scheme are compared with different schemes, BTCS implicit scheme, Crandall method, FTCS scheme and Dufort-Frankel scheme given by [7]. From the table it is clear that the results are in good agreement as compared with the exact ones as well as better than other schemes. Moreover the new scheme is fourth-order accurate except for very small values of h and l when accumulating error is high.

The Example (2) is also solved for $h = 0.01 = l$ for different values of t at $x = 0.25$ and the results are tabulated in Table 3. Table 3 shows that the scheme developed in this paper gives superior results to those computed by using the Crank-Nicolson finite-difference method [12], the implicit finite-difference technique and the parallel techniques [6]. The parallel technique developed in [6] is second-order accurate while the parallel technique developed in this paper is fourth-order accurate.

EXAMPLE (3):- Once again Consider the problem (1)-(4) with
$f(x) = sin(\pi x) + cos(\pi x), \quad 0 < x < 1,$
$\phi(x, t) = 2sin(\pi x), \quad 0 < x < 1,$
$\psi(x, t) = -2cos(\pi x), \quad 0 < x < 1,$
$g_1(t) = 0, \quad 0 < t < 1,$
$g_2(t) = 0, \quad 0 < t < 1,$
and $q(x, t) = (\pi^2 - 1)exp(-t)\{sin(\pi x) + cos(\pi x)\}, \quad 0 < t \leq 1, \quad 0 < x < 1$
which has theoretical solution
$u(x, t) = exp(-t)\{sin(\pi x) + cos(\pi x)\}$ [6].

In this problem the results are computed for $h = l = 0.01$ for different values of t at $x = 0.25$ and the results are presented in Table 4. Table 4 shows that the scheme developed in this paper gives superior results to other schemes namely, the Crank-Nicolson finite-difference method [12], the implicit finite-difference technique and the parallel techniques [6] .

EXAMPLE (4):- Consider the problem (1)-(4) with
$f(x) = x(x - 1) + \delta/6(1 + \delta), \quad 0 < x < 1,$
$\phi(x, t) = -\delta, \quad 0 < x < 1,$
$\psi(x, t) = -\delta, \quad 0 < x < 1,$
$g_1(t) = 0, \quad 0 < t < 1,$
$g_2(t) = 0, \quad 0 < t < 1,$
which has theoretical solution
$u(x, t) = [x(x - 1) + \delta/6(1 + \delta)]exp(-t)$ where $\delta = 0.0144$ [4, 9]

In Example (4) results computed are given in Table 5 and Table 6. In Table 5 results are calculated for $h = 0.01 = l$ at $x = 1$ and for different values of t. From the table it is clear that the analytical solution calculated by using the scheme developed in this paper is good agreement with the exact ones. Also the solution converges towards exact solution as t increases.

In Table 6 results are given for $t = 1$ with $h = l = 0.1, 0.05, 0.025, 0.0125$ and 0.00625 at $x = 0.5$ and $x = 1$. It is clear from the table that results

Table 1: Relative errors at various spatial lengths at $t = 1$ for Example (1)

Spatial length	BTCS	Crandall	FTCS	Dufort-Frankel	New scheme
h=0.0500	7.3×10^{-02}	3.8×10^{-03}	7.5×10^{-02}	7.8×10^{-02}	2.6×10^{-06}
h=0.0250	1.8×10^{-02}	2.1×10^{-04}	1.9×10^{-02}	1.9×10^{-02}	2.1×10^{-07}
h=0.0100	4.4×10^{-03}	1.2×10^{-05}	4.0×10^{-03}	3.9×10^{-03}	6.1×10^{-09}
h=0.0050	1.2×10^{-03}	7.1×10^{-07}	1.0×10^{-03}	1.0×10^{-03}	3.5×10^{-10}
h=0.0025	3.0×10^{-04}	4.3×10^{-08}	2.5×10^{-04}	2.4×10^{-04}	8.0×10^{-11}
h=0.0010	7.5×10^{-05}	2.5×10^{-09}	6.1×10^{-05}	6.0×10^{-05}	1.1×10^{-11}

Table 2: Relative errors at various spatial lengths at $t = 0.1$ for Example (2)

Spatial length	BTCS	Crandall	FTCS	Dufort-Frankel	New scheme
h=0.0500	6.3×10^{-02}	3.9×10^{-03}	6.4×10^{-02}	6.8×10^{-02}	3.0×10^{-07}
h=0.0250	1.5×10^{-02}	2.4×10^{-04}	1.6×10^{-02}	1.7×10^{-02}	1.9×10^{-08}
h=0.0100	4.0×10^{-03}	1.5×10^{-05}	4.1×10^{-03}	4.1×10^{-03}	5.0×10^{-10}
h=0.0050	1.0×10^{-03}	1.0×10^{-06}	1.0×10^{-03}	1.0×10^{-03}	7.9×10^{-12}
h=0.0025	2.4×10^{-04}	6.4×10^{-08}	2.5×10^{-04}	2.6×10^{-04}	7.0×10^{-11}
h=0.0010	6.1×10^{-05}	4.0×10^{-09}	4.0×10^{-05}	3.9×10^{-05}	1.3×10^{-10}

Table 3: Results for u at different values of t for Example (2)

t	Exact u	Error			
		Crank-Nicolson	The implicit	The parallel	New sheme
0.1	.7048055	6.0×10^{-05}	5.2×10^{-05}	3.8×10^{-06}	5.3×10^{-10}
0.2	.6384772	5.2×10^{-05}	4.1×10^{-05}	3.7×10^{-06}	9.7×10^{-10}
0.3	.5795403	9.7×10^{-05}	7.1×10^{-05}	4.6×10^{-06}	1.4×10^{-09}
0.4	.5275993	8.0×10^{-05}	6.5×10^{-05}	5.5×10^{-06}	1.8×10^{-09}
0.5	.4821859	1.2×10^{-05}	8.9×10^{-05}	2.3×10^{-06}	2.3×10^{-09}
0.6	.4427977	1.1×10^{-05}	9.8×10^{-05}	1.0×10^{-06}	2.7×10^{-09}
0.7	.4089274	2.5×10^{-05}	1.4×10^{-05}	1.1×10^{-06}	3.2×10^{-09}
0.8	.3800867	3.8×10^{-05}	2.6×10^{-05}	1.0×10^{-06}	3.7×10^{-09}
0.9	.3558213	5.8×10^{-05}	4.4×10^{-05}	2.1×10^{-06}	4.3×10^{-09}
1.0	.3357223	7.1×10^{-05}	6.4×10^{-05}	1.9×10^{-06}	4.9×10^{-09}

for $x = 1$ are far better than those at $x = 0.5$ also the method is third-order accurate. CPU time taken for the new scheme developed in this paper is also given in the table which shows that the new scheme is very fast.

This problem is also solved by [2] with $\phi(x,t) = \frac{12}{13}$ and $\psi(x,t) = \frac{12}{13}$. It is noted that the increase in the value of δ causes more accuracy.

4 CONCLUSION

It is observed that the results obtained using new scheme are highly accurate as compared to those of other schemes and the method developed is fourth-order accurate in space and time as well as L-acceptable. This technique can be coded easily on serial or parallel computers.

It is worth mentioning that the method using real arithmetic and multiprocessor architecture will save CPU time remarkably, rather than the complex arithmetic based methods.

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Table 4: Results for u at different values of t for Example (3)

t	Exact u	Error			
		Crank-Nicolson	The implicit	The parallel	New sheme
0.1	1.2796330	5.2×10^{-05}	4.3×10^{-05}	4.8×10^{-06}	8.8×10^{-09}
0.2	1.1578600	6.2×10^{-05}	6.0×10^{-05}	4.7×10^{-06}	1.1×10^{-08}
0.3	1.0476750	6.5×10^{-05}	6.4×10^{-05}	3.9×10^{-06}	1.1×10^{-08}
0.4	0.9479756	6.4×10^{-05}	6.3×10^{-05}	4.8×10^{-06}	1.1×10^{-08}
0.5	0.8577639	6.2×10^{-05}	5.9×10^{-05}	5.3×10^{-06}	9.9×10^{-09}
0.6	0.7761369	5.6×10^{-05}	4.8×10^{-05}	3.7×10^{-06}	9.1×10^{-09}
0.7	0.7022777	5.0×10^{-05}	4.9×10^{-05}	2.3×10^{-06}	8.2×10^{-09}
0.8	0.6354471	1.6×10^{-05}	1.5×10^{-05}	1.6×10^{-06}	7.4×10^{-09}
0.9	0.5749763	4.1×10^{-05}	3.3×10^{-05}	1.1×10^{-06}	6.7×10^{-09}
1.0	0.5202601	5.0×10^{-05}	4.7×10^{-05}	1.0×10^{-06}	6.1×10^{-09}

Table 5: Results for $h = 0.01$ at $x = 1$ for Example (4)

t	Exact Solution	Numerical Solution	Absolute Error
0.1	-0.00681710790377	-0.00681710790268	1.09×10^{-12}
0.2	-0.00616837431412	-0.00616837431277	1.35×10^{-12}
0.3	-0.00558137588787	-0.00558137588651	1.35×10^{-12}
0.4	-0.00505023774747	-0.00505023774619	1.28×10^{-12}
0.5	-0.00456964408389	-0.00456964408271	1.18×10^{-12}
0.6	-0.00413478495421	-0.00413478495314	1.07×10^{-12}
0.7	-0.00374130814210	-0.00374130814213	9.71×10^{-13}
0.8	-0.00338527559937	-0.00338527559949	8.80×10^{-13}
0.9	-0.00306312403268	-0.00306312403188	7.96×10^{-13}
1.0	-0.00277162924085	-0.00277162924013	7.20×10^{-13}

Table 6: Results for different spatial lengths at $t = 1$ for Example (4)

Spatial length h	Absolute Errors at $x = 0.5$	Absolute Errors at $x = 1.0$	CPU Time in seconds
0.100000	3.01×10^{-08}	9.96×10^{-08}	0.0310
0.050000	1.61×10^{-09}	1.07×10^{-08}	0.0930
0.025000	6.74×10^{-11}	9.44×10^{-10}	0.3590
0.012500	2.28×10^{-12}	7.16×10^{-11}	1.6410
0.006250	5.06×10^{-14}	4.38×10^{-12}	10.078
0.003125	3.28×10^{-15}	9.01×10^{-13}	96.625

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