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Non Deterministic Recognizability of Fuzzy Languages

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Abstract

We introduce non deterministic monoid recognizability (NDMR) of fuzzy languages and we show its equivalence with the deterministic version. Thus, fuzzy automata over the pairs $(\max, \min)$, $(\max, \Delta_L)$, $(\max, \Delta_D)$ have the same recognition power as NDMR Δ_L, Δ_D, are the Lukasiewicz and drastic intersection respectively.

1 Introduction and Basic Facts

The set X^* of all words over the alphabet X, with the word concatenation as operation, becomes a monoid whose unit element is the empty word e.

A language over X (i.e. a subset of X^*) computed by a finite automaton is called *recognizable*.

Such languages can be characterized in purely algebraic terms: $L \subseteq X^*$ is monoid recognizable iff there exists a finite monoid M and a monoid morphism $h : X^* \to M$ so that $L = h^{-1}(P)$ for some $P \subseteq M$.

The above result was used in [BLB 1, 2] in order to define recognizability in the setup of fuzzy languages.

Precisely, we say that the fuzzy language $\phi : X^* \to [0, 1]$ is *monoid recognizable* (m-recognizable) if there exists a finite monoid M, a monoid morfism $h : X^* \to M$ and a fuzzy subset $a : M \to [0, 1]$ so that $\phi = a \circ h$.

An advantage of this consideration is that does not make use of any (algebraic or topological) structure of the unit interval $[0, 1]$.

Next series of interesting logical equivalences was established in [BLB 1, 2]:

1. The fuzzy language $\phi : X^* \to [0, 1]$ is m-recognizable.

2. The syntactic congruence $\sim_\phi$ on X^* defined by $w \sim_\phi w'$ iff $\phi(\tau_1 w \tau_2) = \phi(\tau_1 w' \tau_2)$ for all $\tau_1, \tau_2 \in X^*$ has finite index.

3. The syntactic monoid $M_\phi = X^* \diagup_{\sim\phi}$ is finite.

4. ϕ has finitely many right derivatives

$$card\left\{\tau^{-1}\phi \diagup \tau \in X^*\right\} < \infty$$

where $\tau^{-1}\phi : X^* \to [0,1]$ is given by

$$\left(\tau^{-1}\phi\right)(w) = \phi\left(\tau w\right), \text{ for all } w \in X^*.$$

5. ϕ has finitely many left derivatives

$$card\left\{\phi\tau^{-1} \diagup \tau \in X^*\right\} < \infty$$

where $\phi\tau^{-1} : X^* \to [0,1]$ is given by

$$\left(\phi\tau^{-1}\right)(w) = \phi\left(w\tau\right), \text{ for all } w \in X^*.$$

6. ϕ is the behavior of a $(\max, \min)$-automaton.

7. ϕ is the behavior of a $(\max, \triangle_L)$-automaton, where $\triangle_L : [0,1]^2 \to [0,1]$ is the Lukasiewicz intersection

$$x \triangle_L y = \max\left(0, x + y - 1\right), x, y \in [0,1].$$

8. ϕ is the behavior of a $(\max, \triangle_D)$-automaton, where $\triangle_D : [0,1]^2 \to [0,1]$ is the drastic intersection

$$x \triangle_D y = x \text{ (if } y = 1\text{)}, \ y \text{ (if } x = 1\text{)}, \ 0 \text{ (else)}.$$

We denote by m-$Rec\,(X)$, the set of all m-recognizable fuzzy languages over X.

However, it should be noticed that m-recognizability cannot capture simple fuzzy languages such as

$$\phi : X^* \to [0,1], \phi(x) = \frac{1}{2^{|w|}}, w \in X^*$$

with $|w|$ standing for the length of the word w.

This led us to introduce and study non deterministic m-recognizability.

2 Non Determinism

Let X be a finite alphabet, $(M, \bullet, e)$ be a finite monoid.

We choose a *t-norm* $\triangle : [0,1]^2 \to [0,1]$ distributive over a *t-conorm* $\bigtriangledown : [0,1]^2 \to [0,1]$, that is the equality

$$x \triangle (y \bigtriangledown z) = (x \bigtriangledown y) \triangle (x \bigtriangledown z)$$

holds for all $x, y, z \in [0,1]$.

Then we say that $(\bigtriangledown, \triangle)$ is a distributive pair.

For instance $(\max, \min)$, $(\max, \triangle_L)$ and $(\max, \triangle_D)$ are distributive pairs.

The set $Fuzzy(M)$ of all fuzzy subsets of M with multiplication defined by the formula

$$(\phi_1 \circ \phi_2)(m) = \mathop{\bigtriangledown}_{m = m_1 \cdot m_2} \phi_1(m_1) \triangle \phi_2(m_2), m \in M$$

becomes a monoid whose unit element is $\widehat{e}$, the characteristic function of the singleton $\{e\}$.

A *non deterministic represantation* is a triple $\Re = (M, h, a)$, where M is a finite monoid, $h : X^* \to Fuzzy(M)$ is a monoid morphism and $a : M \to [0,1]$.

It computes the fuzzy language

$$\phi_\Re : X^* \to [0,1], \phi_\Re(w) = \mathop{\bigtriangledown}_{m \in M} a(m) \triangle h(w)(m).$$

In other words $\phi_\Re = \langle a, - \rangle \circ h$ where $\langle a, - \rangle$ is the inner product operator defined for all $\beta \in Fuzzy(M)$ by

$$\langle a, \beta \rangle = \mathop{\bigtriangledown}_{m \in M} a(m) \triangle \beta(m).$$

We denote by $ndm\text{-}Rec(X, \bigtriangledown, \triangle)$ the set of all fuzzy laguages $\phi : X^* \to [0,1]$ such that $\phi = \phi_\Re$, for some non deterministic represantation $\Re$.

Proposition 1. *It holds*

$$m\text{-}Rec(X) \subseteq ndm\text{-}Rec(X, \bigtriangledown, \triangle).$$

Proof. Assume that $\phi \in m\text{-}Rec(X, \bigtriangledown, \triangle)$ and let $(M, \bullet, e)$ be a finite monoid, $h_1 : X^* \to M$ a monoid morphism and $a : M \to [0,1]$ so that $\phi = a \circ h_1$. For each $m \in M$ we denote by $\widehat{m}$ the characteristic function of the singleton $\{m\}$. It holds $\widehat{m}_1 \cdot \widehat{m}_2 = \widehat{m_1 \cdot m_2}$ (for all $m_1, m_2 \in M$) and thus the mapping

$$\wedge : M \to Fuzzy(M), m \mapsto \widehat{m}$$

is a monoid morphism.

Furthermore, we have $\langle a,-\rangle \circ \wedge = a$. Indeed, for all $m \in M$ we have

$$\begin{aligned}(\langle a,-\rangle \circ \wedge)(m) &= \langle a,-\rangle(\widehat{m}) = \langle a,\widehat{m}\rangle \\ &= \mathop{\bigtriangledown}_{n\in M} a(n) \bigtriangleup \widehat{m}(n) = a(m)\end{aligned}$$

It follows that

$$\phi = a \circ h_1 = \langle a,-\rangle \circ \wedge \circ h_1$$

and thus the non deterministic represantation $\Re = (M, \wedge \circ h_1, a)$ computes ϕ, i.e. $\phi \in$ *ndm-Rec* $(X, \bigtriangledown, \bigtriangleup)$ as wanted. □

Example 2. *Take the monoid $M = \{e\}$ reduced to its unit element e and choose $(\bigtriangledown, \bigtriangleup) = (\max, \bigtriangleup_m)$ with $\bigtriangleup_m$ to be the t-norm $x \bigtriangleup_m y = xy$. Then the monoid $(Fuzzy(e), \bullet)$ is obviously isomorhic to the monoid $([0,1], \bigtriangleup_m)$ and the represantation $\Re = (\{e\}, h, a)$ with*

$$h : X^* \to [0,1], h(w) = \frac{1}{2^{|w|}} \text{ and } a(e) = 1,$$

computes the fuzzy language $\phi(w) = \frac{1}{2^{|w|}}$.It turns out that the inclusion

$$m\text{-}Rec(X) \subset ndm\text{-}Rec(X, \max, \bigtriangleup_m)$$

is proper.

Since in the crisp case, non determinism does not increase the recognition power of the used mecanism, the question is weather a analogous phenomenon appears in the fuzzy case. The answer depends on the used pair $(\bigtriangledown, \bigtriangleup)$. Let us recall that a $(\bigtriangledown, \bigtriangleup)$-*automaton* is a 5-tuple $\mathcal{A} = (Q, X, \delta, I, F)$ where Q is a finite set of states, X is the finite input alphabet, $\delta : X \to FRel(Q)$ is the move function and $I, F : Q \to [0,1]$ are the initial and final fuzzy subsets of Q respectively.

Here, $FRel(Q)$ is the set of all fuzzy relations

$$R : Q \times Q \to [0,1].$$

The composition of any two $R, S : Q \times Q \to [0,1]$ is given by

$$(R \circ S)(p,q) = \mathop{\bigtriangledown}_{r\in Q} R(p,r) \bigtriangleup S(r,q)$$

and obviously structures $FRel(Q)$ into a monoid. Thus δ above is uniquely extended into a monoid morphism $\delta^* : X^* \to FRel(Q)$ via $\delta^*(x_1, \ldots, x_k) = \delta(x_1) \circ \cdots \circ \delta(x_k), x_1, \ldots, x_k \in X, k \geq 0$.

The *behavior* of $\mathcal{A}$ is then the fuzzy language $|\mathcal{A}| : X^* \to [0,1]$ with

$$|\mathcal{A}|(w) = \mathop{\bigtriangledown}_{p,q\in Q} I(p) \bigtriangleup \delta^*(w)(p,q) \bigtriangleup F(q), w \in X^*.$$

Rec $(X, \bigtriangledown, \bigtriangleup)$ stands for the set of all fuzzy languages obtained as behaviors of $(\bigtriangledown, \bigtriangleup)$-*automata* over X.

Theorem 3. *It holds*

$$ndm\text{-}Rec\,(X, \bigtriangledown, \bigtriangleup) \subseteq Rec\,(X, \bigtriangledown, \bigtriangleup)$$

for any distributive pair $(\bigtriangledown, \bigtriangleup)$.

Proof. Let $\Re = (M, h, a)$ be a non deterministic represantation of $\phi \in ndm\text{-}Rec\,(X, \bigtriangledown, \bigtriangleup)$, i.e. $\phi = \phi_{\Re}$. Consider the $(\bigtriangledown, \bigtriangleup)$-*automaton*

$$\mathcal{A} = (M, X, \phi, I = \{e\}, T = a)$$

where e is the unit element of the monoid M, whereas $\delta : X \to FRel\,(M)$ is given by

$$\delta\,(x) = (m.m') = \mathop{\bigtriangledown}_{m'=mn} h\,(x)\,(n)\,, x \in X, m.m' \in M.$$

For all $x_1, \ldots, x_k \in X$ $(k \geq 0)$ we have

$$\delta^*\,(x_1, \ldots, x_k)\,(m, m') = [\delta\,(x_1) \circ \cdots \circ \delta\,(x_k)]\,(m, m')$$

$$= \mathop{\bigtriangledown}_{m_1,\ldots,m_{k-1} \in M} \delta\,(x_1)\,(m, m_1) \bigtriangleup \cdots \bigtriangleup \delta\,(x_k)\,(m_{k-1}, m')$$

$$= \left(\mathop{\bigtriangledown}_{m=m_1 n_1} h\,(x_1)\,(n_1)\right) \bigtriangleup \left(\mathop{\bigtriangledown}_{m_2=m_1 n_2} h\,(x_2)\,(n_2)\right) \bigtriangleup \cdots \bigtriangleup \left(\mathop{\bigtriangledown}_{m_{k-1}=m' n_k} h\,(x_k)\,(n_k)\right)$$

$$= \mathop{\bigtriangledown}_{m'=mn_1\cdots n_k} h\,(x_1)\,(n_1) \bigtriangleup h\,(x_2)\,(n_2) \bigtriangleup \cdots \bigtriangleup h\,(x_k)\,(n_k)$$

$$= \mathop{\bigtriangledown}_{m'=mn} [h\,(x_1) \bullet h\,(x_2) \bullet \cdots \bullet h\,(x_k)]\,(n)$$

$$= \mathop{\bigtriangledown}_{m'=mn} h\,(x_1, \ldots, x_k)\,(n) = \mathop{\bigtriangledown}_{m'=mn} h\,(w)\,(n)\,.$$

Consequently for all $w \in X^*$ we have

$$\begin{aligned} |\mathcal{A}|\,(w) &= \mathop{\bigtriangledown}_{m \in M} \delta^*\,(w)\,(e, m) \bigtriangleup a\,(m) \\ &= \mathop{\bigtriangledown}_{m \in M} h\,(w)\,m \bigtriangleup a\,(m) \\ &= \phi_{\Re}\,(w)\,. \end{aligned}$$

In other words $|\mathcal{A}| = \phi_{\Re}$ and thus $\phi_{\Re} \in Rec\,(X, \bigtriangledown, \bigtriangleup)$ as wanted. □

Theorem 4. *For* $(\bigtriangledown, \bigtriangleup) = (\max, \min)\,, (\max, \bigtriangleup_L)\,, (\max, \bigtriangleup_D)$ *we have the equality*

$$m\text{-}Rec\,(X) = ndm\text{-}Rec\,(X, \bigtriangledown, \bigtriangleup)\,.$$

Proof. In fact, by virtue of proposition 1 and theorem 3 above, we get

$$m\text{-}Rec\,(X) \subseteq ndm\text{-}Rec\,(X, \triangledown, \triangle) \subseteq Rec\,(X, \triangledown, \triangle)\,.$$

According [BLB 2] it holds

$$Rec\,(X, \triangledown, \triangle) = m\text{-}Rec\,(X)$$

for all the pairs $(\triangledown, \triangle)$ of the statement.

The proposed equality follows. □

References

[BLB 1] S. Bozapalidis and O. Louskou-Bozapalidou; On the recognizability of fuzzy languages I (submitted)

[BLB 2] S. Bozapalidis and O. Louskou-Bozapalidou; On the recognizability of fuzzy languages II (submitted)

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