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Using a Certain Linear Ranking Function to Measure the Malmquist Productivity Index with Fuzzy Data and Application in Insurance Organization

G. R. Jahanshahloo[1]

Dept. of Math., Teacher Training University, Tehran, Iran

F. Hosseinzadeh Lotfi

Dept. of Math. Science and Research Branch
Islamic Azad University, Tehran 14515-775, Iran

H. Nikoomaram

Dept. of Management, Science and Research Branch
Islamic Azad University, Tehran 14515-775, Iran

M. Alimardani

Dept. of Math. Science and Research Branch
Islamic Azad University, Tehran 14515-775, Iran

Abstract

Data envelopment analysis is a widely applied approach for measuring the relative efficiency of a set of decision making units ($DMUs$) which use multiple inputs and multiple outputs. The DEA- based malmquist productivity index measures the productivity change over time.In this paper we provide an extension to the DEA- based malmquist productivity index for all (DMUs) with fuzzy data.

[1]Corresponding author: Gholamreza Jahanshahloo, E-mail: Jahanshahlo@yahoo.com, Tel: +98-21-77507772, Fax: +98-21-77602981
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1 Introduction

Using linear programming techniques, data envelopment analysis (DEA) (Charnes et al., 1978) provides a suitable way to estimate a multiple inputs / multiple outputs empirical efficient function as described by Farrell (1957).

Fare, et al. (1992, 1994) developed a DEA-based Malmquist productivity index which measures the productivity change over time. The Malmquist index was first suggested by Malmquist (1953) as a quantity index to be used in the analysis of consumption of inputs, Fare, et al. combined ideas on the measurement of efficiency from Farrell and the measurement of productivity from Caves et al.(1982) to construct a Malmquist productivity index.This index has proven itself to be a good tool for measuring the productivity change of DMUs.[10]

The original DEA-based Malmquist index assumes that inputs and outputs are measured by exact values on a ratio scale. However, this assumption may not be true, in the sense that some inputs and outputs may be only known as in forms of bounded or fuzzy data. since the DEA model is essentially a linear programming, one straightforward idea is to apply the existing fuzzy linear programming (FLP) to the fuzzy DEA problems. In this paper we use a new approach for efficiency measurement by Fuzzy linear programming this approach can be used to treat the Malmquist productivity index for DMUs productivity evaluation with fuzzy data.

The most frequently used DEA model is the CCR model, named after Charnes, Cooper and Rhodes (1978). Consider n decision making units DMU_j ,j=1,...,n, which each DMU consumes inputs levels x_{ij}, i=1,...,m, to produce outputs levels y_{rj}, r=1,...,s. Let is $j = \{1, ..., n\}$and suppose that $X_j = (x_{1j}, ..., x_{mj})^T$ and $Y_j = (y_{1j}, ..., y_{sj})^T$ are the vectors of inputs and outputs values respectively, for DMU_j, in which it has been assumed that $X_j \geq 0, X_j \neq 0$ and $Y_j \geq 0, Y_j \neq 0$. The relative efficiency score of the DMU_o, $o \in \{1, ..., n\}$, is obtained from the following model which is called input-oriented CCR envelopment model.

(CCR)

$$
\begin{array}{lll}
\max & \sum_{r=1}^{s} u_r y_{ro} & \\
\text{s.t.} & \sum_{i=1}^{m} v_i x_{io} = 1; & \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 & \text{j=1,...,n} \\
& u_r, v_i \geq 0, & \text{j=1,...,n}
\end{array}
\tag{1.1}
$$

$(DCCR)$

$$
\begin{array}{lll}
\text{Min} & \theta & \\
\text{s.t.} & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{io}, & \text{i=1,...,m} \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro}, & \text{r=1,...,s} \\
& \lambda_j \geq 0, & \text{j=1,...,n}
\end{array}
\tag{1.2}
$$

It can be proven that $0 < \theta^* \leq 1$ and DMU_o is efficient(technical) in the CCR model if and only if $\theta^* = 1$. Otherwise, the DMU_o is inefficient. From the dualty theorem of linear programming, the optimal objective values of the CCR and DCCR model are equal. Let θ^* be the optimal objective value (efficiency value). Using the constraints $V^T X_o = 1$ and $-V^T X + U^T Y \leq 0$ in (CCR),an efficiency value θ^* of the target DMU falls in the rang of (0,1].

2 Preliminaries

Fare et al. (1992) constructed the DEA-based Malmquist productivity index as the geometric mean of two Malmquist productivity indexes of Caves, et al. (1982), which are defined by a distance function D(.). Caves et al. assume $D^k(k) = 1$, that is their distance function does not reveal inefficiency. By allowing for inefficiency and modeling the technology frontier as piecewise linear, Fare, et al. decomposed their Malmquist productivity index into two components, one measuring the change in the efficiency and the other measuring the change in the frontier technology. The frontier technology determined by the efficient frontier is estimated using DEA for a set of DMUs. However, the frontier technology for a particular DMU under evaluation is only represented by a section of the DEA frontier or a facet.

Malmquist productivity index calculation requires two single period and two mixed period measures. The two single period measures can be obtained by using the DCCR model (Charnes et al., 1978)

$$
\begin{aligned}
D_o^p(k) = \quad & \text{Min} \quad \theta \\
& \text{s.t.} \quad \sum_{j=1}^{n} \lambda_j x_{ij}^p \leq \theta x_{io}^k, \quad \text{i=1,...,m} \\
& \qquad \sum_{j=1}^{n} \lambda_j y_{rj}^p \geq y_{ro}^k, \quad \text{r=1,...,s} \\
& \qquad \lambda_j \geq 0, \quad \text{j=1,...,n}
\end{aligned}
\tag{2.3}
$$

where x_{io}^k is the $i-th$ input and y_{ro}^k is the $r-th$ output for DMU_o in time k and x_{ij}^p is the $i-th$ input and y_{rj}^p is the $r-th$ output for DMU_j in time p. The efficiency $(D_o^t(t))$ determines the amount by which observed inputs can be proportionally reduced, while still producing the given output level. Using $t+1$ instead of t for the above method, we get $D_o^{t+1}(t+1)$, the technical efficiency score for DMU_o in time period $t+1$.

The first of the mixed period measures, which is defined as $D_o^t(t+1)$ for each DMU_o, is computed as the optimal value to the (2.3) linear programming problem, where $p = t$ and $k = t+1$.

Similarly, the other mixed period measure, $D_o^{t+1}(t)$, which is needed in the computation of the Malmquist productivity index, is the optimal value to the (2.3) linear problem, where $p = t+1$ and $k = t$.

Fare et al.'s input-oriented Malmquist productivity index, which measures the productivity change of a particular $DMU_o, o \in J = \{1, ..., n\}$, in time $t+1$ and t is given as:

$$
M_o = \left[\frac{D_o^t(t+1)}{D_o^t(t)} \times \frac{D_o^{t+1}(t+1)}{D_o^{t+1}(t)}\right]^{1/2} \tag{2.4}
$$

It can be seen that the above measure actually is the geometric mean of two Caves, et al.'s Malmquist productivity indexes. Thus, following Caves et al.'s suit, Fare et al. define that $M_o > 1$ indicates the productivity gain; $M_o < 1$ indicates the productivity loss; and $M_o = 1$ means no change in the productivity from time t to $t+1$.

3 Fuzzy linear programming problems

Definition 3.1. *We represent an arbitrary fuzzy number by an ordered pair of functions* $(\underline{u}(r), \overline{u}(r))$, $0 \le r \le 1$, *which satisfy the following requirements:*

- $\underline{u}$(r) is a bounded left continuous nondecreasing function over [0,1].
- $\overline{u}$(r) is a bounded left continuous nonincreasing function over [0,1].
- $\underline{u}(r)$ and $\overline{u}(r)$ are right continuous in 0.
- $\underline{u}(r) \le \overline{u}(r)$,$0 \le r \le 1$.

A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha$, $0 \le r \le 1$.

Theorem 3.1. *for arbitrary fuzzy numbers* $x = (\underline{x}, \overline{x})$, $y = (\underline{y}, \overline{y})$ *and real number* k,

1. $x = y$ *if and only if* $\underline{x}(r) = \underline{y}(r)$ *and* $\overline{x}(r) = \overline{y}(r)$.
2. $x + y = (\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{y}(r))$.
3.

$$kx = \begin{cases} (k\underline{x}, k\overline{x}), & k \ge 0 \\ (k\overline{x}, k\underline{x}), & k < 0 \end{cases} \tag{3.5}$$

Definition 3.2. *The suport of a fuzzy set* $\tilde{A}$ *is a set of elements in* X *for which* $\mu_{\tilde{A}}(x)$*is positive.*

Definition 3.3. *A convex fuzzy set* $\tilde{A}$ *on* $\Re$ *is a fuzzy number if the following coditions hold:*

- *Its membership function is piecewise continuous.*
- *There exist three intervals* $[a, b]$, $[b, c]$ *and* $[c, d]$ *such that* $\mu_{\tilde{A}}(x)$ *is increasing on* $[a, b]$, *equal to* 1 *on* $[b, c]$, *decreasing on* $[c, d]$ *and equal to* 0 *elsewhere.*

Remark 3.1. *In the above defintion, we say interval* $[b, c]$ *is the modal set of fuzzy number* $\tilde{A}$.

Definition 3.4. *Let $\tilde{A} = (a^m, a^u, a^l)$ denote the triangular fuzzy number, where $[a^l, a^u]$ is the suport of $\tilde{A}$ and a^m its modal set.*

Remark 3.2. *We denote the set of all trapezoidal fuzzy numbers by $F(\Re)$. we obtain a triangular fuzzy number, and we show it with:*

$$\tilde{a} = (\underline{a}(r), \overline{a}(r)) = ((a^m - a^l)r + a^l, (a^m - a^u)r + a^u)$$

There are several methods for solving fuzzy linear programming problems; such as Fang (1999),Lai and Hwang (1992),Maleki et al (2000).One of the most convenient of these methods is based on the concept of comparison of fuzzy numbers by use of ranking functions.In fact,an efficient approach for ordering the elements of $F(\Re)$ is to define a ranking function $\tau : F(\Re) \to \Re$ which maps each fuzzy number into the line, where a natural order exists. We define orders on $F(\Re)$ by

1. $\tilde{a} \succeq \tilde{b}$ if and only if $\tau(\tilde{a}) \geq \tau(\tilde{b})$.
2. $\tilde{a} \succ \tilde{b}$ if and only if $\tau(\tilde{a}) > \tau(\tilde{b})$.
3. $\tilde{a} \simeq \tilde{b}$ if and only if $\tau(\tilde{a}) = \tau(\tilde{b})$.

where $\tilde{a}$ and $\tilde{b}$ are in $F(\Re)$.
The following lemma is now immediate.

Lemma 3.1. *Let τ be any linear ranking function.Then*

1. $\tilde{a} \succeq \tilde{b}$ *iff* $\tilde{a} - \tilde{b} \succeq 0$ *iff* $-\tilde{b} \succeq -\tilde{a}$.
2. $\tilde{a} \succeq \tilde{b}$ *and* $\tilde{c} \succeq \tilde{d}$ *iff* $\tilde{a} + \tilde{c} \succeq \tilde{b} + \tilde{d}$.

We restrict our attention to the linear ranking function, that is, a ranking function τ such that

$$\tau(k\tilde{a} + \tilde{b}) = k\tau(\tilde{a}) + \tau(\tilde{b})$$

for any $\tilde{a}$ and $\tilde{b}$ belonging to $\tau(\Re)$ and any $k \in \Re$.

Here, we introduce a linear ranking function that is similar to the ranking function adopted by Maleki (FJMS)(2002). For a fuzzy number $\tilde{a} = (\underline{a}(r), \overline{a}(r))$,we use ranking function as follows:

$$\tau(\tilde{a}) = 1/2 \int_0^1 (\underline{a}(r) + \overline{a}(r))dr. \tag{3.6}$$

This reduces to

$$\tau(\tilde{a}) = 1/2(a^m + 1/2(a^l + a^u)). \tag{3.7}$$

Then, for triangular fuzzy numbers $\tilde{a} == (a^m, a^u, a^l)$ and $\tilde{b} = (b^m, b^u, b^l)$,we have

$$[\tilde{a} \succeq \tilde{b}] \Longleftrightarrow [(a^m + 1/2(a^l + a^u)) \geq (b^m + 1/2(b^l + a^u))]. \tag{3.8}$$

Authors who use ranking function for comparison of fuzzy linear programming

problems usually define a crisp model which is equivalent to the Fuzzy linear programing problem and then use optimal solution of this model as the optimal solution of fuzzy linear programming problems. We now define fuzzy linear programming problems and the corresponding crisp model.

Definition 3.5. *A fuzzy linear programing problem (FLP) is defined as follows:*

$$\begin{aligned} \min \ \ & \tilde{z} \simeq \tilde{c}x \\ s.t. \ \ & \tilde{A}x \succeq \tilde{b} \\ & x \geq 0, \end{aligned} \tag{3.9}$$

where "$\simeq$" and "$\preceq$" mean equality and inequality with respect to the ranking function τ, $\tilde{A} = [\tilde{a_{ij}}]_{m\times n}$, $\tilde{c} = (\tilde{c_1}, ..., \tilde{c_n})$,
$\tilde{b} = (\tilde{b_1}, ..., \tilde{b_m})^T$,$x = (x_1, ..., x_n)$, and $\tilde{a_{ij}}, \tilde{b_i}, \tilde{c_j} \in F(\Re)$ and $x_j \in \Re$ for $i = 1, ..., m; j = 1, ..., n$.

Definition 3.6. *Any x which satisfies the set of constraints of (FLP)is called afeasible solution. Let $\tilde{S}$ be the set of all feasible solution of (FLP).We say that $x^* \in \tilde{S}$ is an optimal feasible solution for FLP iff $\tilde{c}x^* \preceq \tilde{c}x$ for all $X \in \tilde{S}$.*

Definition 3.7. *We say that the real number a corresponds to the fuzzy number $\tilde{a}$, with respect to a given linear ranking function τ, if $a = \tau(\tilde{a})$.*

However the following theorem shows that any FLP can be reduced to a linear programming problem.

Theorem 3.2. *The following linear programming problem (LP) and the FLP in* (3.9) *are equivalent:*

$$\begin{aligned} \min\ & z = cx \\ s.t.\ & Ax \geq b \\ & x \geq 0, \end{aligned} \tag{3.10}$$

where a_{ij}, b_i, c_j are real numbers corresponding to the fuzzy numbers $\tilde{a_{ij}}, \tilde{b_i}, \tilde{c_j}$ with respect to a given linear ranking function τ, respectively.

Proof: By considering the ranking function and definition (2.6) it is to see that every optimal feasible solution of FLP(2.6) is an optimal feasible solution of LP(2.7), on the other hand, every optimal feasible solution of LP(2.7) is an optimal feasible solution of FLP(2.6).

Remark 3.3. *The above theorem shows that the sets of all feasible solutions of FLP and LP are the same. Also if $\bar{x}$is an optimal solution for FLP, then $\bar{x}$ is an optimal solution for LP.*

Corollary 3.1. *If problem LP does not have a optimal solution then FLP does not have a optimal solution either.*

3.1 Data envelopment analysis with fuzzy data

In recent years, the fuzzy set theory has been proposed as a way to quantify imprecise and vague data in DEA models. fuzzy DEA models take the form of fuzzy linear programming model. The CCR model with fuzzy coefficients and its dual are formulated as the following linear programming models:

$(FCCR)$

$$\begin{aligned} \max\quad & \sum_{r=1}^{s} u_r \tilde{y}_{ro} \\ \text{s.t.}\quad & \sum_{i=1}^{m} v_i \tilde{x}_{io} \simeq 1; \\ & \sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \preceq 0 \quad \text{j=1,...,n} \\ & u_r, v_i \geq 0, \quad \text{r=1,...,s,} \quad \text{i=1,...,m} \end{aligned} \tag{3.11}$$

$(FDCCR)$

$$\begin{aligned} &\text{Min} \quad \theta \\ &\text{s.t.} \quad \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \preceq \theta \tilde{x}_{io}, \quad \text{i=1,...,m} \\ &\qquad \sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} \succeq \tilde{y}_{ro}, \quad \text{r=1,...,s} \\ &\qquad \lambda_j \geq 0, \quad \text{j=1,...,n} \end{aligned} \tag{3.12}$$

where $\tilde{X}_o$ is the column vector of fuzzy inputs consumed by the target DMU (DMUo), $\tilde{X}$ is the matrix of fuzzy inputs of all DMUs, $\tilde{Y}_o$ is the column vector of fuzzy outputs consumed by the target DMU (DMUo), $\tilde{Y}$ is the matrix of fuzzy outputs of all DMUs.

The fuzzy CCR models cannot be solved by a standard LP solver like a crisp CCR model because coefficients in the fuzzy CCR model are fuzzy sets. With the fuzzy inputs and fuzzy outputs, the optimality conditions for the crisp DEA model need to be clarified and generalized. This approach by using the concept FLP can be used to solve the above FCCR and FDCCR then.

Theorem 3.3. *The following linear programming problem (DCCR) and the FDCCR are equivalent:*

$(FDCCR)$

$$\begin{aligned} &Min \quad \theta \\ &s.t. \quad \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \preceq \theta \tilde{x}_{io}, \quad i=1,\ldots,m \\ &\qquad \sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} \succeq \tilde{y}_{ro}, \quad r=1,\ldots,s \\ &\qquad \lambda_j \geq 0, \quad j=1,\ldots,n \end{aligned} \tag{3.13}$$

$(DCCR)$

$$\begin{array}{lll} Min & \theta & \\ s.t. & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{io}, & i=1,\ldots,m \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro}, & r=1,\ldots,s \\ & \lambda_j \geq 0, & j=1,\ldots,n \end{array} \tag{3.14}$$

where x_{ij}, y_{rj} *are real numbers corresponding to the fuzzy numbers* $\tilde{x}_{ij}, \tilde{y}_{rj}$ *with respect to a given linear ranking function* τ*, respectively.*

Proof: By considering the ranking function and definition (3.6), it is to see that every optimal feasible solution of FDCCR is an optimal feasible solution of DCCR, on the other hand, every optimal feasible solution of DCCR is an optimal feasible solution of FDCCR.

4 Malmquist productivity index with fuzzy data

In recent years, the fuzzy set theory has been proposed as a way to quantify imprecise and vague data in DEA models. Fuzzy DEA models take the form of fuzzy linear programming model. The fuzzy DCCR models cannot be solved by a standard LP solver like a crisp DCCR model because coefficients in the fuzzy DCCR model are fuzzy sets. With the fuzzy inputs and fuzzy outputs, the optimality conditions for the crisp DEA model need to be clarified and generalized.

In this section, we are in purpose to evaluate the Malmquist productivity index for DMUs with fuzzy data. Therefore, assume that fuzzy numbers, $\tilde{x}^t_{ij}$ and $\tilde{y}^t_{rj}$ are the $i-th$ input and the $r-th$ output for DMU_j in time t.The two single period measures can be obtained by using the FDCCR DEA model :

$$\begin{array}{llll} \overline{D^p_o(k)} = & \text{Min} & \theta & \\ & \text{s.t.} & \sum_{j=1}^{n} \lambda_j \tilde{x}^p_{ij} \preceq \theta \tilde{x}^k_{io}, & \text{i=1,...,m} \\ & & \sum_{j=1}^{n} \lambda_j \tilde{y}^p_{rj} \succeq \tilde{y}^k_{ro}, & \text{r=1,...,s} \\ & & \lambda_j \geq 0, & \text{j=1,...,n} \end{array} \tag{4.15}$$

The efficiency $(\overline{D_o^t(t)})$ determines the amount by which observed inputs can be proportionally reduced, while still producing the given output level. Using $(t+1)$ instead of (t) for the above method, we get $\overline{D_o^{t+1}(t+1)}$, the technical efficiency score for DMU_o in time period $(t+1)$. The first of the mixed period measures, which is defined as $\overline{D_o^t(t+1)}$ for each DMU_o, is computed as the optimal value to the(4.15) fuzzy linear programming problem, where $p = t$ and $k = t + 1$.
Similarly, the other mixed period measure, $\overline{D_o^{t+1}(t)}$, which is needed in the computation of the Malmquist productivity index, is the optimal value to the (4.15) fuzzy linear problem, where $p = t + 1$ and $k = t$.
Now we use ranking function (3.8) to determine x_{ij}^t, x_{ij}^{t+1}, y_{ij}^t and y_{ij}^{t+1} corresponding to the fuzzy numbers $\tilde{x}_{ij}^t$, $\tilde{x}_{ij}^{t+1}$, $\tilde{y}_{ij}^t$ and $\tilde{y}_{ij}^{t+1}$, then we determine the $\overline{D_o^t(t)}$, $\overline{D_o^{t+1}(t+1)}$, $\overline{D_o^t(t+1)}$ and $\overline{D_o^{t+1}(t)}$ by solve corresponding linear programing problem as:

$$\begin{aligned} \overline{D_o^p(k)} = \quad & \text{Min} \quad \theta \\ & \text{s.t.} \quad \sum_{j=1}^{n} \lambda_j x_{ij}^p \leq \theta x_{io}^k, \quad \text{i=1,...,m} \\ & \qquad \sum_{j=1}^{n} \lambda_j y_{rj}^p \geq y_{ro}^k, \quad \text{r=1,...,s} \\ & \qquad \lambda_j \geq 0, \qquad \text{j=1,...,n} \end{aligned} \tag{4.16}$$

The Malmquist productivity index, which measures the productivity change of a particular $DMU_o, o \in J = \{1, ..., n\}$, in time $t + 1$ and t is given as

$$\overline{M}_o = [\frac{\overline{D_o^t(t+1)}}{\overline{D_o^t(t)}} \times \frac{\overline{D_o^{t+1}(t+1)}}{\overline{D_o^{t+1}(t)}}]^{\frac{1}{2}} \tag{4.17}$$

It can be seen that the above measure actually is the geometric mean of two Caves, et al.'s Malmquist productivity indexes. Thus, following Caves et al.'s suit, Fare, et al. define that $\overline{M}_o > 1$ indicates productivity gain; $\overline{M}_o < 1$ indicates productivity loss; and $\overline{M}_o = 1$ means no change in productivity from time t to $t + 1$.

5 Methodology and examples

We evaluate 21 branches of Tehran Social Security Insurance Organization at this section.Each branch uses of four inputs in order to produce four outputs. The lables of inputs and outputs are presented in under table.

	Input	Output
1	The number of personals	The total number of insured persons
2	The total number of computers	The number of insured persons'agreements
3	The area of the branch	The total number of life-pension recievers
4	Administrative expenses	The receipt total sum (Incom)

Table1.The lables of inputs and outputs.

The total data is related to two chronological sections of (2003 and 2004 (A-D)). The total triangular Fuzzy data has been viewed in tables (2), (3), (4) and (5).It is considered that "M" as number middle,"U" as number up and "L" as number low.For example if ($\tilde{I}_{11} = (I_{11}^M, I_{11}^U, I_{11}^L)$ denote a triangular fuzzy number then we use ranking function as follows:

$$\tau(\tilde{I}_{11}) = 1/2(I_{11}^M + 1/2(I_{11}^U + I_{11}^L)). \tag{5.18}$$

After using the ranking function τ , the data is given as crisp numbers and then with applying explained method on the essay, the results are presented in tables (6) and (7).

	Im11	IU11	IL11	IM21	IU21	IL21	IM31	IU31	IL31	IM41	IU41	IL41)
1	98.83	100	96	86.5	87	86	4000	4000	4000	62895029.33	103656656	14730450
2	78.66	81	75	88.83	90	88	2565	2565	2565	71650228.17	95701909	41144517
3	78.5	80	77	87	89	85	1343	1343	1343	42813134.5	60301920	28792550
4	92.33	94	91	94.5	96	93	1500	1500	1500	72295517.33	94569518	24277018
5	103	105	102	97	97	97	3750	3750	3750	77762043	116268609	45571800
6	97.33	100	96	91	92	90	3313	3313	3313	114537397.8	171152176	78011675
7	87	90	85	92	92	92	1500	1500	1500	75597743	135858469	14969393
8	108.5	112	106	88.33	92	84	1600	1600	1600	101976164.3	182951858	61024310
9	97.16	101	94	78	78	78	1920	1920	1920	90515357	124984300	39664990
10	85	88	82	93.16	94	92	2800	2800	2800	89785320.5	168961589	30082208
11	80.16	82	77	92.83	94	92	1630	1630	1630	76900120.67	158581843	48034155
12	89.5	91	89	85	85	85	1127	1127	1127	190788340.2	827138457	35631132
13	101.83	108	94	91.66	92	91	1304	1304	1304	205857963.5	389185592	37568447
14	84	87	82	100.16	101	100	1340	1340	1340	95179649.67	136748402	52060934
15	115.66	118	112	121.83	123	120	2191	2191	2191	91465848.33	190535604	45229750
16	82.83	86	80	100	100	100	2140	2140	2140	78629283.5	96578148	45992927
17	90.66	93	87	91.5	93	91	1231	1231	1231	78946229.17	113530688	47909750
18	99.33	103	97	90	90	90	1960	1960	1960	210592149.3	757564117	52110247
19	81	83	79	81	81	81	3375	3375	3375	65799659.17	109022822	33088649
20	98.16	102	96	91.5	97	87	1603	1603	1603	178472638.7	537844764	45363108
21	91	93	88	92.16	94	90	2930	2930	2930	87029553.67	113930230	66736398

Table 2. The triangular fuzzy Inputs for 21 branches of Insurance Organization at time period (t)

	OM11	OU11	OL11	OM21	OU21	OL21	OM31	OU31	OL31	OM41	OU41	OL41
1	56570.66	57318	55830	36.83	45	30	1336.33	1350	1307	169.83	192	145
2	36800.5	36852	36740	15.66	22	0	8463.33	8571	8385	336.83	486	175
3	38446.16	38783	38004	17.66	27	11	6594.16	6601	6588	181.83	276	113
4	35685.16	36017	35469	30.33	55	10	10820.66	10821	10820	237.66	316	128
5	72446.33	78574	70254	12.33	19	7	8066.16	8752	7536	305	615	82
6	35856.33	37443	32585	93.16	129	47	14557.16	14994	14118	225	392	154
7	45027.66	47270	42900	17.33	27	11	1648.66	1661	1634	146	220	54
8	86221.5	87220	85399	58.16	97	43	10492.66	10775	10206	243	289	179
9	40482.33	44298	36652	178.5	242	81	12099.83	12261	11996	194.83	286	37
10	88898.5	90250	87716	35.66	43	28	646	660	630	107	167	51
11	50400.33	50593	50210	11.83	16	6	10251.66	10256	10247	137.33	295	28
12	48577.83	49489	47727	22.33	30	15	7411.5	7542	7302	200.16	286	85
13	83235.5	89111	78550	20.66	25	13	4879.5	5151	4745	136	224	72
14	28973.83	32943	27978	83.83	325	29	14637.83	14820	14473	275.5	368	190
15	102562.5	103047	102175	39.83	49	31	2438.83	3577	252	212.16	320	120
16	33616	35627	31819	19.16	32	12	2049.83	2147	1963	300.666	522	156
17	53183.66667	55163	51345	62	73	35	10185	10238	10157	151.33	205	85
18	73729	74633	72915	45	52	40	4400.5	4668	4193	233.16	427	112
19	43402.66	44363	42887	25.16	33	11	598.33	628	560	296.16	390	218
20	72152.33	72534	71743	70.66	92	50	11687.66	12569	8762	163.83	240	102
21	63973.33	64541	63182	23.33	32	10	11380.83	11609	11143	269.5	378	122

Table 3. The triangular fuzzy Outputs for 21 branches of Insurance Organization at time period (t)

	Im11	IU11	IL11	IM21	IU21	IL21	IM31	IU31	IL31	IM41	IU41	IL41
1	94.83	97	93	84.5	87	84	4000	4000	4000	78262041.33	147806940	52869780
2	77	79	75	93	95	91	2565	2565	2565	73241407	139078952	54540554
3	76.5	78	75	87	87	87	1343	1343	1343	71482960	133424069	43603089
4	92.83	94	92	93	93	93	1500	1500	1500	497692042.2	2592824900	57579699
5	102.33	105	101	97	97	97	3750	3750	3750	83979470.83	139483237	8425500
6	94.5	95	94	90.5	91	90	3313	3313	3313	124490390	188609133	85596139
7	86.33	89	83	92.33	93	92	1500	1500	1500	77394251.33	103925166	59260272
8	102.83	106	102	92	92	92	1600	1600	1600	136743469.3	256121114	84151656
9	104.66	107	103	104.33	105	103	2500	2500	2500	81318447	151552125	40836156
10	88.5	90	86	95	95	95	2800	2800	2800	58841917.83	82435467	38957251
11	81.66	84	78	93.83	95	92	1630	1630	1630	67658394.67	122078743	46388243
12	89	91	87	85.33	86	85	1127	1127	1127	65516280.33	122657603	33438687
13	114.33	117	111	94.33	95	92	1304	1304	1304	78379833	130400547	31892369
14	86.66	88	85	101	101	101	1340	1340	1340	84203387.33	117282075	48310364
15	113.33	119	108	123	124	122	2191	2191	2191	107027548.2	143628204	77265953
16	79	80	78	100	100	100	2140	2140	2140	136266225	285338884	93515206
17	87.33	89	86	93.5	94	93	1231	1231	1231	67564034.33	103534864	5099980
18	98.33	100	97	90	90	90	1960	1960	1960	89732318.33	132769689	60562473
19	74.66	77	73	85	86	82	3375	3375	3375	72153562.17	113626397	44675390
20	99.16	102		97 97.5	98	96	1603	1603	1603	452222971.8	931530808	70861125
21	89.5	92	87	92	92	92	2930	2930	2930	78813784.17	105707967	57568217

Table 4. The triangular fuzzy Inputs for 21 branches of Insurance Organization at time period $(t+1)$

	OM12	OU12	OL12	OM22	OU22	OL22	OM32	OU32	Ol32	OM42	OU42	OL42
1	58449.33	59603	57668	49.16	71	32	1136.16	1148	1117	211.16	276	189
2	37044.83	37179	36922	21.83	35	14	8795.83	8919	8635	230.83	299	175
3	34438.33	39449	25360	31.66	47	20	6599.16	6604	6588	427.66	585	315
4	36651.83	37023	36247	41	59	21	9406	10821	8080	234.5	329	141
5	71808.66	73005	69071	15.83	33	0	8010.66	8120	7868	382.5	644	222
6	38667.66	39539	37476	104.83	126	73	13781.83	15264	13121	344.5	345	345
7	50189.16	52144	48094	20	31	12	1565.33	1575	1553	391.5	634	261
8	88309.5	92756	84531	68.16	111	0	11802.5	15402	10790	425.33	616	399
9	35972	39341	31554	212.66	268	170	12639	12878	12342	254.83	365	127
10	90320.66	93942	80425	45.83	66	36	717.66	748	673	229.83	312	172
11	48643.83	50965	44305	20.66	29	17	10290.16	10293	10286	134.83	272	79
12	43741.5	49855	39797	31.5	37	26	7851.33	8197	7497	191.66	232	175
13	79290.66	85825	72553	28.33	42	19	4953.16	5289	4279	215.66	324	115
14	32189.16	33637	28855	59.66	116	23	14785.83	15222	14144	360.83	444	301
15	105355.5	108477	103243	58	77	41	2658.5	2668	2649	362.66	468	255
16	34310.66	36205	32259	36.33	49	30	2273	2332	2206	463	589	423
17	58240.83	61760	53934	62.16	73	53	10337.5	10462	10225	201	278	168
18	83197.66	86702	74802	65.83	96	43	4772.66	5010	4568	286	358	211
19	44457	45396	43885	31.16	40	21	617.16	658	587	306.16	431	225
20	69914.33	74218	64989	91	130	67	13219.16 21	13708	12772	212.16	261	141
30	62304.5	64784	60864	30.83	47	15	12275.33	12621	11882	262.5	406	187

Table 5. The triangular fuzzy Outputs for 21 branches of Insurance Organization at time period $(t+1)$

After using of ranking function τ , the datas are given as crisp in tabls 6 and 7.

	Input 1	Input 2	Input 3	Input 4	Output 1	Output 2	Output 3	Output 4	θ^t
1	94.91	85	4000	89300200.67	56572.33	37.16	1332.41	169.16	0.79
2	77	93	2565	85025580	36798.25	13.33	8470.66	333.66	1.00
3	76.5	87	1500	911447170.8	35714.08	31.416	10820.58	229.83	0.86
4	92.91	93	1500	911447170.8	35714.08	31.41	10820.58	229.83	0.87
5	102.66	97	3750	78966919.67	73430.16	12.66	8105.08	326.75	1.00
6	94.5	90.5	3313	79493485.17	45056.33	18.16	1648.08	141.5	1.00
7	86.16	92.41	1500	79493485.17	45056.33	18.16	1648.08	141.5	0.71
8	103.41	92	1600	153439927.2	86265.5	64.08	12114.16	178.16	1.00
9	95	79	1920	100704537.3	40478.66	170	12114.16	178.16	1.00
10	88.25	95	2800	59769138.42	88940.75	35.58	645.5	108	1.00
11	81.33	93.66	1127	71782212.67	48592.91	22.41	7416.75	192.83	1.00
12	89	85.41	1127	71782212.67	48592.91	22.41	7416.75	192.83	1.00
13	114.16	93.91	1304	79763145.5	83533	19.83	4913.75	142	1.00
14	86.58	101	1340	83499803.42	29717.16	130.41	14642.16	277.25	1.00
15	113.41	123	2191	108737313.3	102586.75	39.91	2176.66	216.08	1.00
16	79	100	2140	60940728.17	53218.83	58	10191.25	148.16	1.00
18	98.41	90	1960	93199199.67	73751.5	45.5	4415.5	251.33	1.00
19	74.83	84.5	3375	75652227.83	43513.83	23.58	596.16	300.08	1.00
20	99.33	97.25	1603	476709469.2	72145.41	70.83	11176.58	167.41	0.98
21	62564.25	92	2930	80225938.08	63917.41	22.16	11378.41	259.75	1.00

Table 6. The crisp data for 21 branches of Insuranece Organization at time period (t)

	Input 1	Input 2	Input 3	Input 4	Output 1	Output 2	Output 3	Output 4	θ^{t+1}
1	94.91666667	85	4000	89300200.67	58542.41667	50.33	1134.33	221.83	0.76
2	77	93	2565	85025580	37047.66667	23.16	8786.41	233.91	0.80
3	76.5	87	1343	79998269.5	33421.41667	32.58	6597.58	438.83	1.00
4	92.91	93	1500	911447170.8	36643.41	40.5	9428.25	234.75	0.70
5	102.66	97	3750	78966919.67	71423.33	16.16	8002.33	407.75	1.00
6	94.5	90.5	3313	130796513	38587.58	102.16	13987.16	344.75	1.00
7	86.16	92.41	1500	79493485.17	50154.08	20.75	1564.66	419.5	1.00
8	103.41	92	1600	153439927.2	88476.5	61.83	12449.25	466.41	1.00
9	95	79	1920	100704537.3	35709.75	215.83	12624.5	250.41	1.00
10	88.25	95	2800	59769138.42	88752.08	48.416	714.08	235.91	1.00
11	81.33	93.66	1630	75945943.83	48139.41	21.83	10289.83	155.16	0.95
12	89	85.41	1127	71782212.67	44283.75	31.5	7849.16	197.58	0.82
13	114.16	93.91	1304	79763145.5	79239.83	29.41	4868.58	217.58	1.00
14	86.58	101	1340	83499803.42	31717.58	64.58	14734.41	366.66	1.00
15	113.41	123	2191	108737313.3	105607.75	58.5	2658.5	362.08	1.00
16	79	100	2140	162846635	34271.33	37.91	2271	484.5	1.00
17	87.41	93.5	1231	60940728.17	58043.91	62.58	10340.5	212	1.00
18	98.41	90	1960	93199199.67	81974.83	67.66	4780.83	285.25	1.00
19	74.83	84.5	3375	75652227.83	44548.75	30.83	619.83	317.08	0.90
20	99.33	97.25	1603	476709469.2	69758.91	94.75	13229.58	206.58	1.00
21	89.5	92	2930	80225938.08	62564.25	30.91	12263.41	279.5	1.00

Table 7. The crisp data for 21 branches of Insuranece Organization at time period ($t+1$)

Then with applying explained method on the essay the results are presented in the table 8:

DMU	1	2	3	4	5	6	7	8	9	10	11
$\overline{M}$	0.97	0.76	1.30	0.84	0.91	0.96	1.53	1.08	0.98	1.04	0.96
DUM	12	13	14	15	16	17	18	19	20	21	
$\overline{M}$	0.83	0.94	0.8	1.02	1.04	1.0	0.98	0.83	1.03	0.97	

Table 8. The Malmquist Productivity Indexs.

As it is apparent at the above table, we see the progress in 8 branches and the regress in the 13 remaied branches. It is viewed the most progress is in the branch 8 with due to this DMU was inefficient at the time (t) and had (0.71)

efficiency, nearly. At the time of $(t+1)$,this branch is considered as an efficient DMU. The efficiency has increased about (0.3) in this branch.The analyses of the branches 3 and 6 are similar to the branch 7.
The most regress is in the branch 2 with due to this branch was efficient at the time (t) and was changed to an inefficient DMU at the time of $(t+1)$. The efficiency has decreaed about (0.2).There is a similar analysis about the branch 18.

6 Conclusion

The purpose of this study was to develop the Malmquist productivity index for DMUs with fuzzy data. Since the level of inputs and outputs for DMU_o are not know exactly, we tried by using the concept of FLP to develop a new approach of Malmquist productivity index and applied it to a numerical example.

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