Applied Mathematical Sciences, Vol. 3, 2009, no. 43, 2143 - 2156

General Method for Exponential-Type Equations for Eight- and Nine-Point Prismatic Arrays

G. L. Silver

Los Alamos National Laboratory*
P.O. Box 1663, MS E502
Los Alamos, NM 87545, USA
gsilver@lanl.gov

*Los Alamos National Laboratory is operated by the Los Alamos National Security, LLC for the National Nuclear Security Administration of the U.S. Department of Energy under Contract No. DE-AC52-06NA25396.

Abstract

The results of three-parameter experiments are commonly interpreted by the trilinear equation for eight data in a prismatic array. If a center point estimate is available, the eight- and nine-point arrays can be represented by new exponential-type equations. The equations are easy to generate, they are invariant under data translation, and they estimate curvature coefficients.

Mathematics Subject Classifications: 65D07, 65D17

Keywords: Grid analysis, Interpolation, Shifting operator, Operational calculus, Data treatment

1. Introduction

The trilinear equation is the traditional method for representing data arranged at the eight vertices of a cube. The equation does not utilize a center point datum and it does not estimate curvature coefficients. This paper illustrates methods for developing exponential-type equations for the eight- and nine-point prismatic designs in Fig. 1. The new equations are easy to generate and they estimate curvature coefficients. The methods typically render a variety of interpolating equations for both designs.

2. Representations of trigonometric expressions. Nine-point equations.

The exponential form of cosh(x+y+z) appears in Eq. (1).

$$\cosh(x + y + z) = (1/2)[\exp(x + y + z) + \exp(-x - y - z)] \qquad (1)$$

Let the term “x” represent the product of the partial differentiation operator and a constant (j) as in (j)d/dx. The symbols “y” and “z” are similarly interpreted as (k)d/dy and (l)d/dz, respectively. The three products are substituted into the respective terms on the right-hand side of Eq. (1) as in Eq. (2). The left-hand side of Eq. (2) and the Taylor expansion of its right-hand side can be multiplied by unity as represented by the ratio of an unknown function F(x,y,z)/F(x,y,z). This operation and some of its consequences have been illustrated in Refs. [1-4]. The interpretation of Eq. (2) is Eq. (3).

$$\cosh(x + y + z) = (1/2)[\exp((j)d/dx + (k)d/dy + (l)d/dz) + \exp(-(j)d/dx - (k)d/dy - (l)d/dz)] \qquad (2)$$

$$\cosh(x+y+z)[F(x,y,z)/F(x,y,z)] = [F(x+j,y+k,z+l)+F(x-j,y-k,z-l)] / [2F(x,y,z)] \qquad (3)$$

Eq. (3) is reinterpreted as Eq. (4) by means of Fig. 1. That is, cosh(x+y+z)F(x,y,z) and sinh(x+y+z)F(x,y,z) can be interpreted as relationships among points in a cube. Hereafter, the sign of equality is reinterpreted by the phrase “is replaced by.”

$$\cosh(x+y+z)[F(x,y,z)/F(x,y,z)] = (I + A) / (2E) \qquad (4)$$

In a similar manner, the expression cosh(x+y–z) is replaced by (D+F)/(2E). Likewise, cosh(x–y+z) is replaced by (G+C)/(2E) and cosh(x–y–z) is replaced by (B+H)/(2E). The hyperbolic sine is amenable to analogous treatment. Thus, sinh(x+y+z) is replaced by (I–A)/(2E) while sinh(x+y–z), sinh(x–y+z), and sinh(x–y–z) are replaced by (D–F)/(2E), (G–C)/(2E), and (B–H)/(2E), respectively.

The right hand side of Eq. (1) is subject to a second interpretation as in Eq. (5). In an analogous manner, cosh(x+y–z), cosh(x–y+z), cosh(x–y–z) are replaced by the expressions (1/2)[JK/L+L/(JK)], (1/2)[JL/K+K/(JL)], (1/2)[J/(LK)+LK/J], respectively.

$$\cosh(x + y + z) = (1/2)[JKL + 1/(JKL)] \qquad (5)$$

Eight equations that are based on sums and differences of cosines can now be described. The first equation is illustrated by Eq. (6). The second equation in the series changes the signs of the last terms on the left- and right-hand sides of Eq. (6) from positive to negative. It has the signs of both (B+H)/(2E) and [J/(KL)+KL/J)] changed from plus to minus as in Eq. (7). The orders of the signs of the terms on both sides of the first and second equations are therefore (+,+,+,+) and (+,+,+,–), respectively. The remaining six equations have the signs of the terms on both sides of the equations as (+,+,–,+), (+,+,–,–), (+,–,+,+), (+,–,+,–), (+,–,–,+), (+,–,–,–), respectively.

$$2[(I+A)/(2E) + (D+F)/(2E) + (G+C)/(2E) + (B+H)/(2E)] = [JKL+1/(JKL)] + [JK/L+L/(JK)] + [JL/K+K/(JL)] + [J/(KL)+KL/J] \quad (6)$$

$$2[(I+A)/(2E) + (D+F)/(2E) + (G+C)/(2E) - (B+H)/(2E)] = [JKL+1/(JKL)] + [JK/L+L/(JK)] + [JL/K+K/(JL)] - [J/(KL)+KL/J] \quad (7)$$

Sums and differences of the hyperbolic sine also generate eight new equations in an analogous manner. The first equation in the series is Eq. (8). The remaining seven equations vary the signs of their terms on both sides of the equations. Their signs follow the same pattern as the series of eight equations that are based on the sums and differences of the cosines.

$$2[(I-A)/(2E) + (D-F)/(2E) + (G-C)/(2E) + (B-H)/(2E)] = [JKL-1/(JKL)] + [JK/L-L/(JK)] + [JL/K-K/(JL)] + [J/(LK)-LK/J] \quad (8)$$

There are now eight equations based on the sums and differences of hyperbolic cosines and eight analogous equations based on the sums and differences of hyperbolic sines. All of them contain the letters A .. I representing data at the nine vertices of the prism in Fig. 1. To account for data translation, let each letter A .. I be augmented by adding a term T to it so that A becomes A+T and likewise for the remaining letters. The sixteen equations now contain the nine numbers A .. I and the unknowns J, K, L, and T.

Let the nine data A .. I be positive and real. A subset of four simultaneous equations can be chosen from the sixteen described equations. Let the subset contain at least one equation from the sine group and one equation from the cosine group. The subset can often be solved for J, K, L, and T. The solutions are obtained analytically if the software can do that. Otherwise, the values of J, K, L, T are estimated by a numerical method. All of J, K, L must be real and positive. Unity is not a permissible value for any of J, K, or L. The value of T must be real but it can be positive or negative.

For example, let nine data be obtained by applying the generating function $M^3 - 3M^2 + 10$ to the first nine integers 1 .. 9 assigned to vertices A .. I, respectively, as in Fig. 1. Let Eqs. (6) and (8) be chosen as two of the equations to be solved. Each equation has signs (+,+,+,+) on both sides. Let the third and fourth equations be the same equations with the signs (+,+,+,–) on each side as described above. That is, a new group of four equations is formed by two equations chosen from the cosine group and two equations chosen from the sine group. They are called the basis equations.

When solved analytically, the selected subset of four simultaneous equations yields several sets of solutions for J, K, L, and T but only two of them contain acceptable choices for J, K, L. The proper selection is made by the help of alternative, exponential-type, eight- or nine-point equations applied to the same data. The eight-point reference equation appears as Eq. (14) in Ref. [5]. The nine-point reference equation substitutes the eight-point equation with Eqs. (3)-(5) as listed in Ref. [6]. Comparison of the presently approximated values of J, K, L to the like numbers in either of the reference equations indicates the proper choice: J~1.241, K~1.635, L~3.796. They are not identical to the ones found in the reference equations. The comparison test is based on the assumption that exponential-type equations for the same data in the same configuration have similar values of J, K, L. The assumption applies in many cases.

The selected values of J, K, L, and the eight vertex measurements, are substituted into Eqs. (18)-(26) in Ref. [5]. If E is an estimate, as might be obtained by an alternative approach, a new interpolating equation for the eight-point cube is obtained. It need not reproduce the center point estimate closely. If E is a measurement, a new equation for the nine-point cube is obtained. If the new equation does not reproduce the center point satisfactorily, the equation can be prefixed by a polynomial multiplier as illustrated in Ref. [6]. Alternatively, another new equation for the nine-point prismatic array can often be generated from a new selection of four basis equations. The sets of eight cosine-related and eight sine-related equations permit many possible choices.

In the present case, the interpolating equation rendered by the described method is Eq. (9). Its coefficients have been rounded for simplicity. Eq. (9) reproduces the data at the corners of the cube. Its estimate of E is 61 whereas the true value is 60. The error is less than 2%.

$$\begin{aligned} R = {} & 39.06 - (31.09)(1.241)^{(x+1)} - (23.14)(1.635)^{(y+1)} - (8.630)(3.796)^{(z+1)} \\ & + (14.97)(1.241)^{(x+1)}(1.635)^{(y+1)} + (7.466)(1.241)^{(x+1)}(3.796)^{(z+1)} \\ & + (4.413)(1.635)^{(y+1)}(3.796)^{(z+1)} + (4.952)(1.241)^{(x+1)}(1.635)^{(y+1)}(3.796)^{(z+1)} \end{aligned} \tag{9}$$

The “true” surface is the generating function applied to (5+x/2+y+5z/2). The sums of the squares of the deviations of interpolating equations from the true surfaces make interesting comparisons. That sum is 3015 using the previously published exponential-type, eight-point equation [5]. For the analogous nine-point equation, the sum is 1318 [6]. The same sum, as rendered by Eq. (9), is 1164. By sum-of-squares-of deviations test, Eq. (9) is the preferred interpolating instrument. Neither the nine-point equation nor Eq. (9) was corrected by a polynomial expression as illustrated in Ref. [6].

Let the generating function be changed to $M^3–4M^2+10$. In this case, Eq. (14) in Ref. [5] yields an interpolating equation containing complex-number coefficients. The analogous nine-point equation contains real coefficients [6]. The new interpolating equation that is generated from the chosen subset of four simultaneous equations is Eq. (10). The sum of squares of the deviations is 1833 for the nine-point equation [6], and 1539 for Eq. (10), respectively. The interpolating equation formed from cosine-related Eqs. (6) and (7), and their two arbitrarily selected sine-related analogs, is presently the preferred choice. The error of Eq. (10) at the center point about 3%.

$$\begin{aligned} R = {} & 32.73 - 22.28(1.271)^{(x+1)} - 15.37(1.765)^{(y+1)} - 4.038(5.042)^{(z+1)} \\ & + 8.853(1.271)^{(x+1)}(1.765)^{(y+1)} + (3.432)(1.271)^{(x+1)}(5.042)^{(z+1)} \\ & + (1.793)(1.765)^{(y+1)}(5.042)^{(z+1)} \\ & + (1.885)(1.271)^{(x+1)}(1.765)^{(y+1)}(5.045)^{(z+1)} \end{aligned} \qquad (10)$$

The examples were based on two equations taken from the cosine group with sign designations (+,+,+,+) and (+,+,+,–). Two analogous equations were taken from the sine group to complete the set. These choices were arbitrary. Many equation selections can be solved by analytical as well as by numerical methods, but the numerical approach is usually easier and faster. In difficult cases, choose another subset of four simultaneous equations. The possible choices consist of one equation from the sine or cosine group, and three equations from the other group, or two equations from each group. No method is presently known for finding the optimum selection in a particular case.

Sometimes a particular choice of four simultaneous basis equations fails to yield solutions for J, K, L, and T by the analytical approach. A solution may nevertheless be obtained by that method: change the precision in the calculations and then convert the equations to rational forms. Then try to solve them. The numerical method has the disadvantage that software typically requires specification of the limits of J, K, and L. The limits can be wide. They are easy to adjust but they should not include unity. The limits of T usually do not need to be specified. The numerical method of solving the four simultaneous equations is usually the first approach because it is easy.

Methods for choosing among candidate interpolating equations include comparing their center point estimates to the true values. Alternatively, compare the first partial derivatives dF/dx, dF/dy, and dF/dz of the candidate equations to the like derivatives as rendered by alternative methods of representing the nine-point cube [3-8].

3. The eight-point cube

The eight-point cube can be interpolated by a method analogous to the nine-point method described above. In the eight-point method, five basis equations are selected from the groups of sine- and cosine-related equations. They are solved for the five unknowns J,K,L,T,E. The solution of five simultaneous equations for five unknowns typically presents more difficulties than the solution of four simultaneous equations for four unknowns. The numerical approach to solving the five equations seems to be faster and easier than the analytical method. In many cases, only the ranges of J, K, L need to be specified. Many cases can be solved without specifying the numerical ranges of T and E.

To illustrate the eight-point method, let three of the five basis equations be taken from the cosine group with signs (+,+,+,+), (+,+,–,+), and (+,–,+,+). The remaining two basis equations are taken from the sine group with signs (+,+,+,–) and (+,+,–,–). This selection is arbitrary. Let the generating function M^3 be applied to the numbers 1 .. 4 as A .. D and 6 .. 9 as F .. I in Fig. 1. The five basis equations are solved for J,K,L,T, and E. Eq. (11) is the resulting eight-point equation that interpolates them. Its sum-of-squares-of-deviations from the surface representing the generating function is 1428. Eq. (11) estimates the center point as E=121.2.

Eq. (12) is the analogous nine-point equation based on the same data but with the addition of the center point datum. It is developed from two equations from each of the sine and cosine groups as the basis equations. The two equations from each group have signs [(+,+,+,–), (+,+,–,–)]. The four basis equations are solved simultaneously for J, K, L, and T. The sum-of-squares-of-deviations of Eq. (12) from the surface representing the generating function is 833 and it estimates the center point as E=128.1. The true value is E=125. The nine-point method is superior in this case. It has the advantage of the extra datum at center point E. Other choices of the basis equations yield other results.

$$\begin{aligned} R = {} & 47.13 - (65.25)(1.208)^{(x+1)} - (55.01)(1.436)^{(y+1)} - (24.28)(2.884)^{(z+1)} \\ & + (44.75)(1.208)^{(x+1)}(1.436)^{(y+1)} + (18.94)(1.208)^{(x+1)}(2.884)^{(z+1)} \\ & + (17.89)(1.436)^{(y+1)}(2.884)^{(z+1)} + (16.83)(1.208)^{(x+1)}(1.436)^{(y+1)}(2.884)^{(z+1)} \end{aligned} \qquad (11)$$

$$\begin{aligned} R = {} & 73.49 - (89.14)(1.172)^{(x+1)} - (73.83)(1.407)^{(y+1)} - (35.60)(2.671)^{(z+1)} \\ & + (55.42)(1.172)^{(x+1)}(1.407)^{(y+1)} + (25.68)(1.172)^{(x+1)}(2.671)^{(z+1)} \\ & + (18.16)(1.407)^{(y+1)}(2.671)^{(z+1)} + (26.82)(1.172)^{(x+1)}(1.407)^{(y+1)}(2.671)^{(z+1)} \end{aligned} \qquad (12)$$

4. Discussion of the exponential-type equations

The second column of Table 1 represents the sums-of-squares-of-deviations of an earlier method for representing an eight-point cube: Eq. (14) in Ref. [5]. The third column represents similar results using another eight-point, exponential-type equation that is based on three equations chosen from the cosine group and two equations taken from the sine group as described above. Comparison of the tabulated results in Table 1 suggests that there is little difference between the two methods. That is an artifact of the particular choice of the five basis equations.

The described methods are potentially useful for the eight- and nine-point prismatic arrays. They typically generate several exponential-type equations that reproduce the data at the corner points of the prism. They are exact on simple exponential functions like 2^x and they are invariant under data translation. Both methods estimate the datum at the center point of the prism. The user decides if the estimate is satisfactory, or if the interpolating equation should be prefixed by a polynomial multiplier, or if a new set of simultaneous basis equations should be tested [6].

Table 2 compares the sums-of-squares-of-deviations of three exponential-type surfaces as rendered by three nine-point equations. The entries in the second column are obtained from the nine-point exponential-type method described in Ref. [6]. That nine-point equation was not corrected by a polynomial multiplier [6]. The letter M represents a number at a vertex of the cube. In the present cases, the numbers A .. I are the first nine integers 1-9, respectively. The notation M^2 indicates that every number A .. I was squared to generate the trial data. Thus, M(4)=16. Monotonic data are not typical of laboratory experience, nor are they required by the present methods, but they make convenient examples.

The entries in the third and fourth columns in Table 2 are based on the illustrated nine-point method. The third column lists results obtained from two equations taken the cosine group. One has terms with signs (+,+,+,+) on both sides while the other one has signs (+,+,+,–) on both sides. The remaining two equations are chosen from the sine group with the same sign assignments. This selection is represented by Method (1) in Table 2. The table also lists sums-of-squares-of-deviations obtained by another group of four simultaneous equations. The first two equations are chosen from the cosine group with signs (+,+,+,–) and (+,+,–,–) on both sides, respectively. The remaining two equations are taken from the sine group with the same sign assignments. Results obtained by the second choice are denoted Method (2) in Table 2. All of the equations estimate curvature, something formerly believed to be impossible on eight-point prismatic designs and seldom illustrated with nine-point designs.

The eight-point method described by Eq. (14) in Ref. [5] yields complex-number coefficients in certain cases. However, other exponential-type equations derived from the same data, as illustrated in this manuscript, may not be compromised by the same problem. In some cases, the values of J,K,L may lie outside the arbitrary limits typically needed by numerical methods for solving simultaneous equations. The numerical approach is then certain to fail. More than one set of estimates of J,K,L may satisfy a numerical algorithm within the limits of its precision. Different interpolating equations could be obtained from the same choice of basis equations and the same data.

There are many ways to generate polynomial-, trigonometric-, and exponential-type equations for eight- and nine-point prismatic arrays as in Fig. 1 [5-9]. The variety of alternatives offered by operational methods may interest the experimentalists. It may not be apparent which one of the many equations represents the best selection in a particular case. The problem has not attracted attention. Decisions based on laboratory results may represent the most practical solution to this question. The presence of improper curvature effects, such as unwarranted extrema, is a criterion for rejection of an equation but the criterion is tedious to apply [9]. The easy default is the trilinear equation but its popularity is not evidence that it invariably makes the best use of experimental data. The literature citations are restricted to methods that estimate curvature coefficients for the eight- and nine-point prismatic arrays. They are primarily related to the shifting operator. It is difficult to otherwise find so many simple, easily-applied alternatives. There appear to be more equations for the eight-point cube than for the four-point rectangle [5,7,10].

5. Rotational invariance

There are eight equations in each of the cosine and sine groups. The former contain differences between terms and the latter contain differences between terms and within terms. If a numerically substituted cube is rotated about its center point, the sums and differences on the left-hand sides of the equations are affected but J, K, and L on the right-hand sides are not affected. This asymmetry has a consequence: the interpolated number at an arbitrary point within the cube is not reproduced as the cube is rotated.

Suppose x, y, and z are positive numbers that are the arguments of exponential-type functions such as exp(x). Let those functions of x, y, and z be designated J, K, and L, respectively. If x, y, and z change their signs, J, K, and L become the reciprocals of their former values so they are represented by 1/J, 1/K, and 1/L, respectively. Let L represent the effect of the z-coordinate (the vertical coordinate) that increases from bottom to top in the initial orientation of a particular cubical array. Let the cube be rotated so that the bottom and top faces are interchanged. Now the effect of z decreases from bottom to top. In other words, L in the old orientation becomes 1/L in the new orientation. Similar remarks apply to the effects of the x- and y-coordinates so that they are also represented by J or 1/J, and K or 1/K, depending on how the cube is oriented.

Let the generating function be M^3+10. If this function is applied to the numbers [1,2,3,4,5,6,7,8,9] at vertices A .. I, respectively, the trial data become [11,18,37, ...], respectively. Let one of the four simultaneous, basis equations be taken from the cosine group with signs (+,+,+,+) on both sides and let the second equation be taken from the sine group the same sign assignments. The third and fourth equations are taken one each from the cosine and sine groups with signs assignments (+,+,+,–) on both sides, respectively. The interpolating equation is Eq. (13) in which J~1.191, K~1.460, L~2.698. At the point x=0.9, y=0.8, and z=0.7, the predicted response is R=491.99, nearly.

$$R = 63.84 - (72.06)J^{(x+1)} - (58.15)K^{(y+1)} - (29.23)L^{(z+1)} + (43.14)J^{(x+1)}K^{(y+1)} + (25.49)J^{(x+1)}L^{(z+1)} + (17.82)K^{(y+1)}L^{(z+1)} + (20.16)J^{(x+1)}K^{(y+1)}L^{(z+1)} \quad (13)$$

Let the cube be rotated so that the data at the vertices A .. I become the generating function applied to [8,3,9,4,5,6,1,7,2], respectively. The numbers that were formerly increasing from left to right, the x-direction, are now increasing from front to back, the y-direction. The new K is therefore assigned as the old J. The parameter L was formerly associated with numbers that were increasing from bottom to top. The same numbers are now found in the side planes, the x-direction, but in a decreasing order from left to right. This means the new J is assigned as the reciprocal of the old L. The numbers that were formerly increasing in the y-direction are now decreasing in the z-direction. The new L is therefore the reciprocal of the old K. We now have the new values J~0.3706, K~1.191, L~0.6848. The new interpolating equation for the rotated cube is obtained from the new J, K, L and the new corner-point data by means of Eqs. (18)-(26) in Ref. [5]. It is Eq. (14). Note that the new values of J, K, L are determined by assignments based on their previous values and the new orientation of the cube, not by solving a new subset of four simultaneous basis equations for the rotated cube.

$$R = 63.84 - (212.8)J^{(x+1)} - (72.06)K^{(y+1)} - (124.0)L^{(z+1)} + (185.5)J^{(x+1)}K^{(y+1)} + (276.6)J^{(x+1)}L^{(z+1)} + (91.97)K^{(y+1)}L^{(z+1)} + (312.9)J^{(x+1)}K^{(y+1)}L^{(z+1)} \quad (14)$$

The point with former (x,y,z) coordinates (0.9,0.8,0.7) has new (x,y,z) coordinates (–0.7,0.9,–0.8), respectively. The response at the point with the new coordinates substituted into Eq. (14) is R=491.99, nearly. Both the original and the rotated cubes exhibit the same sums-of-squares-of-deviations from the original and the rotated reference surfaces. The example illustrates the how the exponential-type equations adapt to rotation of the prismatic design in Fig. 1. These remarks apply to the eight- and nine-point methods.

6. Curvature coefficients

Let the numbers 1 .. 4 and 6 .. 9 be placed at vertices A .. D and F .. I, respectively, in Fig. 1. Let trial generating functions be denoted by the letter M. Let those functions be applied to the numbers positioned at vertices A .. I in the eight-point cube in Fig. 1. The coordinate system is –1 .. 1. Taylor expansions of M(5+x/2+y+5z/2) yield the reference values of the curvature (quadratic-term) coefficients. Those coefficients are estimated by eight-point operational equations by means of their Taylor expansions [5].

Table 3 illustrates comparisons of curvature coefficients for seven generating functions that are listed in the first column of the table. The top numbers in each block in the second, third, and fourth columns of Table 3 represent the reference values of the x^2-, y^2-, and z^2-coefficients. The middle numbers in each block are estimates of those numbers as rendered by an operational, polynomial-type equation, Eq. (10) in Ref. [5]. The bottom numbers are the estimates as obtained by the eight-point method described in Section 3 above. The comparisons suggest that polynomial-type equations render better estimates when applied to data generated by polynomial-type functions such as M^2. Taylor expansions of the 8-point exponential-type equations also estimate the quadratic-term coefficients. Generally, those estimates are more accurate when the data are generated by exponential-type functions like 2^M. These observations are not surprising. Laboratory experience may be the easiest way to make decisions about which form is best for routine applications. That experience is useful for making the choice of the basis equations, too.

Table 1. Sums of squares of deviations of two eight-point, exponential-type equations from typical trial surfaces. The data at vertices A-D and F-I in Fig. 1 are the integers 1-4 and 6-9, respectively, treated by the generating functions. The second column of the table lists results as obtained by Eq. (14) in Ref. [5]. The third column lists results obtained by the method described in Section 3 in the text.

Function*	Eight-point [5]	Eight-point (text)
M^2	6.77	6.64
M^3	1566	1428
$M^3 + 2^M$	905	797
sinh(M/4)	0.00316	0.00308
cosh(M/4)	0.00565	0.00544
(M)ln(M+1)	0.153	0.154
$(12/10)^M$	0	0
$(M)(12/10)^M$	0.0788	0.0754

*The function is applied to the integers 1-4 at vertices A-D, and 6-9 at vertices F-I, respectively.

Table 2. Sums-of-squares-of-deviations of nine-point, exponential-type equations from typical trial surfaces. The data at vertices A-I in Fig. 1 are the integers 1-9, respectively. The second column lists results as obtained by the approach in Section 3 of Ref. [6]. (A polynomial multiplier was not applied.) Method (1) uses two equations from the cosine group with signs (+,+,+,+) and (+,+,+,–) on both sides of the equations and two analogous equations from the sine group. Method (2) uses two equations from cosine group with signs (+,+,+,–) and (+,+,–,–) on both sides and two analogous equations from the sine group of basis equations.

Function*	Nine-point [6]	Method (1)	Method (2)
M^2	5.73	5.46	5.50
M^3	1067	976	833
$M^3 + 2^M$	400	333	289
sinh(M/4)	0.00276	0.00263	0.00267
cosh(M/4)	0.00415	0.00375	0.00372
(M)ln(M+1)	0.146	0.147	0.208
$(12/10)^M$	0	0	0
$(M)(12/10)^M$	0.0683	0.0647	0.0601

*The function is applied to the integers 1-9, denoted M, at vertices A-I in Fig. 1.

Table 3. Estimates of quadratic-term coefficients as obtained by Taylor expansions of the generating functions and two operational equations. The top, middle, and bottom numbers in each block represent the reference coefficients, the approximations as rendered by Eq. (10) in Ref. [5], and the approximations as rendered by exponential-type equations prepared as described in Section 3 of the text, respectively. The entries apply to an eight-point prismatic array as in Fig. 1. See Section 6 above.

Function (M)	x^2-coefficient	y^2-coefficient	z^2-coefficient
M^2	0.25 0.25 0.20	1.00 1.00 0.85	6.25 6.25 6.37
M^3	3.75 3.75 2.65	15.0 15.0 10.3	93.8 93.8 86.5
2^M	1.92 6.88 1.92	7.69 22.3 7.69	48.0 54.6 48.0
sinh(M/2)	0.19 0.40 0.20	0.76 1.42 0.78	4.73 4.78 4.73
$(M!)^{(1/2)}$	1.12 14.2 1.36	4.48 35.5 5.73	28.0 53.9 31.4
$\tan(9M^o)$	0.012 0.11 0.024	0.049 0.26 0.13	0.31 0.42 0.52
$\tan^{-1}(M^3)$	$-(4.8)(10^{-4})$ –0.032 –0.0011	–0.0019 –0.050 –0.0066	–0.012 –0.057 –0.022

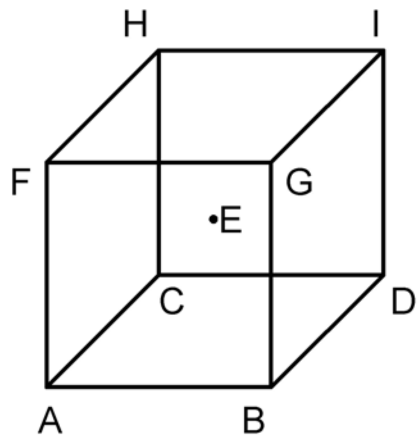

Fig. 1. The nine-point prismatic array.

References

1. G. L. Silver, Operational method of data treatment. J. Comput. Chem. 6 (1985) 229-236.

2. G. L. Silver, Deriving operational equations. Appl. Math. Sci. 2 (2008) 397-406.

3. G. L. Silver, Analysis of three-dimensional grids: Sine-and-cosine equation for the nine-point prismatic array. Appl. Math. Comput. 183 (2006) 617-621.

4. G. L. Silver, Analysis of three-dimensional grids: Interpolating the nine-point prism. The shifting operator as an analytical instrument. Appl. Math. Comput. 186 (2007) 1694-1700.

5. G. L. Silver, Analysis of three-dimensional grids: the eight-point cube. Appl. Math. Comput. 153 (2004) 467-473.

6. G. L. Silver, Analysis of three-dimensional grids: Interpolation of data in prismatic array and the estimation of a missing datum. Appl. Math. Comp. 174 (2006) 1398-1407.

7. G. L. Silver, Analysis of three-dimensional grids: cubes and cubic coefficients. Appl. Math. Comput. 166 (2005) 196-203. Also see Erratum to the citation: Appl. Math. Comput 174 (2006) 1659. Simplified forms of the linear-term coefficients appear as Eqs. (12)-(14) in Appl. Math. Comput. 179 (2006) 553-558 by the same author.

8. G. L. Silver, Analysis of three-dimensional grids: Cubic equations for nine-point prismatic arrays. Appl. Math. Comput. 170 (2005) 752-760.

9. G. L. Silver, Multiple-point cubes. Appl. Math. Sci. 2 (2008) 1229-1238.

10. G. L. Silver, Operational equations for data in rectangular array, Comput. Stat. & Data Analysis 28 (1998) 211-215.

Received: January, 2009