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Common Fixed Point Theorems for a Class Maps in $\mathcal{L}$-Fuzzy Metric Spaces

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Abstract. In this paper we first prove a common fixed point theorem in $\mathcal{L}$-fuzzy metric space. Then we prove fixed point theorems for various compatible maps in $\mathcal{L}$-fuzzy metric spaces.

1. Introduction and Preliminaries

The notion of fuzzy sets was introduced by Zadeh [23]. Various concepts of fuzzy metric spaces were considered in [7, 8, 13, 14]. Many authors have studied fixed theory in fuzzy metric spaces; see for example [3, 4, 11, 12, 16, 17]. In the sequel, we shall adopt the usual terminology, notation and conventions of $\mathcal{L}$-fuzzy metric spaces introduced by Saadati et al. [19] which are a generalization of fuzzy metric sapces [10] and intuitionistic fuzzy metric spaces [18, 20].

Definition 1.1. ([11]) Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice, and U a non-empty set called a universe. An $\mathcal{L}$-fuzzy set $\mathcal{A}$ on U is defined as a mapping $\mathcal{A} : U \longrightarrow L$. For each u in U, $\mathcal{A}(u)$ represents the degree (in L) to which u satisfies $\mathcal{A}$.

Lemma 1.2. ([5, 6]) *Consider the set L^* and the operation $\leq_{L^*}$ defined by:*

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 + x_2 \leq 1\},$$

$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Longleftrightarrow x_1 \leq y_1$ *and* $x_2 \geq y_2$*, for every* $(x_1, x_2), (y_1, y_2) \in L^*$. *Then* $(L^*, \leq_{L^*})$ *is a complete lattice .*

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Classically, a triangular norm T on $([0,1],\leq)$ is defined as an increasing, commutative, associative mapping $T:[0,1]^2\to[0,1]$ satisfying $T(1,x)=x$, for all $x\in[0,1]$. These definitions can be straightforwardly extended to any lattice $\mathcal{L}=(L,\leq_L)$. Define first $0_{\mathcal{L}}=\inf L$ and $1_{\mathcal{L}}=\sup L$.

Definition 1.3. A triangular norm (t-norm) on $\mathcal{L}$ is a mapping $\mathcal{T}:L^2\to L$ satisfying the following conditions:

(i) $(\forall x\in L)(\mathcal{T}(x,1_{\mathcal{L}})=x)$; (boundary condition)
(ii) $(\forall(x,y)\in L^2)(\mathcal{T}(x,y)=\mathcal{T}(y,x))$; (commutativity)
(iii) $(\forall(x,y,z)\in L^3)(\mathcal{T}(x,\mathcal{T}(y,z))=\mathcal{T}(\mathcal{T}(x,y),z))$; (associativity)
(iv) $(\forall(x,x',y,y')\in L^4)(x\leq_L x'$ and $y\leq_L y'\Rightarrow\mathcal{T}(x,y)\leq_L\mathcal{T}(x',y'))$. (monotonicity)

A t–norm $\mathcal{T}$ on $\mathcal{L}$ is said to be continuous if for any $x,y\in\mathcal{L}$ and any sequences $\{x_n\}$ and $\{y_n\}$ which converge to x and y we have

$$\lim_n \mathcal{T}(x_n,y_n)=\mathcal{T}(x,y)$$

For example, $\mathcal{T}(x,y)=\min(x,y)$ and $\mathcal{T}(x,y)=xy$ are two continuous t–norms on $[0,1]$. A t-norm can also be defined recursively as an $(n+1)$-ary operation $(n\in\mathbb{N})$ by $\mathcal{T}^1=\mathcal{T}$ and

$$\mathcal{T}^n(x_1,\cdots,x_{n+1})=\mathcal{T}(\mathcal{T}^{n-1}(x_1,\cdots,x_n),x_{n+1})$$

for $n\geq 2$ and $x_i\in L$.

Definition 1.4. A negation on $\mathcal{L}$ is any decreasing mapping $\mathcal{N}:L\to L$ satisfying $\mathcal{N}(0_{\mathcal{L}})=1_{\mathcal{L}}$ and $\mathcal{N}(1_{\mathcal{L}})=0_{\mathcal{L}}$. If $\mathcal{N}(\mathcal{N}(x))=x$, for all $x\in L$, then $\mathcal{N}$ is called an involutive negation.

Definition 1.5. The 3-tuple $(X,\mathcal{M},\mathcal{T})$ is said to be an *$\mathcal{L}$-fuzzy metric space* if X is an arbitrary (non-empty) set, $\mathcal{T}$ is a continuous t–norm on $\mathcal{L}$ and $\mathcal{M}$ is an $\mathcal{L}$-fuzzy set on $X^2\times\,]0,+\infty[$ satisfying the following conditions for every x,y,z in X and t,s in $]0,+\infty[$:

(a) $\mathcal{M}(x,y,t)>_L 0_{\mathcal{L}}$;
(b) $\mathcal{M}(x,y,t)=1_{\mathcal{L}}$ for all $t>0$ if and only if $x=y$;
(c) $\mathcal{M}(x,y,t)=\mathcal{M}(y,x,t)$;
(d) $\mathcal{T}(\mathcal{M}(x,y,t),\mathcal{M}(y,z,s))\leq_L\mathcal{M}(x,z,t+s)$;
(e) $\mathcal{M}(x,y,\cdot):\,]0,\infty[\,\to L$ is continuous.

Let $(X,\mathcal{M},\mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. For $t\in\,]0,+\infty[$, we define the *open ball* $B(x,r,t)$ with center $x\in X$ and radius $r\in L\setminus\{0_{\mathcal{L}},1_{\mathcal{L}}\}$, as

$$B(x,r,t)=\{y\in X:\mathcal{M}(x,y,t)>_L\mathcal{N}(r)\}.$$

A subset $A\subseteq X$ is called *open* if for each $x\in A$, there exist $t>0$ and $r\in L\setminus\{0_{\mathcal{L}},1_{\mathcal{L}}\}$ such that $B(x,r,t)\subseteq A$. Let $\tau_{\mathcal{M}}$ denote the family of all open subsets of X. Then $\tau_{\mathcal{M}}$ is called the *topology induced by the $\mathcal{L}$-fuzzy metric $\mathcal{M}$*.

Example 1.6. ([21]) Let (X,d) be a metric space. Denote $\mathcal{T}(a,b) = (a_1b_1, \min(a_2+b_2,1))$ for all $a=(a_1,a_2)$ and $b=(b_1,b_2)$ in L^* and let M and N be fuzzy sets on $X^2\times(0,\infty)$ be defined as follows:

$$\mathcal{M}_{M,N}(x,y,t) = (M(x,y,t),N(x,y,t)) = (\frac{t}{t+d(x,y)}, \frac{d(x,y)}{t+d(x,y)}).$$

Then $(X,\mathcal{M}_{M,N},\mathcal{T})$ is an intuitionistic fuzzy metric space.

Example 1.7. Let $X=\mathbb{N}$. Define $\mathcal{T}(a,b) = (\max(0,a_1+b_1-1), a_2+b_2-a_2b_2)$ for all $a=(a_1,a_2)$ and $b=(b_1,b_2)$ in L^*, and let $\mathcal{M}(x,y,t)$ on $X^2\times(0,\infty)$ be defined as follows:

$$\mathcal{M}(x,y,t) = \begin{cases} (\frac{x}{y}, \frac{y-x}{y}) & if \quad x\le y \\ (\frac{y}{x}, \frac{x-y}{x}) & if \quad y\le x. \end{cases}$$

for all $x,y\in X$ and $t>0$. Then $(X,\mathcal{M},\mathcal{T})$ is an $\mathcal{L}$-fuzzy metric space.

Lemma 1.8. ([10]) *Let $(X,\mathcal{M},\mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. Then, $\mathcal{M}(x,y,t)$ is nondecreasing with respect to t, for all x,y in X.*

Definition 1.9. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in an $\mathcal{L}$-fuzzy metric space $(X,\mathcal{M},\mathcal{T})$ is called a *Cauchy sequence*, if for each $\varepsilon\in L\setminus\{0_{\mathcal{L}}\}$ and $t>0$, there exists $n_0\in\mathbb{N}$ such that for all $m\ge n\ge n_0$ $(n\ge m\ge n_0)$,

$$\mathcal{M}(x_m,x_n,t) >_L \mathcal{N}(\varepsilon).$$

The sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be *convergent* to $x\in X$ in the $\mathcal{L}$-fuzzy metric space $(X,\mathcal{M},\mathcal{T})$ (denoted by $x_n \xrightarrow{\mathcal{M}} x$) if $\mathcal{M}(x_n,x,t)=\mathcal{M}(x,x_n,t)\to 1_{\mathcal{L}}$ whenever $n\to+\infty$ for every $t>0$. A $\mathcal{L}$-fuzzy metric space is said to be *complete* if and only if every Cauchy sequence is convergent.

Henceforth, we assume that $\mathcal{T}$ is a continuous t–norm on the lattice $\mathcal{L}$ such that for every $\mu\in L\setminus\{0_{\mathcal{L}},1_{\mathcal{L}}\}$, there is a $\lambda\in L\setminus\{0_{\mathcal{L}},1_{\mathcal{L}}\}$ such that

$$\mathcal{T}^{n-1}(\mathcal{N}(\lambda),...,\mathcal{N}(\lambda)) >_L \mathcal{N}(\mu).$$

For more information see [19].

Definition 1.10. Let $(X,\mathcal{M},\mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. $\mathcal{M}$ is said to be continuous on $X\times X\times]0,\infty[$ if

$$\lim_{n\to\infty}\mathcal{M}(x_n,y_n,t_n) = \mathcal{M}(x,y,t)$$

whenever a sequence $\{(x_n,y_n,t_n)\}$ in $X\times X\times]0,\infty[$ converges to a point $(x,y,t)\in X\times X\times]0,\infty[$ i.e., $\lim_n\mathcal{M}(x_n,x,t)=\lim_n\mathcal{M}(y_n,y,t)=1_{\mathcal{L}}$ and $\lim_n\mathcal{M}(x,y,t_n)=\mathcal{M}(x,y,t)$.

Lemma 1.11. *Let $(X,\mathcal{M},\mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. Then $\mathcal{M}$ is continuous function on $X\times X\times]0,\infty[$.*

Proof. The proof is the same as that for fuzzy spaces (see Proposition 1 of [15]). □

Definition 1.12. Let A and S be mappings from an $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ into itself. Then the mappings are said to be weak compatible if they commute at their coincidence point, that is, $Ax = Sx$ implies that $ASx = SAx$.

Definition 1.13. Let A and S be mappings from an $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ into itself. Then the mappings are said to be compatible if

$$\lim_{n\to\infty} \mathcal{M}(ASx_n, SAx_n, t) = 1_{\mathcal{L}}, \forall t > 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X.$$

Proposition 1.14. ([22]) *If self-mappings A and S of an $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ are compatible, then they are weak compatible.*

Lemma 1.15. ([1, 19]) *Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. Define $E_{\lambda,\mathcal{M}} : X^2 \longrightarrow \mathbb{R}^+ \cup \{0\}$ by*

$$E_{\lambda,\mathcal{M}}(x, y) = \inf\{t > 0 : \mathcal{M}(x, y, t) >_L \mathcal{N}(\lambda)\}$$

for each $\lambda \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ and $x, y \in X$. Then we have

(i) *For any $\mu \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ there exists $\lambda \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that*

$$E_{\mu,\mathcal{M}}(x_1, x_n) \leq E_{\lambda,\mathcal{M}}(x_1, x_2) + E_{\lambda,\mathcal{M}}(x_2, x_3) + \cdots + E_{\lambda,\mathcal{M}}(x_{n-1}, x_n)$$

for any $x_1, ..., x_n \in X$;

(ii) *The sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent w.r.t. $\mathcal{L}$-fuzzy metric $\mathcal{M}$ if and only if $E_{\lambda,\mathcal{M}}(x_n, x) \to 0$. Also the sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy w.r.t. $\mathcal{L}$-fuzzy metric $\mathcal{M}$ if and only if it is Cauchy with $E_{\lambda,\mathcal{M}}$.*

Lemma 1.16. *Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. If*

$$\mathcal{M}(x_n, x_{n+1}, t) \geq_L \mathcal{M}(x_0, x_1, k^n t)$$

for some $k > 1$ and $n \in \mathbb{N}$. Then $\{x_n\}$ is a Cauchy sequence.

Proof. For every $\lambda \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ and $x_n \in X$, we have

$$\begin{aligned}
E_{\lambda,\mathcal{M}}(x_{n+1}, x_n) &= \inf\{t > 0 \ : \ \mathcal{M}(x_{n+1}, x_n, t) >_L \mathcal{N}(\lambda)\} \\
&\leq \inf\{t > 0 \ : \ \mathcal{M}(x_0, x_1, k^n t) >_L \mathcal{N}(\lambda)\} \\
&= \inf\{\frac{t}{k^n} \ : \ \mathcal{M}(x_0, x_1, t) >_L \mathcal{N}(\lambda)\} \\
&= \frac{1}{k^n} \inf\{t > 0 \ : \ \mathcal{M}(x_0, x_1, t) >_L \mathcal{N}(\lambda)\} \\
&= \frac{1}{k^n} E_{\lambda,\mathcal{M}}(x_0, x_1).
\end{aligned}$$

From Lemma 1.15, for every $\mu \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ there exists $\lambda \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$, such that

$$\begin{aligned} E_{\mu,\mathcal{M}}(x_n, x_m) &\leq E_{\lambda,\mathcal{M}}(x_n, x_{n+1}) + E_{\lambda,\mathcal{M}}(x_{n+1}, x_{n+2}) + \cdots + E_{\lambda,\mathcal{M}}(x_{m-1}, x_m) \\ &\leq \frac{1}{k^n} E_{\lambda,\mathcal{M}}(x_0, x_1) + \frac{1}{k^{n+1}} E_{\lambda,\mathcal{M}}(x_0, x_1) + \cdots + \frac{1}{k^{m-1}} E_{\lambda,\mathcal{M}}(x_0, x_1) \\ &= E_{\lambda,\mathcal{M}}(x_0, x_1) \sum_{j=n}^{m-1} \frac{1}{k^j} \longrightarrow 0. \end{aligned}$$

Hence sequence $\{x_n\}$ is a Cauchy sequence. □

2. THE MAIN RESULTS

A class of implicit relation. Let Φ be the set of all continuous functions $\phi : L \longrightarrow L$, such that $\phi(t) > t$ for every $t \in L \setminus \{1_L\}$.

Theorem 2.1. *Let $(X, \mathcal{M}, \mathcal{T})$ be a complete $\mathcal{L}$-fuzzy metric space and assume $S, T, I, J : X \longrightarrow X$ be four mappings,such that $TX \subseteq JX$, $SX \subseteq IX$, $(*)$ and*

$$\mathcal{M}(Tx, Sy, t)) \geq_L \phi\left(\begin{array}{l} \min\{\mathcal{M}(Ix, Tx, kt), \mathcal{M}(Jy, Sy, kt), \mathcal{M}(Ix, Jy, kt)\} \\ \max\{\mathcal{M}(Ix, Sy, kt), \mathcal{M}(Jy, Tx, kt)\} \end{array} \right)$$

for every $x, y \in X$, some $k > 1$ and $T(X)$ or $S(X)$ is a closed subset of X. Suppose in addition that either

(i)*T, I are compatible, I is continuous and S, J are weak compatible, or*

(ii)*S, J are compatible, J is continuous and T, I are weak compatible. Then I, J, T and S have a unique common fixed point.*

Proof. Let $x_0 \in X$ be given . By $(*)$ one can choose a point $x_1 \in X$ such that $Tx_0 = Jx_1 = y_1$, and a point $x_2 \in X$ such that $Sx_1 = Tx_2 = y_2$. Continuing this way, we define by induction a sequence $\{x_n\}$ in X such that

$$\begin{array}{ll} Ix_{2n+2} = Sx_{2n+1} = y_{2n+2} & n = 0, 1, 2, \cdots \\ Jx_{2n+1} = Tx_{2n} = y_{2n+1} & n = 0, 1, \cdots . \end{array}$$

For simplicity, we set

$$d_n(t) = \mathcal{M}(y_n, y_{n+1}, t), \; n = 0, 1, 2, \cdots$$

It follows from assume that for $n = 0, 1, 2, \cdots$.

$$\begin{aligned} d_{2n+1}(t)) &= \mathcal{M}(y_{2n+1}, y_{2n+2}, t) = \mathcal{M}(Tx_{2n}, Sx_{2n+1}, t) \\ &\geq_L \phi\left(\begin{array}{l} \min\{\mathcal{M}(Ix_{2n}, Tx_{2n}, kt), \mathcal{M}(Jx_{2n+1}, Sx_{2n+1}, kt), \mathcal{M}(Ix_{2n}, Jx_{2n+1}, kt)\} \\ \max\{\mathcal{M}(Ix_{2n}, Sx_{2n+1}, kt), \mathcal{M}(Jx_{2n+1}, Tx_{2n}, kt)\} \end{array} \right) \\ &\geq_L \phi(\min\{d_{2n}(kt), d_{2n+1}(kt), d_{2n}(kt)\}, \max\{1_L, 1_L\}) \end{aligned}$$

Now, if $d_{2n+1}(kt) <_L d_{2n}(kt)$, then

$$d_{2n+1}(t) \geq_L \phi(d_{2n+1}(kt)) >_L d_{2n+1}(kt).$$

Hence $d_{2n+1}(t) >_L d_{2n+1}(kt)$, is a contradiction. Therefore $d_{2n+1}(t) \geq_L d_{2n}(kt)$. That is $\mathcal{M}(y_{2n+1}, y_{2n+2}, t) \geq_L \mathcal{M}(y_{2n}, y_{2n+1}, kt)$. So

$$\mathcal{M}(y_n, y_{n+1}, t) \geq_L \mathcal{M}(y_{n-1}, y_n, kt) \geq_L \cdots \geq_L \mathcal{M}(y_0, y_1, k^n t).$$

By Lemma 1.15 sequence $\{y_n\}$ is Cauchy sequence, then it is converges to $a \in X$. That is

$$\lim_{n\to\infty} y_n = a = \lim_{n\to\infty} Jx_{2n+1} = \lim_{n\to\infty} Sx_{2n+1} = \lim_{n\to\infty} Ix_{2n+2} = \lim_{n\to\infty} Tx_{2n}.$$

Now suppose that *(i)* is satisfied. Then $I^2x_{2n} \longrightarrow Ia$ and $ITx_{2n} \longrightarrow Ia$, since T and I are compatible, implies that $TIx_{2n} \longrightarrow Ia$. Now we wish to show that a is common fixed point of I, J, T and S.

(i) a is fixed point of I. Indeed, if $Ia \neq a$ we have
$\mathcal{M}(TIx_{2n}, Sx_{2n+1}, t) \geq_L$

$$\phi(\begin{matrix} \min\{\mathcal{M}(I^2x_{2n}, TIx_{2n}, kt), \mathcal{M}(Jx_{2n+1}, Sx_{2n+1}, kt), \mathcal{M}(I^2x_{2n}, Jx_{2n+1}, kt)\} \\ \max\{\mathcal{M}(I^2x_{2n}, Sx_{2n+1}, kt), \mathcal{M}(Jx_{2n+1}, TIx_{2n}, kt)\} \end{matrix}).$$

Letting $n \to \infty$, (since $Ia \neq a$) yields

$$\begin{aligned} \mathcal{M}(Ia, a, t)) &\geq_L \phi(\begin{matrix} \min\{\mathcal{M}(Ia, Ia, kt), \mathcal{M}(a, a, kt), \mathcal{M}(Ia, a, kt)\} \\ \max\{\mathcal{M}(Ia, a, kt), \mathcal{M}(a, Ia, kt)\} \end{matrix}) \\ &= \phi(\mathcal{M}(Ia, a, kt)) >_L \mathcal{M}(Ia, a, kt), \end{aligned}$$

is a contradiction, hence $Ia = a$.

(ii) a is fixed point of T. Indeed,

$$\mathcal{M}(Ta, Sx_{2n+1}, t) \geq_L \phi(\begin{matrix} \min\{\mathcal{M}(Ia, Ta, kt), \mathcal{M}(Jx_{2n+1}, Sx_{2n+1}, kt), \mathcal{M}(Ia, Jx_{2n+1}, kt)\} \\ \max\{\mathcal{M}(Ia, Sx_{2n+1}, kt), \mathcal{M}(Jx_{2n+1}, Ta, kt)\} \end{matrix})$$

and letting $n \to \infty$, if $Ta \neq a$ gives

$$\begin{aligned} \mathcal{M}(Ta, a, t)) &\geq_L \phi(\begin{matrix} \min\{\mathcal{M}(Ia, Ta, kt), \mathcal{M}(a, a, kt), \quad \mathcal{M}(Ia, a, kt)\} \\ \max\{\mathcal{M}(Ia, a, kt), \mathcal{M}(a, Ta, kt)\} \end{matrix}) \\ &= \phi(\mathcal{M}(Ta, a, kt)) >_L \mathcal{M}(Ta, a, kt) \end{aligned}$$

is a contradiction. Hence , $Ta = a$.

(iii) Since $TX \subseteq JX$ for all $x \in X$, there is a point $b \in X$ such that

$$Ta = a = Jb.$$

We show that b is coincidence point for J and S. Indeed, if $Jb \neq Sb$ we have
$\mathcal{M}(Jb, Sb, t) = \mathcal{M}(a, Sb, t) = \mathcal{M}(Ta, Sb, t)$

$$\geq \phi(\begin{matrix} \min\{\mathcal{M}(Ia, Ta, kt), \mathcal{M}(Jb, Sb, kt), \mathcal{M}(Ia, Jb, kt)\} \\ \max\{\mathcal{M}(Ia, Sb, kt), \mathcal{M}(Jb, Ta, kt)\} \end{matrix}) >_L \mathcal{M}(Jb, Sb, kt),$$

is a contradiction. Thus $Ta = Sb = Jb = a$. Since J and S are weak compatible, we deduce that

$$SJb = JSb \Longrightarrow Sa = Ja.$$

We show that $Ta = Sa$. Indeed, if $Ta \neq Sa$ we have

$$\mathcal{M}(Ta, Sa, t) \geq_L \phi(\begin{matrix} \min\{\mathcal{M}(Ia, Ta, kt), \mathcal{M}(Ja, Sa, kt), \mathcal{M}(Ia, Ja, kt)\} \\ \max\{\mathcal{M}(Ia, Sa, kt), \mathcal{M}(Ja, Ta, kt)\} \end{matrix}) >_L \mathcal{M}(Ta, Sa, kt),$$

is a contradiction, that is, $Ta = Sa$. Therefore

$$Sa = Ta = Ia = Ja = a.$$

Uniqueness, if $b \neq a$ be another fixed point of I, J, T and S, then

$$\begin{aligned} \mathcal{M}(Ta, Sb, t)) &\geq_L \phi(\begin{matrix} \min\{\mathcal{M}(Ia, Ia, kt), \mathcal{M}(a, a, kt), \mathcal{M}(Ia, a, kt)\} \\ \max\{\mathcal{M}(Ia, a, kt), \mathcal{M}(a, Ia, kt)\} \end{matrix}) \\ &>_L \mathcal{M}(a, b, kt) = \mathcal{M}(Ta, Sb, t). \end{aligned}$$

is a contradiction.That is, a is unique common fixed point, and proof of the theorem is complete. □

A class of implicit relation. Let $\{S_\alpha\}_{\alpha\in A}$ and $\{T_\beta\}_{\beta\in B}$ be the set of all self-mappings of a complete $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$.

Theorem 2.2. *Let I, J and $\{T_\alpha\}_{\alpha\in A}, \{S_\beta\}_{\beta\in B}$ be self-mappings of a complete $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ satisfying :*

(i)*there exist $\alpha_0 \in A$ and $\beta_0 \in B$ such that $T_{\alpha_0}(X) \subseteq J(X)$, $S_{\beta_0}(X) \subseteq I(X)$ and $T_{\alpha_0}(X)$ or $S_{\beta_0}(X)$ is a closed subset of X,*

(ii)

$$\mathcal{M}(T_{\alpha_0}x, S_{\beta_0}y, t)) \geq_L \phi(\begin{matrix} \min\{\mathcal{M}(Ix, T_{\alpha_0}x, kt), \mathcal{M}(Jy, S_{\beta_0}y, kt), \mathcal{M}(Ix, Jy, kt)\} \\ \max\{\mathcal{M}(Ix, S_{\beta_0}y, kt), \mathcal{M}(Jy, T_{\alpha_0}x, kt)\} \end{matrix})$$

for every $x, y \in X$, some $k > 1$ and $\phi \in \Phi$.
Suppose in addition that either

(a)*T_{α_0}, I are compatible, I is continuous and S_{β_0}, J are weak compatible, or*

(b)*S_{β_0}, J are compatible, J is continuous and T_{α_0}, I are weak compatible. Then I, J, T_α and S_β have a unique common fixed point.*

Proof. By Theorem 2.1 I, J, S_{α_0} and T_{β_0} for some $\alpha_0 \in A, \beta_0 \in B$ have a unique common fixed point in X. That is there exist a unique $a \in X$ such that $I(a) = J(a) = S_{\alpha_0}(a) = T_{\beta_0}(a) = a$. Let there exist $\lambda \in B$ such that $\lambda \neq \beta_0$ and $\mathcal{M}(T_\lambda a, a, t) < 1_L$ then we have

$$\begin{aligned} \mathcal{M}(a, S_\lambda a, t) &= \mathcal{M}(T_{\alpha_0}a, S_\lambda a, t) \\ &\geq_L \phi(\begin{matrix} \min\{\mathcal{M}(Ia, T_{\alpha_0}a, kt), \mathcal{M}(Ja, S_\lambda a, kt), \mathcal{M}(Ia, Ja, kt)\} \\ \max\{\mathcal{M}(Ia, S_\lambda a, kt), \mathcal{M}(Ja, T_{\alpha_0}a, kt)\} \end{matrix}) \\ &= \phi(\mathcal{M}(a, S_\lambda a, kt)) >_L \mathcal{M}(a, S_\lambda a, kt), \end{aligned}$$

is a contradiction. Hence for every $\lambda \in B$ we have $S_\lambda(a) = a = I(a) = J(a)$. Similarly for every $\gamma \in A$ we get $T_\gamma(a) = a$. Therefore for every $\gamma \in A, \lambda \in B$

we have

$$T_\gamma(a) = S_\lambda(a) = I(a) = J(a) = a.$$

□

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