Applied Mathematical Sciences, Vol. 3, 2009, no. 43, 2113 - 2118

A Property Related to Non-Measurable and Non-Baire Sets

S. Basu

Department of Mathematics, Darjeeling Govt. College
Lebong Cart Road, Darjeeling
sanjibbasu08@gmail.com

Abstract. In an earlier paper [1], the present author (along with D.K. Ganguly) has shown that given any non-measurable (in the sense of Lebesgue) subset of the real line R, these exists a class $\langle E\rangle$ of measurable sets of positive measures such that $M\cap E$ is again non-measurable for every $E\in\langle E\rangle$. In the same paper, they have also presented a Baire-category analogue of the same result. The purpose of the paper is to extend both these results in more general settings.

Mathematics Subject Classification: 28A

Keywords: measure zero cardinal, σ-locally finite base

INTRODUCTION

The following is an well-known result of classical measure theory [2] :

There exists a set M ($\subseteq$P) such that for every Lebesgue measurable set E, $m_*(M\cap E)=0$ whereas $m^*(M\cap E)=m(E)$, where m_*(resp.m^*) are the inner (resp.outer) Lebesgue measure and m is the Lebesgue measure in P.

As an easy consequence of the above result, it follows that the set M is non-measurable such that $M\cap E$ is also non-measurable for every Lebesgue measurable set E with $m(E) > 0$.

In an earlier paper, done with D.K. Ganguly, the present author has shown that the above result can also be proved in such a manner so as to make it applicable for every non-measurable subset of P. The statement of the result is as follows :-

(*) Given any non-measurable subset M of the real line P, there is a corresponding class $\langle E \rangle$ of measurable sets of positive Lebesgue measure (having cardinality same as that of the later) such that $M \cap E$ is again non-measurable for every $E \in \langle E \rangle$.

A set *E* ($\subseteq$P) is said to have the property of Baire if $A = G \Delta P$, where *G* is open and *P* is a set of first category (Δ representing symmetric difference). [3].

In the same paper referred to in the above, the authors have also proved the following Baire category analogue of the above result :-

(**) Given any non-Baire subset *B* of the real line P, there is a corresponding class $\langle E \rangle$ of second category sets having Baire-property (having cardinality same as that of the later) such that $B \cap E$ is again non-Baire for every $E \in \langle E \rangle$.

Now the property of P being a second countable topological space plays an effective role in proving the above two results. Topological spaces, let alone metric spaces in general are not second countable and so it is natural to enquire regarding the extendability of the above two results in such general settings. However, second countability could be generalized with the aid of a notion called 'measure zero cardinal' which has earlier proved itself as being effective in establishing the measure analogue of the well known Banach category theorem and also in extending an well-known decomposition theorem expressing P as the union of a measure zero set and a set of first category [3]. The present paper serves to extend the above two results, the first one by the use of 'measure-zero cardinals' and second one related non-Baire sets with the use of σ-locally finite base. The results are divided in two sections the first one dealing with non-measurability.

THE RESULTS

§1. Let X be a metric space having a base $\mathcal{B}$ and μ^* be the metric outer measure on *X* with $\mu^*(X) < \infty$. The outer measure μ^* is non-atomic provided $\mu^*(\{x\}) = 0$ for any *x* in *X*. A subset of *X* is F_σ (resp. G_δ) if it is the countable union (resp. intersection) of closed (resp. open) sets. A cardinal is said to have measure zero if every finite Borel measure defined for all subsets of a set of that cardinality vanishes identically if it is zero for points [3]. Every cardinal less than the first weakly inaccessible cardinal has measure zero and that (assuming continuum hypothesis) only exceedingly large cardinals can fail to be of measure zero. In fact, *c* has measure zero (assuming continuum hypothesis) and evidently any cardinal less than one of measure zero has measure zero.

Our result states :-

THEOREM. If $\mathcal{B}$ in X has a measure zero cardinal, then for every non-μ^*-measurable set M, there exists $\phi \neq \mathcal{U} \in \mathcal{B}$ such that $M \cap E$ is non-μ^*-measurable for every $E \in \mathcal{M}_{\mu^*}$ with $E \supseteq \mathcal{U}$.

LEMMA (Montegomery) [3]. Let $\{G_\alpha; \alpha \in \Lambda\}$ be an well-ordered family of open sets in X, and for each $\alpha \in \Lambda$, let F_α be a closed subset of $H_\alpha = G_\alpha \setminus \bigcup_{\beta<\alpha} G_\beta$. Then $E = \bigcup_{\alpha\in\Lambda}$ is F_σ.

PROOF OF THE THEOREM. Let $\{U_\alpha ; \alpha \in \Lambda\}$ be an well-ordering of $\mathcal{B}$ and $V_\alpha = U_\alpha \setminus \bigcup_{\beta<\alpha} U_\beta$. We claim that there exists $\alpha_0 \in \Lambda$ such that $M \cap E \notin M_{\mu^*}$ for every $E \supseteq V_{\alpha_0}$.

If possible, on the contrary, let for every $\alpha \in \Lambda$, these exists a set E_α ($\in M_{\mu^*}$) with $E_\alpha \supseteq V_\alpha$ such that $M \cap E_\alpha$ and therefore $M \cap V_\alpha$ is in M_{μ^*}. We set $W_\alpha = M \cap V_\alpha$ ($\alpha \in \Lambda$). Certainly than $W_\alpha \cap W_\beta = \phi$ ($\alpha \neq \beta$) and moreover $M = \bigcup_{\alpha\in\Lambda} W_\alpha$. Now as $\mu^*(X) < \infty$ and μ^* is regular, therefore for each $\alpha \in \Lambda$, we can express $W_\alpha = K_\alpha \cup Z_\alpha$ where K_α is F_σ and $\mu^*(Z_\alpha) = 0$. Again by the same reasoning as given above, there exists a G_δ-set $H_\alpha \subseteq V_\alpha$ such that $H_\alpha \supseteq Z_\alpha$ and $\mu^*(H_\alpha) = 0$.

The set $\bigcup_{\alpha\in\Lambda} K_\alpha$ is F_σ by the above lemma. We now show that $\mu^* \left(\bigcup_{\alpha\in\Lambda} Z_\alpha \right) . = 0$ by showing that $\mu^* \left(\bigcup_{\alpha\in\Lambda} H_\alpha \right) = 0$.

For each $\alpha \in \Lambda$, V_α is F_σ and so $V_\alpha = \bigcup_{n=1}^{\infty} F_{\alpha,n}$ where $F_{\alpha,n}$ is closed. But then $\bigcup_{\alpha\in\Lambda} H_\alpha = \bigcup_{\alpha\in\Lambda} \bigcup_{n=1}^{\infty} (F_{\alpha,n} \cap H_\alpha) = \bigcup_{n=1}^{\infty} \bigcup_{\alpha\in\Lambda} (F_{\alpha,n} \cap H_\alpha)$ where each $F_{\alpha,n} \cap H_\alpha$ is G_δ. Let $T_n = \bigcup_{\alpha\in\Lambda} (F_{\alpha,n} \cap H_\alpha)$ and $E \subseteq \Lambda$. Then $\bigcup_{\alpha\in E} (F_{\alpha,n} \cap H_\alpha)$ being Borel (which can be easily checked), the set function $\gamma(E) = \mu^* \left(\bigcup_{\alpha\in E} (F_{\alpha,n} \cap H_\alpha) \right)$ is defined for all subsets E of Λ which is evidently a finite Borel measure and also non-atomic. As $\mathcal{B}$ possesses a measure zero cardinal, consequently, $\mu^*(T_n)=0$ for each n and therefore $\mu^* \left(\bigcup_{\alpha\in\Lambda} H_\alpha \right) = \mu^* \left(\bigcup_{n=1}^{\infty} T_n \right) = 0$. Thus M becomes the union of an F_σ set and a set of μ^*-measure zero is μ^*-measurable – a contradiction. This establishes our claim.

Now as it is obvious that $\{E \in M_{\mu^*} \,/\, E \supseteq V_{\alpha_0}\} \supseteq \{E \in M_{\mu^*} \,/\, E \supseteq U_{\alpha_0}\}$ the theorem is proved upon setting $U_{\alpha_0} = U$ and $M_{\mu^*}(U) = \{E \in M_{\mu^*} \mid E \supseteq U\}$.

NOTE. Taking a look at the proof of the above theorem a little more closely, one doesn't fail to note that instead of choosing $M_{\mu^*}(U)$, we may also choose the collection $\{E \subseteq X \,/\, E \supseteq U\}$ which is in fact larger and also endowed with a filtered structure.

§2. In a topological space X, a family $\mathfrak{R}$ of subsets of X is termed as 'locally finite' if every point in X has a neighbourhood which meets only a finite number of members of $\mathfrak{R}$. It is called σ (or countably) locally finite if $\mathfrak{R} = \bigcup_{n=1}^{\infty} \mathfrak{R}_n$ where each $\mathfrak{R}_n$ in particular is a locally finite family. For any family $\mathfrak{R}$ of subsets of X, another family $\mathcal{B}$ of subsets of X is said to be a refinement of $\mathfrak{R}$ (or, sometimes is also said to refine $\mathfrak{R}$) if for each B in $\mathcal{B}$, there is an element A of $\mathfrak{R}$ containing the set B. In particularly, $\mathcal{B}$ is called an open refinement provided all the members of $\mathcal{B}$ are open sets.

The following result is an extension of the result proved in [1] in connection with Baire property of sets.

THEOREM. If the topology of X possesses a σ-locally finite basis, then for every non-Baire set B, there exists $\phi \neq U$ which is open in X such that $B \cap E$ is non-Baire for every $E \supseteq U$.

PROOF. Let $\{B_\alpha : \alpha \in \Lambda\}$ be a σ-locally finite basis corresponding to the topology of X. We first show that if F_α $(\subseteq B_\alpha)$ for each α in Λ are sets of first category, then so is $\bigcup_{\alpha \in \Lambda} F_\alpha$.

Now F_α $(\alpha \in \Lambda)$ being sets of first category, we may write $F_\alpha = \bigcup_{n=1}^{\infty} F_{\alpha,n}$, where $F_{\alpha,n}$ are nowhere dense sets in X. Also, as $\{\mathcal{B}_\alpha : \alpha \in \Lambda\}$ is σ-locally finite, $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ where each $\mathcal{B}_n$ is locally finite by definition.

Now, $\bigcup_{\alpha\in\Lambda} F_\alpha = \bigcup_{\alpha\in\Lambda}\left(\bigcup_{n=1}^{\infty} F_{\alpha,n}\right) = \bigcup_{n=1}^{\infty}\left(\bigcup_{\alpha\in\Lambda} F_{\alpha,n}\right)$, so what we wish to show above could be reached once we are able to establish that $\bigcup_{\alpha}\{F_\alpha : B_\alpha \in \mathcal{B}_n\}$ is a set of first category for each n (∈ □).

But $\bigcup_{\alpha}\{F_\alpha : B_\alpha \in \mathcal{B}_n\} = \bigcup_{\alpha\in\Lambda}\{\bigcup_{n=1}^{\infty} F_{\alpha,n} : B_\alpha \in \mathcal{B}_n\} = \bigcup_{n=1}^{\infty}\left\{\bigcup_{\alpha\in\Lambda} F_{\alpha,n} : B_\alpha \in \mathcal{B}_n\right\}$ We claim that $\bigcup_{\alpha}\{F_{\alpha,n} : B_\alpha \in \mathcal{B}_n\}$ is nowhere dense for each n (∈ □).

Let G ($\neq \phi$) be open and x ($\in X$). Then by definition of local finiteness, there exists an open neighbourhood O_x of x which meets only a finite number of B_α's say B_{α_1}, B_{α_2}, …, B_{α_k} belonging to the family $\mathcal{B}_n$. Now $F_{\alpha_1,n}, F_{\alpha_{2,n}}, \ldots, F_{\alpha_{k,n}}$ being nowhere dense sets such that $F_{\alpha_i,n} \subseteq B_{\alpha_i}$, we may select open sets $O_{\alpha_i} (\neq \phi)$ such that $O_{\alpha_i} (\subseteq G \cap O_x)$, $O_{\alpha_j} \subseteq O_{\alpha_i}$ $(i \leq j)$ and $O_{\alpha_i} \bigcap F_{\alpha_{i,n}} = \phi$ (i = 1, 2, …, k ; j = 1, 2, …, k). We set $O = \bigcap_{i=1}^{k} O_{\alpha_i}$. Then O ($\neq \phi$) is open, O ($\subseteq G$) and $O \bigcap \left(\bigcup_{i=1}^{k} F_{\alpha_{i,n}}\right) = \phi$ which proves the claim. Hence $\cup\{F_\alpha : B_\alpha \in \mathcal{B}_n\}$ and therefore $\bigcup_{\alpha\in\Lambda} F_\alpha$ is a set of first category.

We now claim that there exists α_0 ($\in \Lambda$) such that $B \cap E$ is non-Baire for every E ($\supseteq B_{\alpha_0}$).

If possible, on the contrary, for every $\alpha \in \Lambda$, there exists a set E_α ($\supseteq B_\alpha$) such that $B \cap E_\alpha$ and therefore $B \cap B_\alpha$ is a set with Baire-property. We may write $B \cap B_\alpha = (G_\alpha \setminus P_\alpha) \cup Q_\alpha$ where G_α is open, P_α, Q_α ($\subseteq B_\alpha$) are sets of first category and $(G_\alpha \setminus P_\alpha) \cap Q_\alpha = \phi$. From what we have derived above $\bigcup_{\alpha\in\Lambda} Q_\alpha$ is a set of first category. Moreover, as $\left\{\bigcup_{\alpha\in\Lambda}(G_\alpha \setminus P_\alpha) \setminus (\bigcup_{\alpha\in\Lambda} G_\alpha \setminus \bigcup_{\alpha\in\Lambda} P_\alpha)\right\} \subseteq \bigcup_{\beta\in\Lambda}\left(\bigcup_{\alpha\in\Lambda} P_\alpha \setminus P_\beta\right)$ which is a set of first category, it follows that $B = B \cap X = \bigcup_{\alpha\in\Lambda}(B \cap B_\alpha)$ becomes a set having Baire-property which contradicts our choice of the set *B*. This finally proves the theorem.

NOTE. It is interesting to note that although in theorem 1 (dealing with non-measurable sets) we needed an well-ordering of the basis $\mathcal{B}$, no such requirement is essential in proving theorem 2 (dealing with non-Baire sets).

ACKNOWLEDGEMENT. I am thankful to Prof. S.M. Srivastava (Stat Math Unit) of ISI(Cal) for his valuable suggestions.

REFERENCES

1. D.K. Ganguly and S. Basu, A note in non-measurable sets, Soochow J. Math, 20(1) (1994), 57-59.

2. P.R. Halmos, Measure Theory, Van Nostrand, 1950.

3. J.C. Oxtoby, Measure and Category, Springer-Verlag, 1980.

Received: November, 2008