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Application of Adomian Decomposition Method to Study Heart Valve Vibrations

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Abstract

The objective of this paper is to solve the equation of motion of semilunar heart valve vibrations. The vibrations of the closed semilunar valves were modeled with a Caputo fractional derivative of order α. With the help of Laplace transformation, closed-form solution is obtained for the equation of motion in terms of Mittag- Leffler function. We obtained the analytical solution for the nonlinear fractional differential equation using Adomian decomposition method. The simplicity of these solutions makes them ideal for testing the accuracy of numerical methods. These solutions can be of some interest for a better fit of experimental data.

Mathematics Subject Classifications: 51N20, 62J05, 70F99

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1 Introduction

The heart valves are mechanical devices that permit the flow of blood in one direction only. Four sets of valves are of importance to the normal functioning of the heart. Two of these, the atrioventricular valves, guard the opening between the atria and the ventricles. The other two heart valves, the semilunar valves, are located where the pulmonary artery and the aorta arise from right and left ventricles, respectively [18].

The semilunar valves consist of half-moon shaped flaps growing out from the lining of the pulmonary artery and aorta. When these valves are closed, blood fills the spaces between the flaps and the vessel wall. Each flap then looks like a tiny, filled bucket. Inflowing blood smoothes the flaps against the blood vessel walls, collapsing the buckets and thereby opening the valves. Closure of the semilunar valves simultaneously prevents back flow and ensures forward flow of blood in places where there would otherwise be considerable back flow [18].

Wiggers (1915) suggested that there is silent approximation of the semilunar valves and that after vibrations of the closed valve and the column of blood cause the second heart sound. Stein (1981) supported the theory of silent valve closure and suggested that the valvular vibrations are a sufficient cause for production of the second heart sound [9].

Following coaption of the valve leaflets, a pressure difference is developed across the closed valve during the isovolmic relaxation phase, and the leaflets distend slightly toward the ventricle. This causes a pressure reduction within the blood. Subsequent recoil of the valve compresses the blood in the aorta. Vibrations of the leaflets gradually diminishes due damping. The leaflet vibration produces transient pressure gradient in the surrounding blood medium, which subsequently causes vibration of contiguous structures, which are transmitted to the chest wall where they are recognized as audible heart sounds [9].

Blick [2] modeled the aortic valve as a circular membrane of radius a. In this model the aortic valve is assumed to be an elastic, homogeneous membrane secured around a circular edge and undergoing a parabolic displacement. The valve close due to the intraventricular pressure falling below that of the aortic pressure. The force that drives the closed valve to vibrate is the pressure difference Δp that occurs across the valve. The differential equation of motion for forced and damped vibrations of a membrane at any time t, with one-degree-of-freedom equation was expressed as [2]:

$$D_t^2 x(t) + \frac{c}{m} D_t x(t) + \frac{k}{m} x(t) = f(t), \tag{1}$$

where m, c and k represent the effective mass of vibration, the damping force coefficient, and stiffness factor, respectively. $f(t) = \frac{\Delta p \ \pi a^2}{m}$ is an external force. Gotolov [6] introduced the nonlinear model of semilunar heart valve vibrations as

$$D_t^2 x(t) + \frac{c}{m} D_t x(t) + \frac{k}{m} x(t) + \frac{\lambda}{m} x^3(t) = f(t), \tag{2}$$

where λ represents the nonlinear parameter. The fractional derivative approach provides a powerful tool for modeling systems with damping materials. The model based on fractional derivatives has been shown to be one of the most effective approaches [16, 17]. If the fractional derivative model is used to represent the damping characteristic, the equations of motion (1) and (2) assume the form:

$$D_t^2 x(t) + \frac{c}{m} D_t^{\alpha} x(t) + \frac{k}{m} x(t) = f(t), \tag{3}$$

$$D_t^2 x(t) + \frac{c}{m} D_t^\alpha x(t) + \frac{k}{m} x(t) + \frac{\lambda}{m} x^3(t) = f(t), \tag{4}$$

The operator D_t of the conventional model was replaced by D_t^α. There are several definitions of derivative of fractional order. The fractional operator D_t^α based on Riemann-Liouville integral is defined as [7,13]:

$$D_t^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{x(u)}{(t-u)^{\alpha-n+1}} du, \qquad n-1 < \alpha < n$$

where n is an integer number and Γ is the gamma function. Also the Caputo's definition can be written as[7, 13]:

$$D_t^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{x^{(n)}(u)}{(t-u)^{\alpha-n+1}} du, \qquad n-1 < \alpha < n.$$

The main objective of the present paper is the mathematical study and the using of an easy method to obtain closed-form solution for the equation of motion for general value of α for linear case and also to obtain approximate solution to the nonlinear fractional calculus model of the semilunar heart valve vibrations.

2 Adomian decomposition method

The decomposition method does not change the problem into a convenient one for use of linear theory. It therefore provides more realistic solutions. It provides series solutions which generally converge very rapidly in real physical problems. When the solutions are computed numerically, the rapid convergence is obvious. The advantage of the decomposition method relies on the fact that it provides an easily computable scheme and an efficient algorithm. It is well known that the decomposition method decompose the linear term $u(x,t)$ into an infinite sum of components $u_n(x,t)$ defined by [19]

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t).$$

Moreover, the decomposition method identifies the nonlinear term $N(u(x,t))$ by decomposition series

$$N(u(x,t)) = \sum_{n=0}^{\infty} A_n,$$

where A_n are the so-called Adomian polynomials. These polynomials can be calculated for all forms of nonlinearity according to specific algorithms constructed and given in . In this specific nonlinearity, we use the general form of formula for A_n Adomian polynomials as [19]

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N(\sum_{n=0}^{\infty}\lambda^n u_n)\right]_{\lambda=0}, n \geq 0.$$

This formula is easy to set computer code to get as many polynomials as we need in the calculation of the numerical as well as explicit solutions. The first few polynomials in our nonlinearity $N(u) = u^3$ are given by $A_0 = u_0^3$, $A_1 = 3u_0^2 u_1$, $A_2 = 3u_0^2 u_2 + 3u_1^2 u_0$, and so on, the rest of the polynomials can be constructed in a similar manner.

3 Analysis

One of the most efficient and elegant methods to solve Eq(2) is by means of the Laplace transform. The formula for Laplace Transform of Riemann-Liouville fractional derivative is:

$$\int_0^{\infty} e^{-st}\ D_t^{\alpha}x(t)dt = s^{\alpha}x(s) - \sum_{j=0}^{n-1} s^j D_t^{\alpha-j-1}x(0), (n-1 < \alpha < n).$$

where $x(s)$ is the Laplace transform of $x(t)$. The Laplace transform of the Riemann-Liouville fractional derivatives is well known. However, its practical applicability is limited by the absence of the physical interpretation of the limit values of fractional derivatives at $t = 0$. The formula for Laplace Transform of Caputo fractional derivative is

$$\int_0^{\infty} e^{-st}\ D_t^{\alpha}x(t)dt = s^{\alpha}x(s) - \sum_{j=0}^{n-1} s^{\alpha-j-1} D_t^{j}x(0), (n-1 < \alpha < n).$$

Since this formula for the Laplace transform of the Caputo derivative involves the values of the function $f(t)$ and its derivatives at $t = 0$, for which a certain physical interpretation exists, we can expect that it can be useful for solving applied problems leading to linear fractional differential equations with constant coefficients with accompanying initial conditions in traditional form [13, p. 106].

Applying the Laplace transform to equation(3) we obtain:

$$x(s) = \frac{f(s)}{s^2 + \frac{c}{m}s^{\alpha} + \frac{k}{m}}, \tag{5}$$

where $x(s)$ and $f(s)$ are the Laplace transform of $x(t)$ and $f(t)$ respectively. Eq (5) can be written in the form:

$$x(s) = \sum_{j=0}^{\infty} \frac{(-1)^j (\frac{k}{m})^j s^{-\alpha(j+1)} f(s)}{(s^{2-\alpha} + \frac{c}{m})^{j+1}}. \tag{6}$$

Using the convolution theorem, the inversion of Eq(6) takes the form:

$$x(t) = \int_0^t G(t-u) f(u) du, \tag{7}$$

where

$$G(t) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} (\frac{k}{m})^j \; t^{2j+1} E^j_{2-\alpha, 2+\alpha j}(-\frac{c}{m} \; t^{2-\alpha}), \tag{8}$$

and $E_{\alpha,\beta}$ is the two-parameter function of the Mittag-Leffler type defined by the series expansion[13]:

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}.$$

Equation(8) can be written in the form:

$$G(t) = \sum_{j=0}^{\infty} (-1)^j (\frac{k}{m})^j \sum_{i=0}^{\infty} \frac{(i+j)!}{j! \; i!} (-1)^i (\frac{c}{m})^i \frac{t^{2j+2i-\alpha i+1}}{\Gamma(2j+2i-\alpha i+2)}, \tag{9}$$

Measurements during catheterization of the instantaneous pressure gradient across the semilunar valve during diastole indicate that, to the first approximation, the pressure gradient increases linearly with time until a time t_1 is reached following which the pressure gradient remains essentially constant[2]. The deflection of the centerline for time $t < t_1$ can be expressed as

$$x(t) = \frac{\pi a^2 A}{m} \sum_{j=0}^{\infty} (-1)^j (\frac{k}{m})^j \sum_{i=0}^{\infty} \frac{(i+j)!}{j! \; i!} (-1)^i (\frac{c}{m})^i \frac{t^{2j+2i-\alpha i+3}}{\Gamma(2j+2i-\alpha i+4)}, \tag{10}$$

where $A = \frac{d\Delta p}{dt}$ and the deflection of the centerline for $t > t_1$ is given by

$$x(t) = \frac{\pi a^2 B}{m} \sum_{j=0}^{\infty} (-1)^j (\frac{k}{m})^j \sum_{i=0}^{\infty} \frac{(i+j)!}{j! \; i!} (-1)^i (\frac{c}{m})^i \frac{t^{2j+2i-\alpha i+2}}{\Gamma(2j+2i-\alpha i+3)}. \tag{11}$$

where $B = 150$ is the constant value of Δp at $t > t_1$ and $t_1 = 0.0175$ see[9].

The velocity of the centerline deflection of the membrane can be obtained by differentiation of equations (8) and (9). Instead of approximating the pressure difference occurring across the valve by a ramp function, Mazudard [8] used an exponential function $\Delta p = C(1 - e^{-b\;t})$ as a better approximation to the physiological pressure gradient. The deflection of the centerline in this case can be expressed as

$$x(t) = \frac{\pi a^2 C}{m} \sum_{j=0}^{\infty} (-1)^j (\frac{k}{m})^j \sum_{i=0}^{\infty} \frac{(i+j)!}{j! \; i!} (-1)^i (\frac{c}{m})^i \frac{F(t)}{\Gamma(2j+2i-\alpha i+2)}, \tag{12}$$

where

$$F(t) = t^{-i\alpha}(-b)^{-2(i+j)}(\frac{t^2(-b\ t)^{2(i+j)}}{2j+2i-\alpha i+2}+$$
$$\frac{e^{-t\ b}(-t\ b)^{i\alpha}(\Gamma(2j+2i-\alpha i+2)+\Gamma(2j+2i-\alpha i+2,-t\ b))}{b^2}).$$

The power series in equations (10), (11) and (12) embodies the deflection of the centerline of the heart valve.

The value $\alpha = \frac{1}{2}$ was adopted in this section because it has been shown that it describes that it describes the frequency dependence of the damping materials quite satisfactorily [8]. The equation of motion (3) can be written in the form

$$D_t^2 x(t) + 2\eta\omega^{\frac{3}{2}} D_t^{\frac{1}{2}} x(t) + \omega^2 x(t) = f(t),$$

where $2\eta\omega^{\frac{3}{2}} = \frac{c}{m}$ and $\omega^2 = \frac{k}{m}$. The coefficient η is the damping ratio of an oscillator with fractional damping of order $\frac{1}{2}$ and ω is the natural frequency. The exponent $\frac{3}{2}$ was introduced for consistency of dimensions.

4 Nonlinear fractional calculus model

Equation (4) can be written in the form:

$$D_t^2 x(t) = f(t) - \frac{c}{m} D_t^{\alpha} x(t) - \frac{k}{m} x(t) - \frac{\lambda}{m} x^3(t). \tag{13}$$

We adopt Adomian decomposition method for solving equation (13). Applying the inverse operator L^{-1} on both sides of (13), we obtain

$$x(t) = x(0) + t\,x'(0) + L^{-1} f(t) - L^{-1}\left(\frac{c}{m} D_t^{\alpha} x(t) + \frac{k}{m} x(t) + \frac{\lambda}{m} x^3(t)\right). \tag{14}$$

We assume that $x(t) = x_0(t) + x_1(t) + x_2(t) + ...$, to be the solution of Eq(14).When $f(t) = A$, using Adomian decomposition method we get

$$x_0(t) = \frac{A\ t^2}{2} \tag{15}$$

$$x_{n+1}(t) = -L^{-1}\left(\frac{c}{m} D_t^{\alpha} x_n(t) + \frac{k}{m} x_n(t) + \frac{\lambda}{m}\sum_{n=0}^{\infty} A_n\right). \tag{16}$$

In view of (15), (16) we get

$$x_1(t) = -\frac{cAt^{4-\alpha}}{m\Gamma(5-\alpha)} - \frac{kAt^4}{m\Gamma(5)} - \frac{90\lambda A^3 t^8}{\Gamma(9)},$$

and

$$x_2(t) = \frac{Ak^2t^6}{720m^2} + \frac{A^3kt^{10}\lambda}{2688m} + \frac{3A^5t^{14}\lambda^2}{326144} + \frac{Ac^2\ t^{6-2\alpha}}{m^2\Gamma(7-2\alpha)} + \frac{2Ac\ t^{6-\alpha}}{m^2\Gamma(7-\alpha)}$$

$$+\frac{90A^3c\ t^{10-\alpha}\lambda}{m\Gamma(11-\alpha)} + \frac{3A^3c\ t^{10-\alpha}\lambda}{4m(90-19\alpha+\alpha^2)\Gamma(5-\alpha)},$$

other components are determined similarly.

The input pressure gradient was taken to be a ramp function where Δp increased 8570 mm Hg/sec for time less than $t_1 = 0.0175$ sec. The value of Δp was constant at 150 mm Hg per sec for time equal to or greater than 0.0175. The effective stiffness factor for the membrane was assumed to be equal to the measured static value of k which was 6.8×10^6 dyn/cm,the damping force factor $D = 2.8 \times 10^3$ dyn sec cm^{-2} and the effective mass $m = 195$ g[9].

5 Results

The theoretical curve for the membrane vibration is plotted in [2, 8, 9]. It can be seen that the calculated curve has a shape similar to the curve calculated from experimental observations [4]. The initial peak deflection were approximately equal. The peak rise times differed by about 0.01 sec. There was also a similarity between the rates of deflection. Since the fractional derivatives model approximates the physical models more closely than other models[16], it would be expected that the results using the fractional derivatives model would be closer to the actual physiological values for deflection and rate of deflection of the centerline. The simplicity of the present solutions makes them ideal for testing the accuracy of numerical methods. The stiffness factor, k, in valve vibration and sound production explains the reason for an increased amplitude of the pulmonary component of the second sound relative to the aortic component in pulmonary hypertension. Another important factor is the effect of the viscosity factor, c, on the valve vibration. Since anemic patients have reduced viscosities, they are shown to have augmented heart sounds[9]. Also, from the relation $\omega = \sqrt{k/m}$, it is clear that the higher the stiffness of the valve leaflet, the higher will be the natural frequency of vibration. Hence, patients with aortic stenosis can be expected to have higher frequency of the aortic component of the second sound. On the other hand, the increased mass of valve leaflet due to calcification will have only a little effect upon the frequency or amplitude of the second sound because the effective mass, m, appearing in the expression of ω consists largely of the fluid surrounding he valve leaflets[9]. We hope that the above analysis helps to explain some previously unexplained clinical observations by means of factors that are identified to relate to valve vibration and heart sounds that result from it.

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