The Use of Variational Iteration Method and Homotopy Perturbation Method for Painlevé Equation I

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Abstract

In this study, Variational Iteration Method (VIM) and Homotopy Perturbation Method (HPM) are employed to approximate solutions of Ppainlevé equation I, with it's initial conditions. VIM based on Lagrangian multipliers and HPM based on an embedding parameter. Both of these methods have been introduced by He[11-17] to solve approximately differential equations. In this paper we construct approximate polynomials to find approximate solutions of Painlevé equation I. Numerical comparsions are made between VIM and HPM and maple numerical results.

1   Introduction

The Painlevé equations and their solutions arise in parts of pure and applied mathematics and theoretical physics. Painlevé considered wide class of second order equations and classified them to the nature of singularities.Painlevé and

his coworkers found essentially six different equations within the class considered whose solutions are single valued as functions of complex independent variables, except possibly at the fixed singularities of the coefficients. These are known as Painlevé transcendents and have a great variety of interesting properties and applications. Painlevé equations have been investigated by number of researchers using of several techniques. In this work we are going to study Painlevé equation I which is well known as

$$u'' = 6u^2 + x, \tag{1.1}$$

with the initial conditions

$$u(0) = 0, \qquad u'(0) = 1, \tag{1.2}$$

by using of Variational Iteration Method (VIM) and Homotopy Perturbation Method (HPM). VIM and HPM have been introduced by He [7-18]. These methods can be applied successfully to various types of ordinary and partial differential equations. He [11-15] developed VIM for solving linear and nonlinear problems, which arise in different branches of pure and applied sciences. Also, He [8-10,17] introduced HPM, which is developed by combinig the standard homotopy and perturbation method. In these methods the solution is given in an infinite series usually convergent to an accurate solution. We apply these methods to equation (1.1) which has singularities according to initial conditions.

2　Variational Iteration Method (VIM)

To explain the basic idea of variational iteration method, consider the following nonlinear differential equation

$$L(u) + N(u) = g(x), \tag{2.1}$$

where L is a linear operator, N is a nonlinear operator, and g(x) is an inhomogeneous term. According to the variational iteration method, we can construct a correction functional as follows

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi) \ [Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)] \, d\xi, \tag{2.2}$$

where $\lambda(\xi)$ is a general Lagrangian multiplier which can be identified optimally via the variational iteration method. The subscripts n denote the nth approximation,$\tilde{u}$ is considered as a restricted variation,i.e. $\delta \tilde{u}_n = 0$. (2.2) is called the correct functional. Employing the restricted variation in equation

(2.2) makes it easy to compute the Lagrange multiplier. The principles of variational iteration method and its applicability for various kinds of differential equations are given in [2,3,4,11]. Under reasonable choice of u_0, the fixed point of the correction functional (2.2) is considered as an approximate solution of (2.1). The solutions of the linear equations can be obtained by only a single iteration due to the exact identification of the Lagrangian multiplier. The successive approximation u_{n+1} of the solution u will be readily obtained upon the determined Lagrange multiplier and any selective function u_0, consequently, the solution is given by $u = \lim_{n \to \infty} u_n$.

3 Homotopy Perturbation Method (HPM)

To illustrate the homotopy perturbation method, we consider a general equation of the type

$$A(u) - f(r) = 0, \qquad r \in \Omega \qquad (3.1)$$

with the boundary conditions

$$B(u, \frac{\partial u}{\partial n}) = 0, \qquad r \in \Gamma \qquad (3.2)$$

where A is a general differential operator, B a boundary operator, Γ the boundary of the domain Ω and $f(r)$ is a known analytical function. Generally speaking, the operator A can be divided into a linear part L and a nonlinear part N. Now equation (3.1) can be rewritten as:

$$L(u) + N(u) - f(r) = 0. \qquad (3.3)$$

By the homotopy perturbation method, we construct a homotopy as $v(r, p) : \Omega \times [0, 1] \to R$ which satisfies

$$H(v, p) = (1 - p) \left[L(v) - L(u_0) \right] + p \left[A(v) - f(r) \right] = 0, \qquad (3.4)$$

where $p \in [0, 1]$ is an embedding parameter and u_0 is an initial approximation of equation (3.1) which satisfies the boundary conditions. Considering equation (3.4) we will have

$$H(v, 0) = L(v) - L(u_0) = 0, \qquad (3.5)$$

$$H(v, 1) = A(v) - f(r) = 0. \qquad (3.6)$$

The changing process of p from zero to unity is just that of $v(r, p)$ from $u_0(r)$ to $u(r)$. In topology this is called deformation and $L(v) - L(u_0)$ and $A(v) - f(r)$

are called homotopy.According to the homotopy perturbation theory, we can first use the embedding parameter p as a small parameter and assume that vthe solution of equation (3.4) can be written as a power series in p :

$$v = v_0 + pv_1 + p^2v_2 + \cdots \qquad (3.7)$$

setting $p = 1$ one have the approximation solution of equation (3.1) as the following

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots \qquad (3.8)$$

The (3.8) is convergent for most cases. However, the convergent rate deponds on the nonlinear operator $A(v)$.

4 VIM for Painlevé Equation

Using the VIM for equation (1.1), we have the functional

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi) \left[\frac{d^2}{d\xi^2} u_n(\xi) - 6\tilde{u}_n^2(\xi) - \xi \right]d\xi, \qquad (4.1)$$

where $\tilde{u}_n$ is consider as restricted variation,i.e.$\delta\tilde{u}_n = 0$. We find the optimal value of $\lambda(\xi)$ as follows

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta \int_0^x \lambda(\xi) \left[\frac{d^2}{d\xi^2} u_n(\xi) - 6\tilde{u}_n^2(\xi) - \xi \right]d\xi = 0, \qquad (4.2)$$

or

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta \int_0^x \lambda(\xi) \left[\frac{d^2}{d\xi^2} u_n(\xi) - \xi \right]d\xi = 0, \qquad (4.3)$$

which yields

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta\lambda(\xi)u'_n(\xi) \mid_{\xi=x} -\delta\lambda'(\xi)u_n(\xi) \mid_{\xi=x}$$

$$+ \delta \int_0^x \lambda''(\xi)u_n(\xi)d\xi + \delta \int_0^x \lambda(\xi)\xi d\xi = 0, \qquad (4.4)$$

by the stationary conditions we find

$$1 - \lambda'(\xi) = 0 \mid_{\xi=x} \qquad (4.5)$$

$$\lambda(\xi) = 0 \mid_{\xi=x} \qquad (4.6)$$

$$\lambda''(\xi) = 0 \mid_{\xi=x}, \tag{4.7}$$

so the optimal value for $\lambda(\xi)$ is

$$\lambda(\xi) = \xi - x. \tag{4.8}$$

Applying $\lambda(\xi)$ into functional (4.1), we can find an iteration formula as

$$u_{n+1}(x) = u_n(x) + \int_0^x (\xi - x) \left[\frac{d^2}{d\xi^2} u_n(\xi) - 6\tilde{u}_n^2(\xi) - \xi \right] d\xi. \tag{4.9}$$

Choosing $u_0(x) = x + \frac{1}{6}x^3$ we find approximate solutions for the problem (1.1) - (1.2) in the following forms:

$$u_1 = x + \tfrac{1}{6}x^3 + \tfrac{1}{2}x^4 + \tfrac{1}{15}x^6 + \tfrac{1}{336}x^8$$

$$u_2 = x + \tfrac{1}{6}x^3 + \tfrac{1}{2}x^4 + \tfrac{1}{15}x^6 + \tfrac{1}{7}x^7 + \tfrac{1}{336}x^8 + \tfrac{1}{40}x^9 + \tfrac{1}{60}x^{10} + \tfrac{71}{46200}x^{11}$$

$$+ \tfrac{1}{330}x^{12} + \tfrac{1}{26208}x^{13} + \tfrac{187}{764400}x^{14} + \tfrac{1}{100800}x^{16} + \tfrac{1}{5757696}x^{18}$$

$$u_3 = x + \tfrac{1}{6}x^3 + \tfrac{1}{2}x^4 + \tfrac{1}{15}x^6 + \tfrac{1}{7}x^7 + \tfrac{1}{336}x^8 + \tfrac{1}{40}x^9 + \tfrac{1}{28}x^{10} + \tfrac{71}{46200}x^{11}$$

$$+ \tfrac{23}{3080}x^{12} + \tfrac{893}{131040}x^{13} + \tfrac{5219}{8408400}x^{14} + \tfrac{1543}{970200}x^{15} + \tfrac{960077}{1009008000}x^{16}$$

$$+ \tfrac{91061}{571771200}x^{17} + \tfrac{15034573}{61751289600}x^{18} + \tfrac{2651251}{28756728000}x^{19} + \tfrac{60943}{2130128000}x^{20}$$

$$+ \tfrac{17671403}{720431712000}x^{21} + \tfrac{214967}{38846808000}x^{22} + \tfrac{36866671}{11391826446000}x^{23} + \tfrac{1429522453}{1215128154240000}x^{24}$$

$$+ \tfrac{6673}{26013487500}x^{25} + \tfrac{119120971}{735869534400000}x^{26} + \tfrac{1835681}{140484183840000}x^{27}$$

$$+ \tfrac{37104619}{2574691951104000}x^{28} + \tfrac{3611683}{8449863502080000}x^{29} + \tfrac{247344709}{2851182893195200000}x^{30}$$

$$+ \tfrac{1}{12007195200}x^{31} + \tfrac{127657}{3573341291520000}x^{32} + \tfrac{1}{13278997315584}x^{33}$$

$$+ \tfrac{1187}{1210325276160000}x^{34} + \tfrac{1}{60939454466400}x^{36} + \tfrac{1}{7768399149858816}x^{38}.$$

The desired solution is $u = \lim u_n$ when $n \to \infty$. Some values for $u_3(x)$ are obtained in table (1). From this table a comparsion between our solution and Maple numerical results can be made.

x	VIM	Maple results
0.1	0.1002167477	0.1002167980
0.2	0.2021394527	0.2021395810
0.3	0.3086307490	0.3086309826
0.4	0.4239862788	0.4239865878
0.5	0.5543399110	0.5543405192
0.7	0.8992296944	0.8992504369
0.9	1.481778951	1.4825252525
1.0	1.959421042	1.9631292204
1.2	3.815497429	3.8908998547
1.4	9.212665846	10.726462766
1.5	15.87244591	23.751855594
1.6	29.40300724	90.421617620
1.7	58.27134634	37524.936553

Table 1: Comparsion between VIM and Maple results

The Painlevé equation (1.1) has a singularity almost at $x = 1.7051622$ according to initial conditions (1.2), so our approximate polynomial has not desired accuracy in points close to singularity. In the next section we construct an other polynomial to approximate solutions of equation (1.1) by HPM.

5   HPM for Painlevé Equation

In this section we apply the homotopy perturbation method for finding an approximate solution of painlevé equation I. Consider the general form of the equation

$$F(u) = g(x), \tag{5.1}$$

where F is a nonlinear operator, and expand it into the following equation

$$L(u) + N(u) = g(x), \tag{5.2}$$

where the linear term is represented by $L(u)$ and L is a linear differential operator and easily invertible. The nonlinear term is represented by $N(u)$. L^{-1} is defined as n-fold integration for

$$L = \frac{d^n}{dx^n}. \tag{5.3}$$

As an example for $L = \frac{d^2}{dx^2}$, we will have $L^{-1} = \int_0^x \int_0^x [\cdot] dx dx$ and

$$(L^{-1}L)(u) = u - u(0) - xu'(0). \tag{5.4}$$

From (5.2) we have

$$L(u) = g(x) - N(u), \tag{5.5}$$

$$L^{-1}(L(u)) = L^{-1}g(x) - L^{-1}N(u), \tag{5.6}$$

and in this case, one can obtain u as follows

$$u = u(0) + xu'(0) + L^{-1}g(x) - L^{-1}N(u), \tag{5.7}$$

applying this result for the equation (1.1), we obtain

$$u = u(0) + xu'(0) + \frac{1}{6}x^3 + 6\int_0^x \int_0^x u^2 dx dx, \tag{5.8}$$

by the initial conditions (1.2), we have

$$u = x + \frac{x^3}{6} + 6\int_0^x \int_0^x u^2 dx dx. \tag{5.9}$$

According to the homotopy perturbation method, we construct the following homotopy. Using equation (3.4) we find

$$H(u, p) = u - x - \frac{x^3}{6} - 6p\int_0^x \int_0^x u^2 dx dx = 0. \tag{5.10}$$

Inserting $\sum_{i=0}^{\infty} p^i u_i$ instead u in (5.9) and comparing the coefficients of the same powers of p, we obtain

$p^0: \quad u_0 = x + \frac{1}{6}x^3$

$p^1: \quad u_1 = \frac{1}{2}x^4 + \frac{1}{15}x^6 + \frac{1}{336}x^8$

$p^2: \quad u_2 = \frac{1}{7}x^7 + \frac{1}{40}x^9 + \frac{71}{46200}x^{11} + \frac{1}{26208}x^{13}$

$p^3: \quad u_3 = \frac{1}{28}x^{10} + \frac{23}{3080}x^{12} + \frac{5219}{8408400}x^{14} + \frac{3551}{144144000}x^{16} + \frac{95}{224550144}x^{18}$

$p^4: \quad u_4 = \frac{3}{364}x^{13} + \frac{131}{64680}x^{15} + \frac{19867}{95295200}x^{17} + \frac{163469}{14378364000}x^{19} + \frac{163451}{491203440000}x^{21}$

$\qquad\qquad + \frac{145806371}{395015136480000000}x^{23}$

$p^5: \quad u_5 = \frac{37}{20384}x^{16} + \frac{489}{952952}x^{18} + \frac{1367141}{21727305600}x^{20} + \frac{32425891}{75285113904000}x^{22}$

$\qquad\qquad + \frac{4047099827}{23087434930560000}x^{24} + \frac{145806371}{395015136480000000}x^{26}$

$$+\frac{74849}{1737058836344832}x^{28}.$$

$\vdots$

Then the series solution is given as $u(x) = u_0 + u_1 + u_2 + \cdots$. Some numeric values of $u(x)$ is given in table (2). One can construct a better approximate polynomial by changing our homotopy. We revised the homotopy (5.10) as follows

$$H(u,p) = u - x - p\frac{x^3}{6} - 6p \int_0^x \int_0^x u^2 dx dx = 0. \qquad (5.11)$$

Applying $\sum_{i=0}^{\infty} p^i u_i$ in (5.11) and comparing the same powers of p, we have

$p^0:\quad u_0 = x$

$p^1:\quad u_1 = \frac{1}{6}x^3 + \frac{1}{2}x^4$

$p^2:\quad u_2 = +\frac{1}{15}x^6 + \frac{1}{7}x^7$

$p^3:\quad u_3 = \frac{1}{336}x^8 + \frac{1}{40}x^9 + \frac{1}{28}x^{10}$

$p^4:\quad u_4 = \frac{71}{46200}x^{11} + \frac{23}{3080}x^{12} + \frac{3}{364}x^{13}$

$p^5:\quad u_5 = \frac{1}{26208}x^{13} + \frac{5219}{8408400}x^{14} + \frac{131}{64680}x^{15} + \frac{37}{20384}x^{16}$

$p^6:\quad u_6 = \frac{3551}{144144000}x^{16} + \frac{19867}{95295200}x^{17} + \frac{489}{952952}x^{18} + \frac{75}{193648}x^{19}$

$p^7:\quad u_7 = \frac{95}{224550144}x^{18} + \frac{163469}{14378364000}x^{19} + \frac{1367141}{21727305600}x^{20} + \frac{33067}{266826560}x^{21}$

$$+\frac{219}{2711072}x^{22}$$

$\vdots$

Then the series solution is given by

$$u(x) = x + \frac{1}{6}x^3 + \frac{1}{2}x^4 + \frac{1}{15}x^6 + \frac{1}{7}x^7 + \frac{1}{336}x^8 + \frac{1}{40}x^9 + \frac{1}{28}x^{10} + \frac{71}{46200}x^{11} + \frac{23}{3080}x^{12}$$

$$+\frac{31}{3744}x^{13} + \frac{5219}{8408400}x^{14} + \frac{131}{64680}x^{15} + \frac{1856357}{1009008000}x^{16} + \frac{19867}{95295200}x^{17}$$

$$+\frac{181219}{352864512}x^{18} + \frac{5732219}{14378364000}x^{19} + \frac{1367141}{21727305600}x^{20} + \frac{33067}{266826560}x^{21}$$

$$+\frac{219}{2711072}x^{22} + \cdots$$

In table (2) a comparsion between numerical solutions of homotopy (5.10) and homotopy (5.11) and numerical solutions which obtained by the variational iteration method is given. A graphical comparsion is shown in figure (1).
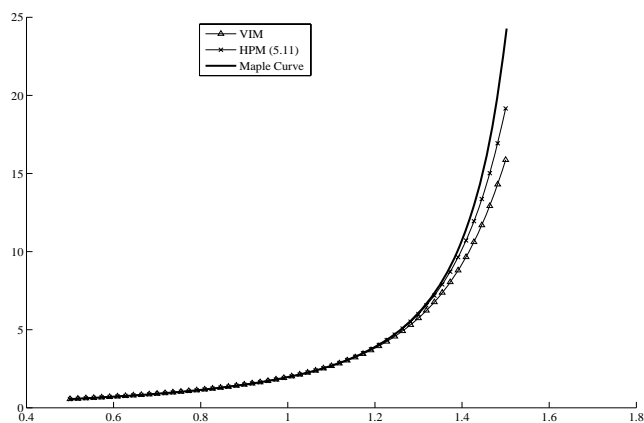
Figure 1: Comparsion between VIM, HPM and Maple results

x	VIM	HPM (5.10)	HPM (5.11)	Maple results
0.1	0.1002167477	0.1002167477	0.1002167477	0.1002167980
0.2	0.2021394527	0.2021394527	0.2021394527	0.2021395810
0.3	0.3086307490	0.3086307492	0.3086307492	0.3086309826
0.4	0.4239862788	0.4239862896	0.4239862896	0.4239865878
0.5	0.5543399110	0.5543401181	0.5543401190	0.5543405192
0.7	0.8992296944	0.8992493820	0.8992499224	0.8992504369
0.9	1.481778951	1.482444377	1.482517737	1.4825252525
1.0	1.959421042	1.962452186	1.963039368	1.9631292204
1.2	3.815497429	3.859792958	3.882059380	3.8908998547
1.4	9.212665846	9.657339885	10.15837961	10.726462766
1.5	15.87244591	17.11647598	19.16051446	23.751855594
1.6	29.40300724	32.60464299	40.27336198	90.421617620
1.7	58.27134634	65.78149064	92.4976098	37524.936553

Table 2 : Comparsion between VIM, HPM and Maple numerical results

6 Conclusion

In this paper the HPM and VIM applied to finding the approximate solutions of Painlevé equation I with initial conditions. The numerical solutions are compared with the numerical solutions from Maple in table (1), table (2) and figure (1). The results showed that the homotopy perturbation method is more powerful than variational iteration method and we will achieve to a desired approximation to the solution by simple calculations by using the homotopy formula (5.11). However, in this paper we showed application of HPM and VIM methods for a problem which has no exact solutions.

MAPLE has been used for computations in this paper.

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