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Fuzzy Congruences on Fuzzy Algebras

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Abstract

A notion of congruence on fuzzy algebras is introduced and the isomorphism theorem is established.

1 Introduction

A basic problem in fuzzifying algebras is to find out a notion of congruence so that the Isomorphism Theorem in classical Universal Algebra is valid.

It is the scope of this short paper to introduce such an appropriate notion. We start with the definition of a non-deterministic (ND) algebra.

A *ranked alphabet* is a pair $(\Gamma, rank)$ consisting of a finite set Γ equipped with a function $rank : \Gamma \to \mathbb{N}$ (natural numbers). The set $\Gamma_k = rank^{-1}(k)$, $k \in \mathbb{N}$ is the set of functional symbols of rank k.

A non-deterministic Γ-algebra is a structure $\mathcal{A} = (A, \alpha)$ where A is a set (the carrier of $\mathcal{A}$) and

$$\alpha = (\alpha_f : A^k \to \mathcal{P}(A))_{f \in \Gamma_k, k \geqslant 0}$$

is a set of multifunctions called *structural operations* of $\mathcal{A}$.

An equivalence $\sim$ on A is a *congruence* on $\mathcal{A}$ whenever for all $q_i, q_i' (1 \leqslant i \leqslant k)$ and $f \in \Gamma_k$,

$$q_1 \sim q_1' \ , \ \dots \ , q_k \sim q_k' \ \text{ implies}$$
$$\alpha_f(q_1, ..., q_k) \cap [q] \neq \emptyset \ \text{ iff } \ \alpha_f(q_1', ..., q_k') \cap [q] \neq \emptyset$$

for all $\sim$-classes $[q] \in {}^{A}/_{\sim}$.

The quotient set ${}^{A}/_{\sim}$ is converted into a non-deterministic Γ-algebra by defining its structural operations

$$(\alpha_\sim)_f : ({}^{A}/_{\sim})^k \to \mathcal{P}\left({}^{A}/_{\sim}\right) \qquad , f \in \Gamma_k \ , \ k \geqslant 0$$

by the formula

$$(\alpha_\sim)_f([q_1], ..., [q_k]) = \{[q] / \alpha_f(q_1, ..., q_k) \cap [q] \neq \emptyset\}.$$

The definition is clearly independent from the representatives used. It is the above notion of congruence we are going to fuzzify.

Throughout this note $\vartriangle$ denotes a t-norm distributive over a t-conorm $\triangledown$ on $[0,1]$ (cf. [2]). For an arbitrary index set I and any family $a_i \in [0,1]$, $i \in I$ we set

$$\bigtriangledown_{i \in I} a_i = \bigtriangledown_{i \in I'} a_i$$

the supremum running over all finite subsets I' of I. Obviously $a \vartriangle (\bigtriangledown_{i \in I} a_i) = \bigtriangledown_{i \in I} a \vartriangle a_i$.

2 Fuzzy Congruences

A *fuzzy* Γ-*algebra* is a pair $\mathcal{A} = (A, \alpha)$ formed by a set A and a family of structural operations

$$\alpha = (\alpha_f : A^k \to Fuzzy(A))_{f \in \Gamma_k, k \geqslant 0}$$

where $Fuzzy(A)$ denotes the set of all fuzzy subsets of A.

We extend α_f into a function

$$\overline{\alpha}_f : Fuzzy(A)^k \to Fuzzy(A)$$

by setting

$$\overline{\alpha}_f(\varphi_1, ..., \varphi_k) = \bigtriangledown_{q_1,\dots,q_k \in A} \varphi_1(q_1) \vartriangle ... \vartriangle \varphi_k(q_k) \vartriangle \alpha_f(q_1, ..., q_k).$$

The *underlying* non deterministic Γ-algebra $U(\mathcal{A}) = (A, U(\alpha))$ is given by

$$U(\alpha)_f(q_1, ..., q_k) = supp\, \alpha_f(q_1, ..., q_k)$$

supp(φ) standing for the *support* of the fuzzy set $\varphi : A \to [0,1]$

$$supp(\varphi) = \{q/q \in A\ ,\ \varphi(q) \neq 0\}.$$

Given fuzzy Γ-algebras $\mathcal{A} = (A, \alpha)$ and $\mathcal{B} = (B, \beta)$ any function $h : A \to B$ commuting with structural operations

$$\overline{h}(\alpha_f(q_1, ..., q_k)) = \beta_f(h(q_1), ..., h(q_k))$$

is called a *morphism* of fuzzy Γ-algebras.

Here $\overline{h} : Fuzzy(A) \to Fuzzy(B)$ is the linear extension of h, i.e.

$$\overline{h}(\varphi) = \bigtriangledown_{q \in A} \varphi(q) \vartriangle h(q).$$

An equivalence relation $\sim$ on the set A is a *fuzzy congruence* on $\mathcal{A}$, if

$$q_1 \sim q_1' \ , \ \dots \ , \ q_k \sim q_k' \ \text{ and } \ f \in \Gamma_k \ , \ k \geqslant 1$$

implies

$$\mathop{\triangledown}_{r \in [q]} \alpha_f(q_1, ..., q_k)(r) = \mathop{\triangledown}_{r' \in [q]} \alpha_f(q_1', ..., q_k')(r')$$

for any $\sim$-class $[q] \in {}^{A}/_{\sim}$.

The quotient set ${}^{A}/_{\sim}$ can be converted into a fuzzy Γ-algebra if the structural operation

$$(\alpha_\sim)_f : \left({}^{A}/_{\sim}\right)^k \to Fuzzy\left({}^{A}/_{\sim}\right) \ , \ f \in \Gamma_k \ , \ k \geqslant 0$$

is defined by

$$(\alpha_\sim)_f([q_1], ..., [q_k])([q]) = \mathop{\triangledown}_{r \in [q]} \alpha_f(q_1, ..., q_k)(r).$$

This formula is consistent by the definition of fuzzy congruence. Moreover the canonical function

$$h_\sim : A \to {}^{A}/_{\sim} \quad , \quad h_\sim(q) = [q]$$

is a morphism of fuzzy Γ-algebras since for all $q_1, ..., q_k \in A$ and $f \in \Gamma_k \ , \ k \geqslant 0$ we have

$$\begin{aligned}\overline{h}_\sim(\alpha_f(q_1, ..., q_k))([q]) &= \mathop{\triangledown}_{h_\sim(r) = [q]} \alpha_f(q_1, ..., q_k)(r) \\ &= \mathop{\triangledown}_{r \in [q]} \alpha_f(q_1, ..., q_k)(r) \\ &= (\alpha_\sim)_f([q_1], ..., [q_k])([q])\end{aligned}$$

for all $[q] \in {}^{A}/_{\sim}$. Thus

$$\overline{h}_\sim(\alpha_f(q_1, ..., q_k)) = (\alpha_\sim)_f(h_\sim(q_1), ..., h_\sim(q_k)).$$

Fact. *If $\sim$ is a fuzzy congruence on $\mathcal{A} = (A, \alpha)$ then $\sim$ is a congruence on its underlying non-deterministic Γ-algebra $U(\mathcal{A}) = (A, U(\alpha))$.*

For this, we have to show that for any $\sim$-class $[q]$, it holds

$$supp\, \alpha_f(q_1, ..., q_k) \cap [q] \neq \emptyset \ \text{ iff } \ supp\, \alpha_f(q_1', ..., q_k') \cap [q] \neq \emptyset$$

provided that $q_i \sim q_i'$ $(1 \leqslant k \leqslant k)$ and $f \in \Gamma_k \ , \ k \geqslant 1$.

Indeed, if $r \in [q]$ is such that $\alpha_f(q_1, ..., q_k)(r) \neq 0$, then

$$\mathop{\triangledown}_{r' \in [q]} \alpha_f(q'_1, ..., q'_k)(r') = \mathop{\triangledown}_{r \in [q]} \alpha_f(q_1, ..., q_k)(r) \neq 0$$

so that for some $r' \in [q]$ we have $\alpha_f(q'_1, ..., q'_k)(r') \neq 0$, as wanted. □

Our next task will be to establish the well known, in Universal Algebra, Isomorphism Theorem (cf. [1]).

Theorem 1. *Let $h : \mathcal{A} \to \mathcal{B}$ be an epimorphism of fuzzy Γ-algebras. Then the kernel $\sim_h$ of h defined by*

$$q_1, q_2 \in A \qquad , \qquad q_1 \sim_h q_2 \ \textit{ iff } \ h(q_1) = h(q_2)$$

is a fuzzy congruence on $\mathcal{A}$.

Moreover

$$A/_{\sim_h} \xrightarrow{\sim} \mathcal{B}.$$

Proof. Assume that

$$q_1 \sim_h q'_1 \ , \ ... \ , \ q_k \sim_h q'_k \ \text{ and } \ f \in \Gamma_k \ , \ k \geqslant 1$$

that is

$$h(q_1) = h(q'_1) \ , \ ... \ , \ h(q_k) = h(q'_k).$$

Then

$$\begin{aligned} \overline{h}(\alpha_f(q_1, ..., q_k)) &= \beta_f(h(q_1), ..., h(q_k)) \\ &= \beta_f(h(q'_1), ..., h(q'_k)) \\ &= \overline{h}(\alpha_f(q'_1, ..., q'_k)). \end{aligned}$$

Thus, for every $\sim_h$-class $[q]$ we have

$$\overline{h}(\alpha_f(q_1, ..., q_k))([q]) = \overline{h}(\alpha_f(q'_1, ..., q'_k))([q])$$

or

$$\mathop{\triangledown}_{h(r)=[q]} \alpha_f(q_1, ..., q_k)(r) = \mathop{\triangledown}_{h(r')=[q]} \alpha_f(q'_1, ..., q'_k)(r')$$

or

$$\mathop{\triangledown}_{r \in [q]} \alpha_f(q_1, ..., q_k)(r) = \mathop{\triangledown}_{r' \in [q]} \alpha_f(q'_1, ..., q'_k)(r')$$

and thus $\sim_h$ is a fuzzy congruence.

Now the function

$$h_1 : {}^{A}/_{\sim_h} \to B \qquad , \qquad h_1([q]) = h(q)$$

is clearly a well defined bijection. It is also a morphism of fuzzy algebras. Indeed we have

$$\begin{aligned}
\overline{h}_1((\alpha_{\sim_h})_f([q_1], ..., [q_k]))(b) &= \bigtriangledown_{h_1([q])=b} (\alpha_{\sim_h})_f([q_1], ..., [q_k])([q]) \\
&\overset{(\star)}{=} (\alpha_{\sim_h})_f([q_1], ..., [q_k])([q]) \\
&= \bigtriangledown_{h(r)=b} \alpha_f(q_1, ..., q_k)(r) \\
&= \overline{h}(\alpha_f(q_1, ..., q_k)) \\
&= \beta_f(h(q_1), ..., h(q_k)) \\
&= \beta_f(h_1([q_1]), ..., h_1([q_k]))
\end{aligned}$$

where $(\star)$ holds because h_1 is a bijection.

The proof of our theorem is completed. □

Corollary 1. *Let $h : \mathcal{A} \to \mathcal{B}$ be an epimorphism of non-deterministic Γ-algebras. Then $\sim_h$ is a congruence on $\mathcal{A}$ and moreover*

$$^{A}/_{\sim_h} \xrightarrow{\sim} \mathcal{B}.$$

References

[1] P. Cohn, *Universal Algebra*, D. Reidel, Dord Recht, 1981

[2] G. S. Klir, B. Yuan, *Fuzzy Sets and Fuzzy Logic Theory and Applications*, Prentice Hall Upper Saddle River, N.J., 1995

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