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The Evaluation of the Policy of Portfolio Problems

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Abstract

In this note, the authors try to evaluate a portfolio strategy by comparing its expected utility with that of the optimal strategy in the market with consumption, and obtain some interesting results. This research can be regarded as the natural generalization of the work posed by M.B. Haugh, L. Kogan and J. Wang.

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1 Introduction

The study of optimal portfolio choice problem has been an active fields over the past several decades. In particular, round about 1950, since the formidable papers [1, 2, 3, 4, 5, 6, 7] were published in succession, these works stimulate such research fields to present a scene of prosperity, and demonstrate the abnormal importance of this topic. It is well known that the optimal portfolio choice problem is one of the main subjects in finance, and the theory of portfolio selections is becoming more and more perfect after the development [8, 9, 10, 11, 12, 13, 14, 15] of more than

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fifty years. However, the analytical solutions to the optimal portfolio problems are not obtained generally in the practical market. What we can do in general is to get the approximations of this optimal portfolio problem. Therefore, it has some economic significance and academic value to evaluate the difference between the approximate solution and the optimal solution. In [16], M.B. haugh, L.Kogan and J.Wang studied this problem without consumption and obtained some remarkable results. Following this research direction, we continue to study the similar optimal portfolio choice problem with consumption, and arrive at some interesting results. These conclusions can be regarded as the generalization of works in [16].

This manuscript is arranged as follows. In section 2, some necessary notations and terminologies are given; The section 3 pays its attention on studying the optimal portfolio choice with consumptions.

2 Preparations

This section pays mainly attention on the market model without consumption and the duality method, meanwhile develops the duality method of this model.

2.1 The Investment Market Model Without Consumptions

This chapter is going to introduce the optimal portfolio selection problem with portfolio restricted condition in incomplete market.

Investment Probability Spaces

Thinking of complete probability spaces, (Ω, Γ, P), with fixed time interval $[0, T]$, where $T > 0$. Define the $N-$dimension standard Brownian motion in the probability space as $w = \{w_n(t) : t \in [0, T], n = 1, 2, \cdots, N\}$. The filter constructed from w is $F = \{\Gamma_t : t \in [0, T]\}$, it's an increasing σ-field of Γ. Suppose F is complete, i.e. Γ_0 includes a zero measure set of the measure P and $\Gamma_T = \Gamma$. The financial market M is composed of one risk-free bond and N stocks, at the time t the interest rate of the no-risk bond is r_t, the price is P_{0t}, the price vector of N stocks is $P_t = (P_{1t}, \cdots, P_{Nt})$. With no loss of generality, we can suppose the N stocks don't send bonus. The instant return of asset is decided by the $M-$dimensional state variables X_t, which satisfy

$$dP_{0t} = P_{0t} r(t) dt \quad P_{00} = 1 \tag{2.1.1}$$

$$dP_t = P_t[\mu_p(X_t)dt + \Sigma_p(X_t)dw_t] \tag{2.1.2}$$

$$dX_t = \mu_x(X_t)dt + \Sigma_x(X_t)dw_t \tag{2.1.3}$$

where μ_p and μ_x are respectively N and M dimensional drifting vector, $\sum_p$ and $\sum_x$ are respectively $N \times N$ and $M \times N$ diffusion matrix. Let $r(t) \geq -k, \forall 0 \leq t \leq T$ and $\epsilon' \sum_p(X_t)^T \epsilon \geq \delta \|\epsilon\|^2, \forall (t, \epsilon) \in [0, T] \times \mathbb{R}^N$, where $k \geq 0, \delta > 0$ and assume

$E_0 \int_0^T r(s)ds < \infty$. Let the instant risk price of N stocks be a "relative risk process" given by

$$\eta_t = \Sigma_{pt}^{-1}(\mu_{pt} - r(t)I) \tag{2.1.4}$$

where $I = (1, 1, \cdots, 1)'$, and it satisfy the square integrability: $E_0 \int_0^T \|\eta_t\|^2 ds < \infty$.

Constraints Conditions For Portfolios

This work considers that an economic agent can make the decision at time $t \in [0, T]$ as follow: the proportion of stock i in total assets $A(t)$ is $\pi_i(t)$. Obviously, this decision is decided by instant information amount Γ_t, doesn't include any expectation for future, meanwhile it guarantees these market behavior of economic agents will not effect market price of bond. When we ensure $\pi(t) = (\pi_1(t), \cdots, \pi_N(t))'$, one can know the fund invested in no-risk bond is $A(t)[1 - \pi' I]$. From (2.1.1),(2.1.2), we get that, at time t, the wealth process $A(t)$ satisfies

$$\begin{aligned} dA(t) &= A(t)\pi'(t)(\mu_{pt}dt + \Sigma_{pt}dW_t) + A(t)(1 - \pi'(t)I)r(t)dt & (2.1.5)\\ &= r(t)A(t)dt + A(t)\pi'(t)\Sigma_{pt}dW_{0t} & (2.1.6)\end{aligned}$$

where $W_{0t} \doteqdot W_t + \int_0^t \eta_s ds$ is a standard Browain motion. For a given portfolio policy process: $\pi = \{\pi(t), 0 \le t \le T\} \in \mathbb{R}^N$, $\{\Gamma_t\}$ is a progressive measurable process and satisfies $E_0 \int_0^T \|\pi_t\|^2 dt < \infty$. For a given portfolio policy, we can get the wealth process satisfying (2.1.5) is of $A \equiv A^{a,\pi}$, where$a \in (0, \infty)$ is the initial wealth. For the initial wealth $a \in (0, \infty)$, the portfolio-consumption policy (π) satisfying $A^{a,\pi} \ge 0, \forall 0 \le t \le T$ is said to be feasible. The set of all feasible policies (π) is defined as $\ell_0(a)$. Suppose that the investment proportion vector $\pi(t)$ is restricted in a given closed convex constrained set K including zero element, i.e., there holds

$$\pi(t) \in K \tag{2.1.7}$$

Utility Functions

Assume that the utility function $U : (0, \infty) \to \mathbb{R}$ is a strict concave increasing continuous function and satisfies

$$\begin{cases} \nabla U(0+) \triangleq \lim\limits_{x \downarrow 0} \nabla U(x) = \infty \\ \nabla U(\infty) \triangleq \lim\limits_{x \to \infty} \nabla U(x) = 0 \end{cases} \tag{2.1.8}$$

Let $I(\cdot) : (0, \infty) \to (0, \infty)$ be the inverse function of function $\nabla U(\cdot)$, it's strictly decreasing continuous, and satisfies

$$I(0+) = \infty, I(\infty) = 0 \tag{2.1.9}$$

Suppose that the Legend-Fenchel transform of $U(-x)$ is

$$\widetilde{U} \doteqdot \max_{x>0}[U(x) - xy] = U(I(y) - yI(y)), 0 < y < \infty \tag{2.1.10}$$

$\widetilde{U}$ is a strictly deceasing and strictly convex function satisfying

$$\begin{cases} \widetilde{U}(y) = -I(y), 0 < y < \infty \\ \widetilde{U}(x) \doteqdot \min_{y>0}[\widetilde{U}(x) + xy] = \widetilde{U}(\nabla U(x)) + x\nabla U(x), 0 < x < \infty \end{cases} \tag{2.1.11}$$

It is not hard to see that for any $x, y > 0$ there holds the following

$$U(I(y)) \geq U(x) + y[I(y) - x] \tag{2.1.12}$$

$$\widetilde{U}(\nabla U(x)) + x[\nabla U(x) - y] \leq \widetilde{U}(y) \tag{2.1.13}$$

Objective Functions

For the feasible policy (π) in feasible policy set $\ell_0(a)$, the total expected utility

$$J(a;\pi) \doteqdot E_0 U_1(A^{a,\pi}(T)) \tag{2.1.14}$$

Define a set $\ell'_0(a) = \{\pi \in \ell_0(a) : E_0 U_2^-(A^{a,\pi}(T)) < \infty\}$ (where $X^- = max(-X, 0)$). The constrained optimal portfolio selection problem (P_0) of this model can be defined

$$(P_0) \quad \begin{cases} V(a) \doteqdot \sup_{(\pi)\in\ell'(a)} J(a;\pi), a \in (0,\infty) \\ \text{s.t.} dP_{0t} = P_{0t} r(t)dt \quad P_{00} = 1 \\ dP_t = P_t[\mu_p(X_t)dt + \sum_p(X_t)dW_t] \\ dX_t = \mu_x(X_t)dt + \sum_x(X_t)dW_t \\ dA(t) = A(t)\pi'(t)(\mu_{pt}dt + \sum_{pt} dW_t) + A(t)(1-\pi'(t)I)r(t)dt \end{cases} \tag{2.1.15}$$

where $\ell'(a) \doteqdot \{(\pi) \in \ell'_0(a) : \pi(t, W) \in K\}$, K is the set of (2.1.7).

2.2 The Duality Methods

The idea of duality methods is through the linear function $\delta(\cdot)$ defined on portfolio policy set $K \in \mathbb{R}^N$, we can get the dual space of original space $\mathbb{R}^N$; we can also get a set D satisfying some conditions in the dual space. That is, for any $\nu \in D$, we can get the new supposed market model relative to original market model, and contact constrained optimization in original market model with unconstrained optimization in the new market model, the stochastic process $\{\nu\}$ here acts same as the Lagrange multiplier in constrained optimization. This method is introduced by Cvitanic, Karatzas [18] and Schroder, Skiadas[19].

Support Functions on Portfolio Policy Restricted Set K

Given a nonempty closed convex portfolio policy restricted set $K \in \mathbb{R}^N$, define a support function on convex set K as

$$\delta(x) \doteqdot \sup_{\pi\in K}\{(\pi' x) : \mathbb{R}^N \to \mathbb{R}^N \cup \{+\infty\}\} \tag{2.2.1}$$

The effective domain of the support function is

$$\begin{cases} \tilde{K} \doteqdot \{x \in \mathbb{R}^N : \delta(x) < \infty\} \\ \quad = \{x \in \mathbb{R}^N : \exists \beta \in \mathbb{R}, -\pi' x \leq \beta, \forall \pi \in K\} \end{cases} \tag{2.2.2}$$

The effective domain is a convex cone, and $\delta(x)$ is continuous on the effective domain. It is easy to see that there holds the following

1. Nonnegative homogeneity: $\delta(\alpha\nu) = \alpha\delta(\nu), \forall \nu \in \mathbb{R}^N, \alpha \geq 0$;
2. Subadditivity: $\delta(\nu + \eta) \leq \delta(\nu) + \delta(\eta), \forall \nu, \eta \in \mathbb{R}^N$;
3. $\pi \in K \Leftrightarrow \delta(\nu) + \pi'\nu \geq 0, \forall \nu \in \tilde{K}$;
4. Boundedness $\delta(\nu) \geq c_0, \exists c_0 \in \mathbb{R}$.

New Market Model and it's Unconstrained Optimal Selection Problems

Suppose
$\hbar = \{\nu = (\nu(t), 0 \leq t \leq T) : \nu(t) \in \mathbb{R}^N, \nu$ is a recurrent measurable process of $\{\Gamma_t\}\}$, and define the norm on this space by $\|\nu\|^2 \doteqdot E_0 \int_0^T \|\nu(t)\|^2 dt < \infty$. Then, one can define an inner product as $< \nu_1, \nu_2 > \doteqdot E_0 \int_0^T \nu_1'(t)\nu_2(t)dt$, in this setting, the space $\hbar$ is a Hilbert space. A set D of $\mathcal{F}_t$adapted $\mathbb{R}^N$ valued process to be defined as $D \doteqdot \{\nu \in \ell : E_0 \int_0^T \delta(\nu(t))dt < \infty\}$. Obviously, for any $\nu \in D$, we can conclude that $\nu(t, w) \in \tilde{K}$. For any given $\nu \in D$, one can get a new finance market model M_ν, likewise there are a no-risk bond and N exchangeable stocks. But in the new market, the price process of these bonds has changed accordingly

$$dP_{0t}^{\nu} = P_{0t}^{\nu}[r(t) + \delta(\nu_t)]dt \quad P_{00} = 1 \tag{2.2.3}$$

$$dP_t^{(\nu)} = P_t^{(\nu)}[(\mu_p(X_t) + \nu(t) + \delta(\nu(t)))dt + \Sigma_p(X_t)dW_t] \tag{2.2.4}$$

The risk market price and no-risk return rate in the new market model are respectively

$$\eta_t^{(\nu)} = \Sigma_{pt}^{-1}[\mu_{pt} + \nu(t) + \delta(\nu(t))I - (r_t + \delta(\nu(t)))I] = \eta_t + \Sigma_{pt}^{-1}\nu(t) \tag{2.2.5}$$

$$\gamma_0^{(\nu)}(t) \doteqdot \exp\{-\int_0^t [r(s) + \delta(\nu(s))]ds\} \tag{2.2.6}$$

The wealth process in new market model satisfies

$$\begin{cases} dA^{(\nu)}(t) = A^{(\nu)}(t)\pi_t'\{(\mu_{pt} + \nu(t) + \delta(\nu(t)))dt + \Sigma_{pt}dW_t\} \\ +A^{(\nu)}(t)(1 - \pi_t'I)(r(t) + \delta(\nu(t)))dt \\ = [(r(t) + \delta(\nu(t)))A^{(\nu)(t)}]dt + A^{(\nu)}(t)\pi'\Sigma_{pt}dW_{0t}^{(\nu)} \\ = r(t)A^{(\nu)}(t)dt + A^{(\nu)}(t)[\delta(\nu(t)) + \pi_t'\nu(t)]dt + A^{(\nu)}(t)\pi'\Sigma_{pt}dW_{0t} \end{cases} \tag{2.2.7}$$

where $A^{(\nu)}(0) = a, W_{0t}^{(\nu)} \doteqdot W_t + \int_0^t \eta_s^{(\nu)}ds$. Likewise we can define feasible policy set in the new market model M_ν as $\ell_0^{(\nu)}(a) = \{\pi : A^{(\nu)}(t) \geq 0, \forall 0 \leq t \leq T\}$. Similar to $\ell_0'(a)$, we can define $\ell_0'^{(\nu)}(a) = \{\pi \in \ell_0^{(\nu)}(a) : E_0 U_2^-(A^{(\nu)a,\pi}(T)) < \infty\}$. Then, the unconstrained optimization $(P_0^{(\nu)})$ in the new market can be defined as

$$(P_0^{(\nu)}) \quad \begin{aligned} & V_0^{(\nu)}(a) \doteqdot \sup_{\pi \in \ell'^{(\nu)}(a)} J(a;\pi), a \in (0,\infty) \\ & \text{s.t. } dP_{0t}^{(\nu)} = P_{0t}^{(\nu)}[r(t) + \delta(\nu_t)]dt \quad P_{00} = 1 \\ & dP_t^{(\nu)} = P_t^{(\nu)}[(\nu_p(_t) + \nu(t) + \delta(\nu(t)))dt + \Sigma_p(X_t)dW_t] \\ & dA^{(\nu)}(t) = r(t)A^{(\nu)}(t)dt + A^{(\nu)}(t)[\delta(\nu(t)) + \pi_t'\nu(t)]dt + A^{(\nu)}(t)\pi'\Sigma_{pt}dW_{0t} \\ & A^{(\nu)}(0) = a \end{aligned}$$

Thinking of constrained optimization problem (P_0) in original market model and unconstrained optimization problem $(P_0^{(\nu)})$ in the new market model as follows:

In (P_0), the wealth process satisfies

$$dA(t) = A(t)\pi'(t)(\mu_{pt}dt + \Sigma_{pt}dW_t) + A(t)(1 - \pi'(t)I)r(t)dt \tag{2.2.8}$$

In $(_0^{(\nu)})$, the wealth process satisfies:

$$\begin{aligned} dA^{(\nu)}(t) &= A^{(\nu)}(t)\pi'(t)[(\mu_{pt} + \nu(t) + \delta(\nu(t)))dt + \Sigma_{pt}dW_t] + A^{(\nu)}(t)(1 - \pi'(t)I)r(t)dt \\ &= r(t)A^{(\nu)}(t)dt + A^{(\nu)}(t)[\delta(\nu(t)) + \pi'(t)\nu(t)]dt + A^{(\nu)}(t)\pi'\Sigma_{pt}dW_{0t} \end{aligned}$$

Compare the two processes above, we can find they very similarly, there is just something excessive on right side of second equation: $A^{(\nu)}(t)[\delta(\nu(t)) + \pi'(t)\nu(t)]dt$. Through the define of the support function, we can obviously get that: $A^{(\nu)}(t)[\delta(\nu(t)) + \pi'(t)\nu(t)] \geq 0$. So, for a fixed $(\pi, c) \in \ell'(a)$, we know $0 \leq A(t) \leq A^{(\nu)}(t), \forall 0 \leq t \leq T$. Then, we can get $(\pi, c) \in \ell_0'^{(\nu)}(a)$ and $E_0U_2(A^{a,\pi,c}(T)) \leq E_0U_2(A^{(\nu)a,\pi,c}(T))$. So we can get $\ell_0'(a) \subset \ell_0'^{(\nu)}(a)$.

2.3 The Lower Bound of Optimal Portfolio Without Consumption

Similar to the duality method above, we can get the lower bound of problem (P_0) as follow: Firstly, the lower support function on set K can be defined

$$\delta'(x) \doteqdot \inf_{\pi \in K}\{(-\pi'x) : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}\} \tag{2.3.1}$$

The effective domain of the lower support function: $\tilde{K}' \doteqdot \{x \in \mathbb{R}^N : \delta'(x) > -\infty\} = \{x \in \mathbb{R}^N : \exists \beta \in \mathbb{R}, -\pi'x \geq \beta, \forall \pi \in K\}$. For Hilbert space $\hbar$, one can define similarly the set D' of $\mathcal{F}_t-$adapted $\mathbb{R}^N$ valued process to be $D' \doteqdot \{\nu \in \hbar : E_0 \int_0^T \delta'(\nu(t))dt \geq -\infty\}$. For a given $\nu' \in D'$, we can get a new finance market model $M_{\nu'}$, and in this new market model, no-risk interest rate and expected return rate of risk bond have changed:

$$r^{(\nu')}(t) = r(t) + \delta'(\nu'(t)) \tag{2.3.2}$$

$$\mu^{(\nu')}(X_t) = \mu_p(X_t) + \nu'(t) + \delta'(\nu'(t)) \tag{2.3.3}$$

Then, in new market, the price processes of these bonds have changed relatively:

$$dP_{0t}^{(\nu')} = P_{0t}^{(\nu')}[r(t) + \delta'(\nu'(t))]dt, \quad P_{00}^{(\nu')} = 1 \tag{2.3.4}$$

$$dP_t^{(\nu')} = P_t^{(\nu')}[(\mu_p(X_t) + \nu'(t) + \delta'(\nu'(t)))dt + \Sigma_p(X_t)dW_t] \tag{2.3.5}$$

Then, the risk market price and no-risk interest rate in new market model are respectively:

$$\eta_t^{(\nu')} = \Sigma_{pt}^{-1}[\mu_{pt} + \nu'(t) + \delta'(\nu'(t))I - (r(t) + \delta'(\nu'(t)))I] = \eta_t + \Sigma_{pt}^{-1}\nu'(t) \tag{2.3.6}$$

$$\gamma_0^{(\nu')}(t) \doteqdot \exp\{-\int_0^t [r(s) + \delta'(\nu'(s))]ds\} \tag{2.3.7}$$

The wealth process in new market satisfies

$$\begin{aligned} &dA^{(\nu')}(t) = A^{(\nu')}(t)\pi_t'\{(\mu_{pt} + \nu'(t) + \delta'(\nu'(t)))dt + \Sigma_{pt}dW_t\} \\ &+A^{(\nu')}(t)(1 - \pi_t'I)(r(t) + \delta'(\nu'(t)))dt \\ &= [(r(t) + \delta(\nu(t)))A^{(\nu)(t)}]dt + A^{(\nu)}(t)\pi'\Sigma_{pt}dW_{0t}^{(\nu)} \\ &= r(t)A^{(\nu')}(t)dt + A^{(\nu')}(t)[\delta'(\nu'(t)) + \pi_t'\nu'(t)]dt + A^{(\nu')}(t)\pi'\Sigma_{pt}dW_{0t} \end{aligned} \tag{2.3.8}$$

where $A^{(\nu')}(0) = a$.

By virtue of (2.1.5) and (2.3.8), we arrive at

$$\frac{dA^{(\nu')}(t)}{A^{(\nu')}(t)} - \frac{dA(t)}{A(t)} = [\delta'(\nu'(t)) + \pi'(t)\nu'(t)]dt$$

By (2.3.1), one knows that: if $\pi \in K$, the equation above is nonpositive, then $A(t) \geq A^{(\nu')}(t) \geq 0, \forall 0 \leq t \leq T$. That is, $V(a) \geq V_0^{(\nu')}(a), \forall \nu \in D'$.

3 The Portfolio Selection Models With Consumptions

3.1 The Portfolio Market Model With Consumptions

The consumption policy process $C = \{C(t) : 0 \leq t \leq T, C(t) \geq 0\}$ is $\mathcal{F}_t-$progressive measurable process, and satisfies $E_0 \int_0^T C(t)dt < \infty$. From (2.1.1)-(2.1.3), we know that the wealth process $A(t)$ at time t in this model satisfies

$$\begin{aligned} dA(t) &= A(t)\pi'(t)(\mu_{pt}dt + \Sigma_{pt}dW_t) + A(t)(1 - \pi'(t)I)r(t)dt - C(t)dt \\ &= [r(t)A(t) - C(t)]dt + A(t)\pi'\Sigma_{pt}dW_{0t} \end{aligned} \tag{3.1.1}$$

Define the index local martingale to be

$$Z_0(t) \doteqdot \exp[-\int_0^t \eta_s' dW_s - \frac{1}{2}\int_0^t \|\eta_s\|^2 ds] \tag{3.1.2}$$

The discount process is defined as $\gamma_0(t) \doteqdot \exp\{-\int_0^t r(s)ds\}$ Let state price density process be $H_0(t) \doteqdot \gamma_0(t)Z_0(t)$, and the wealth process be given $A \equiv A^{a,\pi,C}$. The so-called feasibility of policy is said to be, for the initial wealth $a \in (0,\infty)$, the portfolio-consumption policy satisfying $A^{a,\pi,C}(t) \geq 0, \forall 0 \leq t \leq T$. All feasible policy set (π, C) is defined as $\ell_0(a)$. We now define the final wealth and instant consumption utility continuous function $U_2(t,\cdot) : [0,T] \times (0,\infty) \to \mathbb{R}$, for a fixed time t, $U_2(t,\cdot)$ satisfies the ordinary conditions of utility functions in section 2. Then in this market model M, the unconstrained optimization problem (P_1) can be rewritten as:

$$(P_1)\quad \begin{aligned} & V_0(a) \doteqdot \sup_{(\pi,C)\in\ell_0'(a)} J(a;\pi,C), a \in (0,\infty) \\ & s.t.\ \ dP_{0t} = P_{0t}r(t)dt \qquad P_{00} = 1 \\ & dP_t = P_t[\mu_p(X_t)dt + \Sigma_p(X_t)dW_t] \\ & dX_t = \mu_x(X_t)dt + \Sigma_x(X_t)dW_t \\ & dA(t) = A(t)\pi'(t)(\mu_{pt}dt + \Sigma_{pt}dW_t) + A(t)(1-\pi'(t)I)r(t)dt - C(t)dt \end{aligned}$$

where $V_0(a) < \infty, \forall a \in (0,\infty)$, and the functional $J(a;\pi,C)$ is defined by $J(a;\pi,C) \doteqdot \{(\pi,C) \in \ell_0(a) : E_0\int_0^T U_2(t,C(t))dt + E_0U_1(A^{a,\pi,C}(T))\}$ is the total utility function. The set $\ell_0'(a) = \{(\pi,C) \in \ell_0(a) : E_0\int_0^T U_2^-(t,C(t))dt + E_0U_1^-(A^{a,\pi,C}(T)) < \infty\}$. For the nonempty closed convex portfolio policy restricted set $K \subseteq \mathbb{R}^N$, the constrained optimization problem (P_2) can be rewritten as

$$(P_2)\quad \begin{aligned} & V(a) \doteqdot \sup_{(\pi,C)\in\ell'(a)} J(a;\pi,C), a \in (0,\infty) \\ & s.t.\ \ dP_{0t} = P_{0t}r(t)dt \qquad P_{00} = 1 \\ & dP_t = P_t[\mu_p(X_t)dt + \Sigma_p(X_t)dW_t] \\ & dX_t = \mu_x(X_t)dt + \Sigma_x(X_t)dW_t \\ & dA(t) = A(t)\pi'(t)(\mu_{pt}dt + \Sigma_{pt}dW_t) + A(t)(1-\pi'(t)I)r(t)dt - C(t)dt \end{aligned}$$

where $\ell'(a) \doteqdot \{(\pi,C) \in \ell_0'(a) : \pi(t,W) \in K\}$.

3.2 Evaluation of the Given Investment-Consumption Policy

This section use duality method to evaluate the difference between a given policy and the optimal policy.

Transformation of Unconstrained Optimization Problem

Theorem 3.2.1[17] (π, C) *is the optimal solution of problem* (P_1) *if and only if it's a solution of the following optimization problem*

$$V_0(a) \doteqdot \sup_{(\pi,C)\in\ell_0'(a)} J(a;\pi,C), \quad a \in (0,\infty)$$
$$s.t. \;\; E_0[H_0(T)A(T) + \int_0^T H_0(s)C(s)ds] \leq a$$

From (2.1.4) (3.1.1) (3.1.2), and using Itô formula for $\gamma_0(t)A$, then we know

$$d\gamma_0(t)A(t) = \{-\gamma_0(t)r(t)A(t) + [A(t)r(t) - C(t)]\gamma_0(t)\}dt + A(t)\gamma_0(t)\pi'(t)\Sigma_{pt}dW_{0t}$$

That is,

$$d\gamma_0(t)A(t) = -C(t)\gamma_0(t)dt + A(t)\gamma_0(t)\pi'(t)\Sigma_{pt}dW_{0t}$$

Since $\gamma_0(0)A(0) = a$, thus we get

$$\gamma_0(t)A(t) + \int_0^{\prime} \gamma_0(s)C(s)ds = a + \int_0^t A(s)\gamma_0(s)\pi'(s)\Sigma_{ps}dW_{0s} \tag{3.2.1}$$

By using $Z_0(t) \doteqdot \exp[-\int_0^t \eta_s^t dW(s) - \frac{1}{2}\int_0^t \|\eta_s\|^2 ds]$, we know

$$dZ_0(t) = Z_0(t)[-\eta_t' dW(s) - \frac{1}{2}\|\eta_t\|^2 dt] \tag{3.2.2}$$

Similarly by virtue of Itô formula, we have

$$dH_0(t)A(t) = -H_0(t)C(t)dt + A(t)H_0(t)[\pi'(t)\Sigma_{pt} - \eta_t']dW_t$$

From $H_0(0)A(0) = a$, one knows

$$H_0(t)A(t) + \int_0^t H_0(s)C(s)ds = a + \int_0^t A(s)H_0(s)[\pi'(s)\Sigma_{ps} - \eta_s']dW_s \tag{3.2.3}$$

Since the left side of (3.2.3) is a continuous local martingale, and when $(\pi, C) \in \ell_0(a)$, this local martingale is also nonnegative, so a supermartingale, then we know

$$E_0[H_0(T)A(T) + \int_0^T H_0(s)C(s)ds] \leq a \tag{3.2.4}$$

Evaluate A Given Portfolio-Consumption Policy (π, C)

We can always get the optimal objective function value in original market model problem (P_2) through the upper bound of the objective function in a supposed market model problem $(P_1^{(\nu)})$, then when we don't know the objective function, one can use the result above to evaluate a given portfolio-consumption policy (π, C), the key-point is finding out the corresponding stochastic process $\{\nu_t\}$. That is finding out the relationship among optimal policy set $(\pi^{\nu^0}, C^{\nu^0}) \equiv (\pi^0, C^0)$, the objective function $V_0^{(\nu^0)}(a)$ and the process $\{\nu^0\}$ in the problem $P_1^{(\nu)}$.

From Theorem 3.2.1, the problem $(P_1^{(\nu)})$ can be rewritten as

$$\begin{aligned} V_0^{(\nu)}(a) &\doteqdot \sup_{\{C_{0\le t\le T}, A^{a,\pi,C}(T)\}} E_0 \int_0^T U_2(t, C(t))dt + E_0 U_1(A^{a,\pi,C}(T)) \\ s.t.\ \ & E_0[\int_0^T C(t)H_0^{(\nu)}(t)dt + A(T)H_0^{(\nu)}(T)] \le a,\ \ a \in (0,\infty) \end{aligned} \tag{3.2.5}$$

By using the dynamic programming, we know

$$\begin{aligned} V_t = V_t^{(\nu^0)}(a) &\doteqdot \sup_{\{C_{t\le s\le T}, A^{a,\pi,C}(T)\}} E_t[\int_t^T U_2(s, C(s))ds + U_1(A^{a,\pi,C}(T))] \\ s.t.\ \ & E_t[\int_t^T C(s)\frac{H_0^{(\nu^0)}(s)}{H_0^{(\nu^0)}(t)}ds + A(T)\frac{H_0^{(\nu^0)}(T)}{H_0^{(\nu^0)}(t)}] \le A(t). \end{aligned} \tag{3.2.6}$$

Construct the Lagrange function of the optimization problem above by

$$\begin{aligned} L(C_{t\le s\le T}, A(T), \lambda_t) \ \doteqdot\ & -E_t[\int_t^T U_2(s, C(s))ds + U_1(A^{a,\pi,C}(T))] - \lambda_t\{A(t) \\ & -E_t[\int_t^T C(s)\frac{H_0^{(\nu^0)}(s)}{H_0^{(\nu^0)}(t)}ds - A(T)\frac{H_0^{(\nu^0)}(T)}{H_0^{(\nu^0)}(t)}]\} \end{aligned} \tag{3.2.7}$$

The first order optimization condition is as follows

$$-E_t[\nabla U_1(A^{a,\pi,C}(T))] = -\lambda_t E_t[\frac{H_0^{(\nu^0)}(T)}{H_0^{(\nu^0)}(t)}]$$

Then we arrive at

$$\nabla U_1(A^{a,\pi,C}(T)) = \lambda_t \frac{H_0^{(\nu^0)}(T)}{H_0^{(\nu^0)}(t)} \tag{3.2.8}$$

On the other hand, according to the enveloping condition, the partial differential of the objective function to the instant wealth satisfies $\frac{\partial V_t}{\partial A(t)} = \lambda_t$ and by using the fact that there holds the following $\frac{\partial V_t}{\partial C(t)} = 0, \frac{\partial V_t}{\partial A(T)} = E_t[\nabla U_1(A(T))]$ and $\frac{\partial V_t}{\partial A(t)} = \frac{\partial V_t}{\partial C(t)}\frac{\partial C(t)}{\partial A(t)} + \frac{\partial V_t}{\partial A(T)}\frac{\partial A(T)}{\partial A(t)} = E_t[\nabla U_1(A(T))]E_t[\lambda_t \frac{H_0^{(\nu^0)}(t)}{H_0^{(\nu^0)}(T)}]$. From these statements, we obtain

$$\frac{H_0^{(\nu^0)}(T)}{H_0^{(\nu^0)}(t)} = \frac{\nabla U_1(A(T))}{\frac{\partial V_t}{\partial A(t)}} \tag{3.2.9}$$

For $\forall s \geq t$, similarly, one gets

$$\frac{H_0^{(\nu^0)}(T)}{H_0^{(\nu^0)}(s)} = \frac{\nabla U_1(A(T))}{\frac{\partial V_s}{\partial A(s)}} \tag{3.2.10}$$

(3.2.9) divides by (3.2.10), then there holds

$$\frac{H_0^{(\nu^0)}(s)}{H_0^{(\nu^0)}(t)} = \frac{\frac{\partial V_s}{\partial A(s)}}{\frac{\partial V_t}{\partial A(t)}} \tag{3.2.11}$$

Taking a logarithm of the equation above and by using the arbitrary of s with $s \geq t$, we have

$$dH_0^{(\nu^0)}(t) = d\ln(\partial V_t/\partial A(t)) \tag{3.2.12}$$

Suppose that the objective function V_t is smooth enough, and by virtue of (3.2.9), we know

$$H_0^{(\nu^0)}(t) = \exp[-\int_0^t \eta_s^{(\nu^0)\prime} dW(s) - \frac{1}{2}\int_0^t \|\eta_s^{(\nu^0)}\|^2 ds - \int_0^t [r(s) + \delta(\nu^0)(s)]ds] \tag{3.2.13}$$

That is to say, there holds

$$d\ln H_0^{(\nu^0)}(t) = \eta_t^{(\nu^0)\prime} dW(t) - \frac{1}{2}\|\eta_t^{(\nu^0)}\|^2 - r(t) - \delta(\nu^0)(t)]dt \tag{3.2.14}$$

and

$$d\ln(\partial V_t/\partial A(t)) = (\partial V_t/\partial A(t))^{-1}[\frac{\partial^2 V_t}{\partial A(t)^2} dA(t) + \frac{\partial^2 V_t}{\partial A(t)\partial X_t} dX_t] \tag{3.2.15}$$

Substituting (2.1.15) and (3.1.1) into equation (3.2.15), one arrives at

$$\begin{aligned}
& d\ln(\partial V_t/\partial A(t)) \\
&= (\frac{\partial V_t}{\partial A(t)})^{-1}\{\frac{\partial^2 V_t}{\partial A(t)^2}[A(t)\pi'(t)(\mu_{pt}dt + \Sigma_{pt}dW_t) + A(t)(1-\pi'(t)I)r(t)dt - C(t)dt] \\
&+ \frac{\partial^2 V_t}{\partial A(t)\partial X_t}[\mu_x(X_t)dt + \Sigma_x(X_t)dW_t]\} \\
&= (\frac{\partial V_t}{\partial A(t)})^{-1}\{\frac{\partial^2 V_t}{\partial A(t)^2}[A(t)\pi'(t)\mu_{pt} + A(t)(1-\pi'(t)I)r(t)dt - C(t)]dt] \\
&+ \frac{\partial^2 V_t}{\partial A(t)\partial X_t}\mu_x(X_t)dt + \frac{\partial^2 V_t}{\partial A(t)^2}A(t)\pi^{0\prime}(t)\Sigma_{pt}dW_t + \frac{\partial^2 V_t}{\partial A(t)\partial X_t}\Sigma_x(X_t)dW_t\}
\end{aligned} \tag{3.2.16}$$

From (3.2.12), we know that the coefficients of dW_t are equal, in other words, one has the following formula

$$\eta_t^{(\nu^0)} = -A(t)(\frac{\partial V_t}{\partial A(t)})^{-1}\frac{\partial^2 V_t}{\partial A(t)^2}\Sigma_{pt}^T\pi^{0\prime}(t) - (\frac{\partial V_t}{\partial A(t)})^{-1}\Sigma_x^T(X_t)\frac{\partial^2 V_t}{\partial A(t)\partial X_t} \tag{3.2.17}$$

In this way, given an optimal policy $(\tilde{\pi}(t), \tilde{C}(t))$, we can compute estimation value $\tilde{V}_t$ of the objective function, that is to say we can compute the conditional expectation of the final wealth utility and the instant consumption utility integration in the setting of $(\tilde{\pi}(t), \tilde{C}(t))$, and then construct a process $\hat{\nu}_t$ according to the equation above, we can get the upper bound of the original problem. As a result, if the estimation value $\tilde{\nu}_t$ of the objective function is smooth enough, then we can change V_t and $\pi^0(t)$ in (3.2.17) into $\tilde{\nu}_t$ and $\tilde{\pi}_t$, it is easy to see that there holds

$$\eta_t^{(\tilde{\nu})} = -A(t)(\frac{\partial \tilde{V}_t}{\partial A(t)})^{-1}\frac{\partial^2 V_t}{\partial A(t)^2}\Sigma_{pt}^T\tilde{\pi}'(t) - (\frac{\partial \tilde{V}_t}{\partial A(t)})^{-1}\Sigma_x^T(X_t)\frac{\partial^2 \tilde{V}_t}{\partial A(t)\partial X_t} \quad (3.2.18)$$

Thus we know

$$\tilde{\nu}(t) = \Sigma_{pt}(\eta_t^{(\tilde{\nu})} - \eta_t) \quad (3.2.19)$$

where $\eta_t^{(\tilde{\nu})}$ is the risk market price in the new market model, but we can't guarantee that both it and its relative process $\tilde{\nu}_t$ belong to the set D, then we can find a process $\hat{\eta}_t$ to approximate $\tilde{\eta}_t$, and we require $\hat{\eta}_t$ relative to $\hat{\nu}_t$ is bounded, i.e., we can introduce two restriction conditions

$$\|\hat{\eta} - \eta\| \leq B_1$$

$$\delta(\hat{\nu}) \leq B_2$$

where B_1, B_2 are nonnegative constants and can be infinity.

Now we can conclude that the model stated above is the following optimal problem

Assume that $\hat{\eta}_t, \hat{\nu}_t$ are solutions of the following problem (P_3)

$$(P_3) \quad \begin{aligned} &\min_{\hat{\eta},\hat{\nu}} \|\hat{\eta}_t - \tilde{\nu}_t\| \\ &s.t. \;\; \delta(\hat{\nu}) \leq \infty \\ &\hat{\nu}(t) = \Sigma_{pt}(\eta_t^{(\hat{\nu})} - \eta_t) \\ &\|\hat{\eta} - \eta\| \leq B_1 \\ &\delta(\hat{\nu}) \leq B_2 \end{aligned} \quad (3.2.20)$$

Examples

Example If a portfolio policy K doesn't allow oversell and only the first L stocks are allowed to be traded, i.e. $K = \{\pi : \pi \geq 0, \pi_i = 0, L < i \leq N\}$. Given a π, we can compute estimation function value $\tilde{V}_t$, then from (3.2.18), we can construct process $\tilde{\eta}_t$. From the characteristic of K, if $\hat{\nu}$ satisfies $\hat{\nu}_i \geq 0, 1 \leq i \leq L$, then $\delta(\hat{\nu}) = \sup_{\pi\in K}(-\pi'\hat{\nu}) \leq 0$, so the condition in problem (P_3) is tenable obviously. Since the system of equations following

$$\begin{aligned}
&\min_{\hat{\eta},\hat{\nu}} \|\hat{\eta}_t - \tilde{\nu}_t\| \\
&s.t.\ \ \hat{\eta} = \eta + \Sigma_{pt}\hat{\nu} \\
&\hat{\nu}_i \geq 0, 1 \leq i \leq L \\
&\|\hat{\eta} - \eta\| \leq A^2 \\
&\eta_t = \Sigma_{pt}^{-1}(\mu_{pt} - r(t)I)
\end{aligned}$$

By a direct computation, one can obtain the solutions $\hat{\eta}_t, \hat{\nu}_t$ to the system of equations above.

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