

# Numerical Solution of an Inverse Diffusion Problem

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## **Abstract**

In this paper, we propose an algorithm for numerical solving an inverse nonlinear diffusion problem. The algorithm is based on the linearized nonlinear terms by Taylor's series expansion, removed the time-dependent terms by Laplace transform, and so, the results at a specific time can be calculated without step-by-step computations in the time domain. Finite difference technique used for discretize problem. In additional, the least-squares scheme is proposed to correct diffusion coefficient. In the present study, the expression of diffusion coefficient is unknown a priori. To show the efficiency and accuracy of the present method a test problem will be studied.

**Mathematics Subject Classification:** 35R30

**Keywords:** Nonlinear inverse diffusion problem, Finite difference method, Least-squares method

## 1 Introduction

Inverse heat conduction problems are encountered in various branches of science and engineering. Mechanical, aerospace, chemical engineers, mathematicians, astrophysicists, geophysicists, statisticians, and specialists of many other disciplines are all interested in inverse problems, each with different applications in mind. In the field of heat transfer, the use of inverse analysis for the determination of thermal properties such as thermal conductivity of solid by utilizing the transient temperature measurements taken within the medium has numerous practical applications. The governing heat conduction equation and conditions become:

For  $T > 0$  let  $Q_T = \{(x, t) : x \in (0, 1), t \in (0, T)\}$ , and seek both the function  $u(x, t)$  and  $a(u(x, t)) > 0$  for the initial boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( a(u) \frac{\partial u}{\partial x} \right), \quad 0 < x < 1, 0 < t < T, \quad (1)$$

$$u(x, 0) = \varphi(x), \quad 0 < x < 1, \quad (2)$$

$$-a(u(0, t)) \frac{\partial u(0, t)}{\partial x} = g_0(t), \quad 0 < t < T, \quad (3)$$

$$-a(u(1, t)) \frac{\partial u(1, t)}{\partial x} = g_1(t), \quad 0 < t < T, \quad (4)$$

$$u(0, t) = f(t), \quad 0 < t < T, \quad (5)$$

where  $\varphi(x)$ ,  $g_0(t)$ ,  $g_1(t)$  and  $f(t)$  are continuous known functions. We consider the problem (1)-(4) as a direct problem, where  $a(u)$  will be determined from the overspecified data (5). It is evident that if  $a(u)$  is given, then the problem (1)-(5) is overdetermined, i.e., for arbitrary data  $\varphi(x)$ ,  $g_0(t)$ ,  $g_1(t)$  and  $f(t)$  there may be no function  $u(x, t)$  such that all of the conditions (1)-(5) are satisfied. On the other hand, for any given coefficients, there will exist a unique solution  $u(x, t)$  of the direct problem. For an unknown function  $a(u)$  we must therefore provide additional information (5) to provide a unique solution  $(u, a(u))$  to the inverse problem (1)-(5), [1]. Inverse problems and nonlinear inverse problems including equation (1) have been previously treated by many authors who considered certain special case of this type of problem [1-8].

**Theorem 1.1** *If  $\varphi(x)$ ,  $g_0(t)$  and  $g_1(t)$  are continuous functions, then the problem (1)-(4) has a unique solution.*

**Proof.** See Refs. [1].

*For an unknown function  $a(u)$  we must therefore provide additional information (5) to provide a unique solution  $(u, a(u))$  to the inverse problem (1)-(5).*

**Theorem 1.2** *The problem (1)-(5) has a unique solution if  $\varphi(x), g_0(t)$  and  $g_1(t)$  are continuous functions,  $\varphi(0) = f(0)$  and  $a(u)$  satisfies the following conditions:*

- I)  $a(s) \in C[R_1, R_2]$ ,  
 where  $R_1 = \min u(x, t), R_2 = \max u(x, t)$ .
- II)  $a(s) > 0$ .

**Proof.** See Refs. [1].

## 2 Overview of the method

The application of the present numerical method will find a solution of problem (1)-(5), by using the following steps:

step 1) Linearized the nonlinear terms in equations (1), (3) and (4) by used Taylor’s series expansion. Therefore, we obtain

$$\frac{\partial}{\partial x} \left( a(u) \frac{\partial u}{\partial x} \right) = a(\bar{u}) \frac{\partial^2 u}{\partial x^2}, \tag{6}$$

where  $\bar{u} = ( \bar{u}_0, \bar{u}_1, \dots, \bar{u}_N )$  denotes the previously iterated solution. Similarly

$$-a(u(0, t)) \frac{\partial u(0, t)}{\partial x} = -a(\bar{u}(0, t)) \frac{\partial u(0, t)}{\partial x}. \tag{7}$$

and

$$-a(u(1, t)) \frac{\partial u(1, t)}{\partial x} = -a(\bar{u}(1, t)) \frac{\partial u(1, t)}{\partial x}. \tag{8}$$

step 2) Removed time dependent terms using the Laplace transform. The Laplace transform of a real function  $\zeta(t)$  and its inversion formula are defined as

$$\tilde{\zeta}(s) = \mathcal{L}(\zeta(t)) = \int_0^\infty \exp(-st) \zeta(t) dt,$$

and

$$\zeta(t) = \mathcal{L}^{-1}(\tilde{\zeta}(s)) = \frac{1}{2\pi i} \int_{\nu-i\infty}^{\nu+i\infty} \exp(st) \tilde{\zeta}(s) ds,$$

where  $s = \nu + i\omega, \nu, \omega \in R$ . The Laplace transform of equations (6),(7), and (8) give

$$a(\bar{u}) \frac{\partial^2 \tilde{u}}{\partial x^2} = s\tilde{u} - p(x), \quad 0 < x < 1, \tag{9}$$

$$-a(\bar{u}) \frac{\partial \tilde{u}}{\partial x} = G_0(s), \quad x = 0, \tag{10}$$

$$-a(\bar{u}) \frac{\partial \tilde{u}}{\partial x} = G_1(s), \quad x = 1, \tag{11}$$

where  $\tilde{u}$ ,  $\frac{\partial \tilde{u}}{\partial x}$ ,  $\frac{\partial^2 \tilde{u}}{\partial x^2}$ ,  $G_0(s)$  and  $G_1(s)$  are Laplace transform of  $u$ ,  $\frac{\partial u}{\partial x}$ ,  $\frac{\partial^2 u}{\partial x^2}$ ,  $g_0(t)$  and  $g_1(t)$  respectively.

step 3) In this step, we use central finite difference approximation for discretizing problem (9)-(11). Therefore

$$a(\bar{u}_\mu) \frac{\tilde{u}_{\mu+1} - 2\tilde{u}_\mu + \tilde{u}_{\mu-1}}{h^2} = s\tilde{u}_\mu - p(\mu h), \quad \mu = 0, 1, \dots, N, \quad (12)$$

$$-a(\bar{u}_0) \frac{\tilde{u}_1 - \tilde{u}_{-1}}{2h} = G_0(s), \quad x = 0, \quad (13)$$

$$-a(\bar{u}_N) \frac{\tilde{u}_{N+1} - \tilde{u}_{N-1}}{2h} = G_1(s), \quad x = 1. \quad (14)$$

Problem (12)-(14) may be written in the following matrix form

$$A\tilde{U} = B. \quad (15)$$

Note that equation (15) is a linear equation.

step 4) The Gaussian elimination algorithm is used to solve  $\tilde{U}$ .

step 5) The numerical inversion of the Laplace transform technique is applied to invert the transformed result to the physical quantity  $U^t = (u_0 \ u_1 \ \dots \ u_N)$ .

step 6) The unknown function  $a(u)$  approximated as

$$a(u) = a_0 + a_1 u + a_2 u^2 + \dots + a_q u^q, \quad (16)$$

where  $a_0, a_1, \dots, a_q$  are constants which remain to be determined simultaneously.

step 7) To minimize the sum of the squares of the deviations between  $u_0(t)$  (calculated) and  $f(t)$ , at the specific times  $t = t_r$ , we use least squares method. The error in the estimate is

$$E(a_0, a_1, \dots, a_q) = \sum_{j=0}^N (u_0(t_j) - f(t_j))^2, \quad (17)$$

which remain to be minimized. The estimated values of  $a_i$  are determined until the value of  $E(a_0, a_1, \dots, a_q)$  is minimum.

### 3 Numerical experiment

In this section, we are going to demonstrate some numerical results for diffusion coefficient in the inverse problem (1)-(5). All the computations are performed on the PC. However, to further demonstrating the accuracy and efficiency of this method, the present problem is investigated and one example is illustrated.

**Example 3.1** *In this example, let us consider the following inverse nonlinear diffusion problem*

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( a(u) \frac{\partial u}{\partial x} \right), \quad 0 < x < 1, 0 < t < T, \quad (18)$$

$$u(x, 0) = x, \quad 0 < x < 1, \quad (19)$$

$$-a(u(0, t)) \frac{\partial u(0, t)}{\partial x} = -1 - t, \quad 0 < t < T, \quad (20)$$

$$-a(u(1, t)) \frac{\partial u(1, t)}{\partial x} = -2 - t, \quad 0 < t < T, \quad (21)$$

$$u(0, t) = t, \quad 0 < t < T. \quad (22)$$

*For solving this problem the diffusion coefficient  $a(u)$  define as the following form*

$$a(u) = a_0 + a_1 u,$$

*For determine  $a_0$  and  $a_1$  we use*

$$E(a_0, a_1) = \sum_{j=0}^N (u_0(t_j) - f(t_j))^2,$$

*therefore the coefficients can be obtained. The above procedures are repeated until*

$$\left( \sum_{j=0}^N (e_j)^2 \right)^{1/2} \leq \varepsilon,$$

*where  $\varepsilon = 0.01$ , where*

$$e_j = u_0(t_j) - f(t_j), \quad j = 0, 1, \dots, N, \quad (23)$$

*Tables 1 and 2 show the values of  $U^t$  in  $x = ih$  and  $t = jk$ . The estimated values of  $a_0$  and  $a_1$  are  $a_0 = 1.0446$  and  $a_1 = 1.0356$ .*

	Numerical	Exact	Numerical	Exact
$i$	$u_{i,1}$	$u_{i,1}$	$u_{i,2}$	$u_{i,2}$
0	0.012356	0.010000	0.021102	0.020000
1	0.202267	0.210000	0.226587	0.220000
2	0.400156	0.410000	0.421451	0.420000
3	0.600023	0.610000	0.601455	0.620000
4	0.800085	0.810000	0.802174	0.820000

Table 1.

	Numerical	Exact	Numerical	Exact
$i$	$u_{i,3}$	$u_{i,3}$	$u_{i,4}$	$u_{i,4}$
0	0.029862	0.030000	0.038953	0.040000
1	0.228626	0.230000	0.256277	0.240000
2	0.428145	0.430000	0.424452	0.440000
3	0.619943	0.630000	0.602843	0.640000
4	0.805731	0.830000	0.809527	0.840000

Table 2.

## 4 Conclusion

The present study, successfully applies the numerical method involving the Laplace transform technique and the finite difference method in conjunction with the least-squares scheme to a nonlinear inverse problem. Owing to the application of the Laplace transform, the present method is not a time-stepping procedure. Thus the unknown diffusion coefficient at any specific time can be predicted without any step-by-step computations from  $t = t_0$ . We also apply other different sets of the initial guesses, such as  $\{a_0, a_1, \dots, a_q\} = \{0.2, 0.2, \dots, 0.2\}$ ,  $\{0.7, 0.7, \dots, 0.7\}$  and  $\{1.1, 1.1, \dots, 1.1\}$ , results show that the effect of the initial guesses on the accuracy of the estimates is not significant for the present method.

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