Applied Mathematical Sciences, Vol. 3, 2009, no. 40, 1991 - 2001

Flows Driven by a Combination of Source/Sink

Part 1: Exterior Creeping Flows

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Abstract

In this paper, the two-dimensional creeping flows outside a circular cylinder induced by source/sink with various boundary conditions is studied.
Analytical solutions for the flow field are obtained by straight forward application of the Fourier method. The streamline patterns are sketched for a number of special cases where the boundary conditions is varying from no slip to perfect slip boundary conditions. Some interesting flow patterns are observed in the parameter space which may have potential significance in studies of various flows including flows in journal bearing, mixing flows, etc.
We also investigate into the way the streamline topologies change as the parameters are varied.

Mathematics Subject Classification: 76D07, 76D99

1 Introduction

The study of separation in low Reynolds numbers hydrodynamics is a widely investigated subject due to its relevance to engineering ([3] and [4]) and physiological [8] applications. Separation in slow viscous flow at zero Reynolds number (creeping flows) is significant in pertaining to the application of fluid theory because if it occurs in creeping flows then it is almost certain to persist at non-zero Reynolds numbers. Support of this phenomenon can be found in [10]. Therefore, the study of creeping flows is valuable for understanding the onset of separation in low Reynolds number hydrodynamics. It has been observed in low Reynolds number flow that some solid particles in the fluid are trapped by an attached singularity and cannot be flushed out of the corresponding separation region. Hence, they may eventually adhere to points on

the boundary where separation occurs. Examples of this are the entrapment of dirt in bearings and the formation of plaque in blood arteries.
The study of slow viscous flow with a rough surface is important in the development of low Reynolds number hydrodynamics due to the fact that in reality it is impossible to have a perfectly smooth surface immersed in an actual fluid motion. There are industrial and biomedical applications in the study of fluid flow with a rough surface. The industrial applications include the cleansing of a rough surface which has been exposed to contaminants, and the effect of roughness on lubrication. One of the important engineering applications is the study of the gas cetrifuge, as used for separation of uranium isotopes (see for example [6], [7]).
The paper is organized as follows. In section 2, the basic equations are given and using the Stokes stream function the problem is reformulated. The boundary conditions are then derived in terms of stream function and a brief discussion on the no-slip constraints is provided. In section 3, the general solution is derived by the use of Fourier expansion method. The solutions to various singularity driven flow problems are presented in section 4 and 5. The basic singularities considered here are source and sink. The effect of boundary conditions on the flow fields is discussed in each case. The flow description is illustrated in different situations through streamline plots. The effect of primary singularity locations on the fluid velocity on the surface is discussed briefly. The concluding remarks of the present analysis are presented in section 6.

2 Mathematical preliminaries

For two-dimensional steady and slow viscous flow the (non-dimensional) governing equation of motion are

$$R(\mathbf{q}.\nabla)\mathbf{q} = -\nabla p + \nabla^2\mathbf{q}, \tag{1}$$

$$\nabla.\mathbf{q} = 0, \tag{2}$$

where $\mathbf{q}$ is the velocity field, p is the pressure field, $R = \frac{UL}{\nu}$ is the Reynolds number. The velocity vector in Navier-Stokes equations (1) and continuity equation (2) is subjected to the no-slip condition $\mathbf{q} = \mathbf{0}$ on a rigid fixed boundary. Now if the ratio between the magnitudes of the convection and diffusion terms in the Navier-Stokes equations is bounded within the flow region, then as the Reynolds number approaches zero (1) and (2) becomes

$$\nabla^2\mathbf{q} = \nabla p. \tag{3}$$

$$\nabla.\mathbf{q} = 0. \tag{4}$$

Any solution of (3) and (4) for p and $\mathbf{q}$ is called a creeping flows.
The two-dimensional flows discussed in this paper can be described by a single scalar function. If the fluid motion lies in a plane descibed by the Cartesian coordinates (x, y) then the continuity equation (2) is satisfied by representing the velocity field $\mathbf{q}$ as

$$\mathbf{q} = curl(\psi \mathbf{k}), \tag{5}$$

where $\mathbf{k}$ is the unit vector normal to the xy-plane and $\psi = \psi(x, y)$ is a function of x and y called the stream function. By substituting (5) into (3) and eleminating the pressure field, the biharmonic equation is obtained as

$$\nabla^4 \psi = 0, \tag{6}$$

The boundary conditions can be expressed as

(i). Normal velocity is zero on the boundary i.e., $q_r = 0$ on $r = a$.

(ii). Tangential velocity is proportional to the tangential stress at the surface of the cylinder i.e., $q_\theta = \frac{\lambda}{\mu} T_{r\theta}$, where

$$T_{r\theta} = \mu[\frac{1}{r}\frac{\partial q_r \partial}{\partial \theta} + r\frac{\partial}{\partial r}\frac{q_\theta}{r}], \tag{7}$$

Is the tangential stress and $\lambda \geq 0$ is the slip coefficient. In terms of stream function, these conditions become

$$\psi = 0 \quad \texttt{and} \quad \frac{\partial \psi}{\partial r} = \lambda r \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial \psi}{\partial r}, \quad \texttt{on} \quad r = 1. \tag{8}$$

In the present paper, we investigate the flows induced by a combination of line source and sink in the presence of a circular cylinder.

3 Method of solution

There are classical and numerical methods to solve (6) subject to the boundary conditions (8). The most suitable and commonly used technique is the Fourier expansion method. In this method, the given flow is expanded in a Fourier series with known Fourier coefficients. Then, the unknown coefficients of the perturbed flow (also written in a Fourier series) are computed with the aid of the boundary conditions. The other methods applicable to the present problem include the image method ([2], [1]) and the boundary integral equation method [9]. We adopt the Fourier representation technique for our problem and make an attempt to sum the resulting series solution. To this end, we write the given flow field in the absence of the cylinder as

$$\psi_0 = \sum_{n=0}^{\infty}[\alpha_n r^n + \beta_n r^{n+2}] f_n(\theta). \tag{9}$$

Here, $f_n(\theta) = a_n \cos n\theta + b_n \sin n\theta$, and α_n, β_n, a_n, b_n are known constants. The solution satisfying (6)in the presence of a circular cylinder can be taken as

$$\psi = \psi_0 + \sum_{n=0}^{\infty}[\frac{A_n}{r^n} + \frac{B_n}{r^{n-2}}] f_n(\theta). \tag{10}$$

where A_n and B_n are unknown constants to be determined. We remark that the constants α_0 and β_0 corresponding to $n = 0$ may be adjusted by choosing appropriate ones. Applying the boundary conditions (8) we obtain

$$\frac{A_n}{a^n} = (-1 + \frac{n}{n\lambda_1 + 1 - \lambda_1})\alpha_n a^n + (-1 + \frac{(n+1)(1-\lambda_1)}{n\lambda_1 + 1 - \lambda_1})\beta_n a^{n+2}. \tag{11}$$

$$\frac{B_n}{a^{n-2}} = -(\frac{n}{n\lambda_1 + 1 - \lambda_1})\alpha_n a^n - (\frac{(n+1)(1-\lambda_1)}{n\lambda_1 + 1 - \lambda_1})\beta_n a^{n+2}. \tag{12}$$

where $\lambda_1 = 2\lambda/(1+2\lambda)$, $0 \leq \lambda_1 \leq 1$. The corresponding coefficients for a rigid cylinder with no-slip and perfect-slip boundary conditions may be obtained from (11) and (12) by setting $\lambda_1 = 0$ and $\lambda_1 \to 1$ respectively

4 Source and sink

We now employ the solution scheme derived in the previous section. Substitution of equations (9), (11) and (12) in (10) yields the Fourier series expansion for the stream function for a source and sink in the presence of a cylinder.
The stream function of the flow induced by a source of strength Γ_1 located at $r = c$, $\theta = 0$ is given by

$$\psi_0 = \Gamma_1 \tan^{-1}\Big(\frac{r\sin\theta}{c - r\cos\theta}\Big). \tag{13}$$

For $r > c$, above expression may be expanded in the form

$$\psi_0 = \Gamma_1 \sum_{n=1}^{\infty} \frac{r^n}{nc^n}\sin(n\theta). \tag{14}$$

From (11) and ((12), we obtain

$$\alpha_n = \frac{\Gamma_1}{nc^n}, \quad \beta_n = \gamma_n = 0 \quad \delta_n = 1 \quad \texttt{for all} \quad n \geq 1. \tag{15}$$

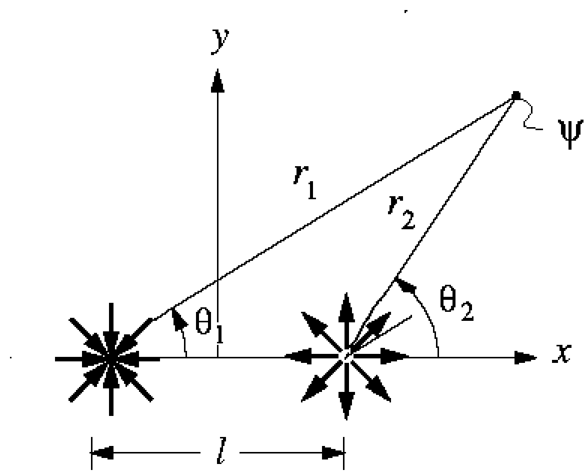

Figure 1: Diagram illustrating the geometry of the source and the sink ($l = c + \widehat{c}$).

The coefficients A_n and B_n for a source flow after the introduction of a cylinder are

$$A_n == \Gamma_1(-1 + \frac{n}{n\lambda_1 + 1 - \lambda_1})\frac{a^{2n}}{nc^n}\alpha_n, \tag{16}$$

$$B_n = -\Gamma_1(\frac{1}{n\lambda_1 + 1 - \lambda_1})\frac{a^{2n-2}}{c^n}\alpha_n. \tag{17}$$

The stream function for the source-sink combination becomes

4.1 no-slip boundary condition

By setting $\lambda_1 = 0$, Fourier series solution with the constants given in (16) and (17) may be summed to yield the following closed form expression:

$$\psi = \Gamma_1[\tan^{-1}(\frac{r\sin\theta}{c - r\cos\theta}) - \tan^{-1}(\frac{\sin\theta}{rc - \cos\theta}) + r(1-r^2)\frac{c\sin\theta}{r^2c^2 - 2rc\cos\theta + 1}]. \tag{18}$$

By using standard techniques for sink of strength Γ_2 at $r = \widehat{c}$ and $\theta = \pi$ we get

$$\psi = \Gamma_2[\tan^{-1}(\frac{r\sin\theta}{\widehat{c} + r\cos\theta}) - \tan^{-1}(\frac{\sin\theta}{r\widehat{c} + \cos\theta}) + r(1-r^2)\frac{\widehat{c}\sin\theta}{r^2\widehat{c}^2 + 2r\widehat{c}\cos\theta + 1}]. \tag{19}$$

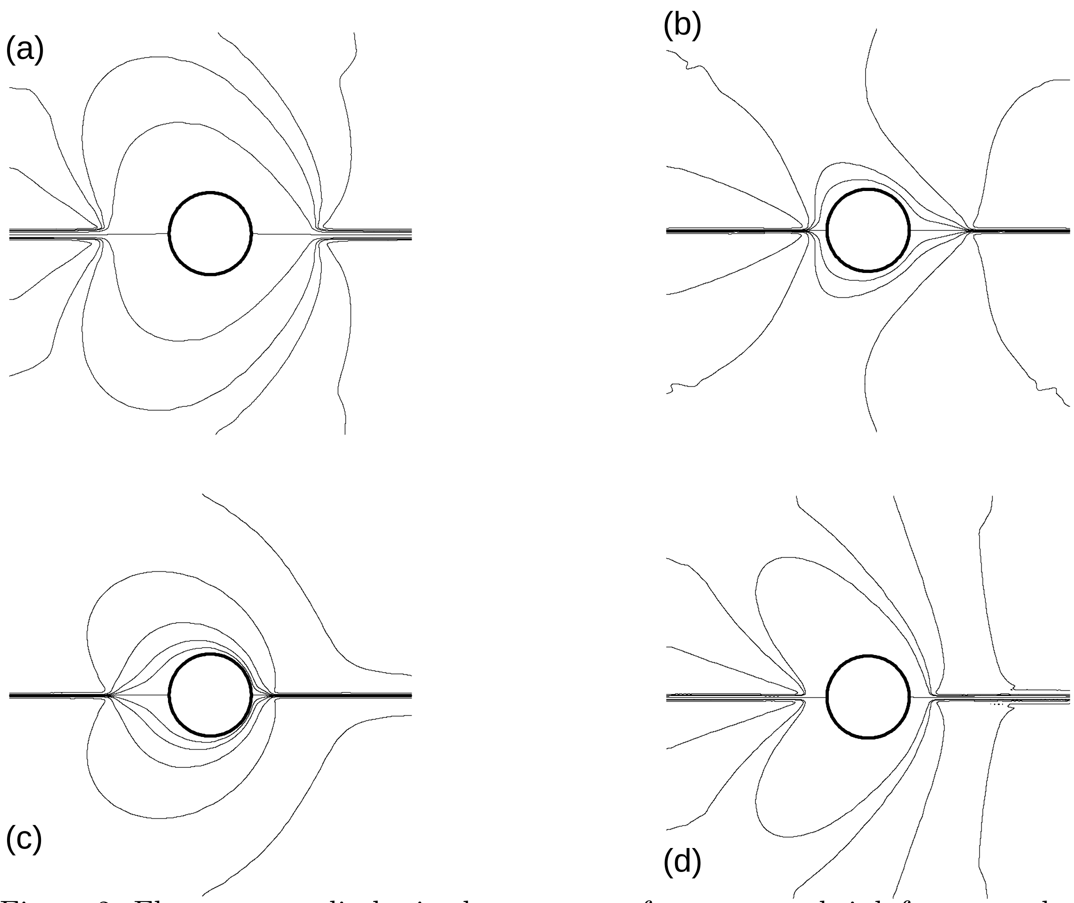

Figure 2: Flow past a cylinder in the presence of a source and sink for strength $\Gamma_1 = -\Gamma_2 = 1$ and $\lambda_1 = 0$ for different position (a)$c = \widehat{c} = 2.5$ (b) $c = 2.5, \widehat{c} = 1.5$(c) $c = 1.5, \widehat{c} = 2.5$ (d) $c = 1.5, \widehat{c} = 1.5$.

Figures 2(a)-(d)show the symmetry which can be simple as reflection across the x-axis. The only difference between (a)-(d) is the shift of the overall phase.

4.2 boundary condition when $\lambda_1 = 0.4,\ 0.5,\ 0.6$.

For general $0 < \lambda_1 < 1$, Fourier series solution with the constants given in (16) and (17) may be summed by standard techniques, to get

$$\psi = \Gamma_1\big[\tan^{-1}(\frac{r\sin\theta}{c - r\cos\theta}) - \tan^{-1}(\frac{\sin\theta}{rc - \cos\theta}) + \frac{1-\lambda_1}{\lambda_1}(1-r^2)I_{si}\big]. \quad (20)$$

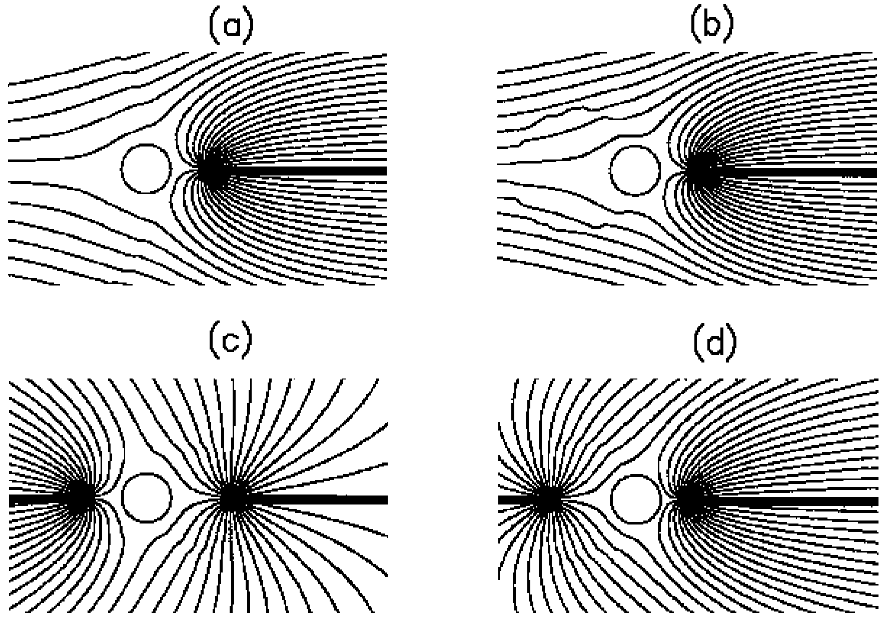

Figure 3: Flow past a cylinder in the presence of (a) a single source at $c = 2.5,\ \Gamma_1 = 1$ and $\lambda_1 = 0.2$ (b) a single source at $c = 2.5,\ \Gamma_1 = 1$ and $\lambda_1 = 0.4$ (c) a source and sink at $c = 3.5,\ \widehat{c} = 2.5,\ \Gamma_1 = -\Gamma_2 = 1$ and $\lambda_1 = 0.6$ (d) a source and sink at $c = 2.0,\ \widehat{c} = 3.5,\ \Gamma_1 = -\Gamma_2 = 1$ and $\lambda_1 = 0.7$.

where

$$I_{si} = r^{\frac{(1-\lambda_1)}{\lambda_1}} \int_0^{1/r} \rho^{\frac{1}{\lambda_1}-2}\big[-\tan^{-1}(\frac{\rho\sin\theta}{c-\rho\cos\theta}) + \frac{\rho\ c\ \sin\theta}{\rho^2 - 2c\rho\cos\theta + c^2}\big]\ d\rho$$

and similarly for sink at $r = \widehat{c}$ and $\theta = \pi$

$$\psi = \Gamma_2\big[\tan^{-1}(\frac{r\sin\theta}{\widehat{c} + r\cos\theta}) - \tan^{-1}(\frac{\sin\theta}{r\widehat{c} + \cos\theta}) + \frac{1-\lambda_1}{\lambda_1}(1-r^2)I_{so}\big]. \qquad (21)$$

where

$$I_{so} = r^{\frac{(1-\lambda_1)}{\lambda_1}} \int_0^{1/r} \rho^{\frac{1}{\lambda_1}-2}\big[-\tan^{-1}(\frac{\rho\sin\theta}{\widehat{c}+\rho\cos\theta}) + \frac{\rho\ \widehat{c}\ \sin\theta}{\rho^2 + 2\widehat{c}\rho\cos\theta + \widehat{c}^2}\big]\ d\rho$$

We see in Figure 3(a)-(b) that the slip parameter has little effect on the qualitative features of the flow fields in the Stokes regime which is weakly depend on the value of λ_1.

In Figures 3, the flows induced by a single source and a combination of source and sink in the presence of a cylinder are shown for various values of λ_1. In the case of single source, expected flow patterns emerge and these do not change qualitatively for different values of the slip parameter (see Figures 3(a)-(b)). In the case of the source located to the right of the cylinder at $(c, 0)$, and the sink is to the left of the cylinder at $(\widehat{c}, \pi)$. Two typical scenarios are shown in Figures 3(c)-(d). The location of the sink is fixed in both of these plots. In Figure 3(c), the sink is closer than the source to the cylinder whereas in Figure 3(d) the source is closer than the sink to the cylinder. It is rather striking that there is no direct transfer of fluid from the source to the sink in the finite plane. This is because the cylinder blocking the flow in Stokes regime. It may be seen from the representative plots Figures 3(c)-(d) that all the fluid expelled from the source goes away to infinity and all that drawn into the sink comes from infinity. Similar features were noticed in the case of no slip boundary. Eddies do not appear in any of these cases as expected.

4.3 perfect-slip boundary condition

By taking the limit $\lambda_1 \rightarrow 1$, in (20) the solution may be reduces to

$$\psi = \Gamma_1 \big[\tan^{-1}\big(\frac{r \sin\theta}{c - r\cos\theta}\big) - \tan^{-1}\big(\frac{\sin\theta}{rc - \cos\theta}\big)\big]. \tag{22}$$

and similarly for sink equation (21)

$$\psi = \Gamma_2 \big[\tan^{-1}\big(\frac{r \sin\theta}{\widehat{c} + r\cos\theta}\big) - \tan^{-1}\big(\frac{\sin\theta}{r\widehat{c} + \cos\theta}\big)\big]. \tag{23}$$

Figure (4) show flow patterns for a perfect-slip boundary condition when a sink and a source are located at distances $x = -2.5$ and $x = 2.5$, respectively. It is found that a symmetric appears with their centers aligned on the $y-$axis. It is shown in Figure (4)(b)-(d) the flow seems to contain a recirculating zone surrounded by fluid flowing from the source to the sink. In figures (4)(c) and (4)(d) if we increase the strength of a source or a sink the eddies pushes towards the cylinder. This result confirm Lord Rayleigh [5] work who found no secondary flow in the case of a source and a sink located diagonally opposite on the surface of a cylinder. One can easily modify this formulae for the cases shown in Figures (2),(3) and (4) and can draw inferences about some of the qualitative features of the various flow patterns directly from this formulae.

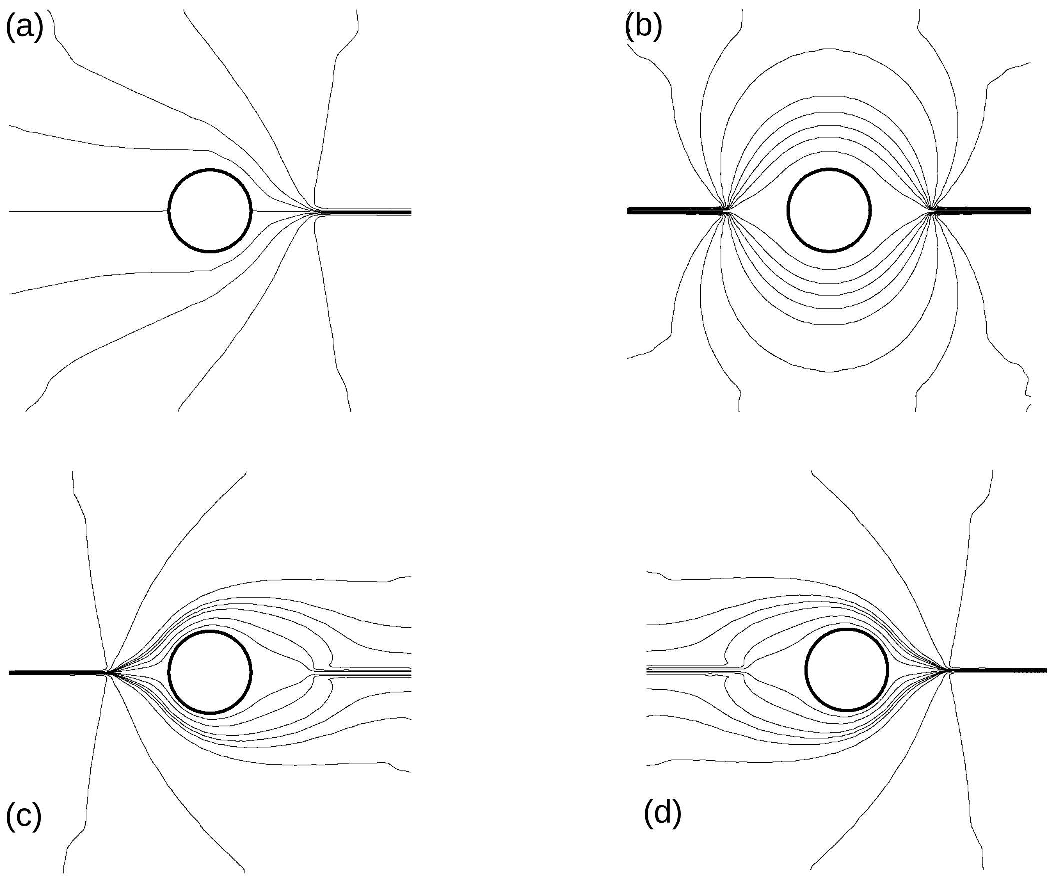

Figure 4: Flow past a cylinder in the presence of a source and sink at $c = \widehat{c} = 2.5$ and $\lambda_1 = 1$ (perfect boundary condition) for different strength (a)$\Gamma_1 = 1, \Gamma_2 = 0$ (b) $\Gamma_1 = -\Gamma_2 = 1$(c) $\Gamma_1 = 1, \Gamma_2 = -5$ (d) $\Gamma_1 = 5, \Gamma_2 = -1$.

5 CONCLUSION

Exact analytical solution are presented for creeping flow induced by source and sink in the presence of a circular cylinder with various boundary conditions. The closed form solution given here in each case contains an integral involving the non-dimensional slip parameter λ_1. The flow fields do not change with the slip parameter also eddies do not appear in any of these cases as expected. Finally the solutions and flow patterns presented here clearly demonstrate the effect of λ_1 on source and sink driven flows.

6 ACKNOWLEDGMENT

This research has been fully supported by Sultan Qaboos University under grant **IG/SCI/DOMS/09/07**.

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Received: January, 2009