

New Multi-step Runge-Kutta Method

O. Y. Ababneh, R. Ahmad and E. S. Ismail

School of Mathematical Sciences, Faculty of Science and Technology
Universiti Kebangsaan Malaysia
ababnehukm@yahoo.com

Abstract

In this article, a new class of Runge-Kutta methods for initial value problems $y' = f(x, y)$ are introduced, this method replace evaluations of f with approximations of f' and use the harmonic mean in the main formula. If f' is approximated to sufficient accuracy from past and current evaluations of f , the resulting multi-step Runge-Kutta method can be considered as replacing functional evaluations with approximations of f' . Here is presented an $O(h^3)$ method which requires only two evaluations of f . The stability of the method is analyzed. Numerical examples with excellent results are shown to verify that this new method is superior to existing multi-step method like the third order Adams-Bashforth.

Mathematics Subject Classifications: 51N20, 62J05, 70F99

Keywords: ODE solver; Multi-step Runge-Kutta; Third-order method; Harmonic mean

1 Introduction

Different authors have effected to increase the efficiency of RK methods. Thus, numerous variations of two-step explicit RK methods suggested now. Byrne and Lambert [1] proposed 3rd- and 4th-order two-step Pseudo RK methods. later Byrne [2] proposed methods that reduce a conservative bound on the error, and Gruttke [3] introduced a 5th-order Pseudo RK method. Costabile [4] proposed the Pseudo RK methods of II species which involve several parameters to develop local accuracy. Jackiewicz et al. [5] considered the two-step Runge-Kutta TSRK methods up to order 4. Jackiewicz and Tracogna [6] deliberated a more general class of these TSRK methods and derived order conditions up to order 5 via Albrecht's method. lately, Wu [7] has introduced a set of two-step methods which make use of the derivative of the differential equation. We consider initial value problems expressed in autonomous form.

Starting with the non-autonomous form, we assume that $f(x, y)$ is a continuous function with domain D in \mathfrak{R}^{n+1} where $x \in \mathfrak{R}$, $y \in \mathfrak{R}^n$ and $(x, y) \in D$. is on assuming that

$$\|f(x, y_1) - f(x, y_2)\|_2 \leq L \|y_1 - y_2\|_2$$

for all $(x, y_1), (x, y_2) \in D$, the problem $y' = f(x, y)$, $y(x_0) = y_0$ with $(x_0, y_0) \in D$ has a unique solution.

This new method uses new terms with higher order derivatives of f in the Runge-Kutta stages k_i terms are to achieve a higher order of accuracy without the increase of evaluations of f , but with the addition of approximations of f' and the harmonic mean in the main formula.

2 A third-order multi-step Runge-Kutta based on harmonic mean

In [11] it was proposed new Runge-Kutta methods by introduce additional terms by assigning

$$k_2 = hf(y_n + a_{21}k_1 + ha_{22}f_y(y_n)k_1)$$

since $f_x = 0$, $y'' = f' = f_y f$ and $k_1 = hf(y_n)$ we have $hf_y(y_n)k_1 = h^2 f'(y_n) = h^2 f'_n = h^2 f'_1$, k_2 can be replaced by $k_2 = hf(y_n + a_{21}k_1 + a_{22}h^2 f'_1)$.

In addition, it was proposed two methods to utilize f_y , i.e to use the exact f' or approximate f' to some given accuracy by using the current and previous evaluations of f . In this study, the approximate f' is utilized to introduce a third-order multi-step Runge-Kutta method with two function evaluations and the harmonic mean in the main formula.

In [8, 9, 10] new Runge-Kutta methods were established by using the harmonic mean in the functional values. It may noted that the harmonic mean of two quantities x_1, x_2 is given by $\frac{2x_1x_2}{x_1+x_2}$.

For the third order, we the following method is proposed in this paper by OR3 (the authors and the order of accuracy).

$$y_{n+1} = y_n + \left[2 \frac{k_1 k_2}{k_1 + k_2} \right], \quad (1)$$

where the stages are given by

$$\begin{aligned} k_1 &= hf(y_n), \\ k_2 &= hf(y_n + a_{21}k_1 + a_{22}h(f(y_n) - f(y_{n-1}))). \end{aligned} \quad (2)$$

Traditionally, The Taylor series expansion of k_1 and k_2 would be substituted into (1) to obtain an expression of y_{n+1} in terms of the function, its derivatives and the parameters a_{21} and a_{22} , the equations obtained are matched with Taylor series expansion of (y_{n+1}) through terms of order h^3 . We get $a_{21}=1$, and there is no any order condition to choose a_{22} so, we take $a_{22} = \frac{3}{2}$.

For the third-order method above, an approximation of $O(h)$ is enough to retain the third order in the computed y_{n+1} . This can be done by comparing the computations using the exact f' with that using a first-order approximation. Let

$$\bar{y}_n = y_n + k_1 + \frac{3}{2}hf_y k_1 \text{ or } \bar{y}_n = y_n + k_1 + \frac{3}{2}h^2 f'_n$$

and let

$$\widehat{y}_n = y_n + k_1 + \frac{3}{2}h^2 \widehat{f}'_n, \text{ where } \widehat{f}' = \frac{f(y_n) - f(y_{n-1})}{h}.$$

$$\text{Since } f(y_{n-1}) = f(y_n) - hf'(y_n) + O(h^2)$$

then

$$\widehat{f}' = f'(y_n) + O(h) = f'_n + O(h).$$

Thus

$$\widehat{y}_n = y_n + k_1 + \frac{3}{2}h^2 f'_n + O(h^3).$$

On replacing

$$y_{n+1} = y_n + \frac{k_1 h f(\bar{y}_n)}{k_1 + h f(\bar{y}_n)}$$

by

$$y_{n+1} = y_n + \frac{k_1 h f(\widehat{y}_n)}{k_1 + h f(\widehat{y}_n)},$$

we get

$$y_{n+1} = k_1 h \left(\frac{f(\widehat{y}_n)}{k_1 + h f(\widehat{y}_n)} - \frac{f(\bar{y}_n)}{k_1 + h f(\bar{y}_n)} \right),$$

$$\text{and by Lipschitz condition } \|f(\widehat{y}_n) - f(\bar{y}_n)\|_2 \leq L \|\widehat{y}_n - \bar{y}_n\|_2,$$

$$\|\widehat{y}_{n+1} - y_{n+1}\|_2 \leq k_1 h L \|\widehat{y}_n - \bar{y}_n\|_2.$$

Thus

$$\|\widehat{y}_{n+1} - y_{n+1}\|_2 \leq O(h^4).$$

The global third order is preserved since y_{n+1} is in agreement with the true solution in all third order terms.

Definition:[12] A scheme is said to be consistent, if the difference equation of the computation formula exactly approximates the differential equation it intends to solve.

Theorem 1: *The explicit Multi-step Runge-Kutta method (OR3) is Consistent.*

Proof: To prove that Eq.1 is consistent, subtract y_n on both sides of Eq.1, then we have

$$y_{n+1} - y_n = \left(2 \frac{k_1 k_2}{k_1 + k_2} \right) \quad (3)$$

But

$$\begin{aligned} k_1 &= hf(y_n), \\ k_2 &= hf(y_n + k_1 + \frac{3}{2}h(f(y_n) - f(y_{n-1}))), \end{aligned} \quad (4)$$

therefore

$$y_{n+1} - y_n = h \left(2 \frac{f(y_n)f(y_n + hf(y_n) + \frac{3}{2}h(f(y_n) - f(y_{n-1})))}{f(y_n) + f(y_n + hf(y_n) + \frac{3}{2}h(f(y_n) - f(y_{n-1})))} \right) \quad (5)$$

Dividing all through by h and taking limit as h tends to zero on both sides to have

$$\lim_{h \rightarrow 0} \frac{y_{n+1} - y_n}{h} = \left(2 \frac{f(y_n)f(y_n)}{2f(y_n)} \right) \quad (6)$$

$$y'(y_n) = f(y_n) \quad (7)$$

Hence, the method is consistent.

Theorem 2: *The explicit Multi-step Runge-Kutta method (OR3) is Convergent.*

Proof: Since the scheme is one-step method and it has been proved to be consistent then it is convergent by Lambert [13].

3 The Stability analysis of OR3

On applying (1) to the scalar test equation

$$y' = \lambda y, \quad \lambda \in C, \quad \text{Re}(\lambda) < 0, \tag{8}$$

and writing $z = h\lambda$, then $k_1 = hf(y_n) = h\lambda y_n = zy_n$ and

$$\begin{aligned} k_2 &= z(y_n + k_1 + \frac{2}{3}(hf_n - hf_{n-1})) \\ &= z(y_n + zy_n + \frac{2}{3}zy_n - \frac{2}{3}zy_{n-1}) \\ &= (z + \frac{5}{3}z^2)y_n - \frac{2}{3}z^2y_{n-1}. \end{aligned}$$

Substituting into the formula for y_{n+1} however, a point z is in the stability region if the solutions to this difference equation are bounded as $n \rightarrow \infty$. This means that the quadratic equation

$$m^2 - \left[\frac{(6 + 11z + 10z^2)m^2 - (2z - 4z^2)m}{(6 + 5z)m - 2z} \right] = 0 \tag{9}$$

has both roots in the unit disc. The boundary of the stability region is calculated by substituting $m = \exp(i\theta)$, where θ is an angle that goes from 0 to 2π . By using previous values for the approximation, one has created a multi-step Runge-Kutta method. Fig (1) shows the stability region for the new third-order which is the union of two regions. Fig(2) shows the stability for multi-step Adams-Bashforth methods. The figures show that the stability region for the new method is larger than the third-order Adams-Bashforth method.

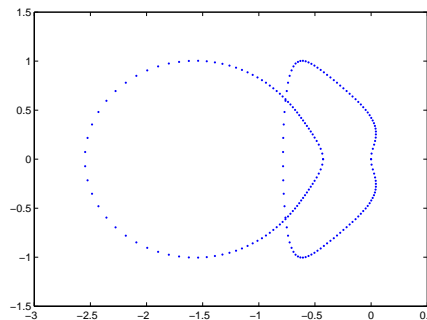


Figure 1: The stability region of the OR3 method.

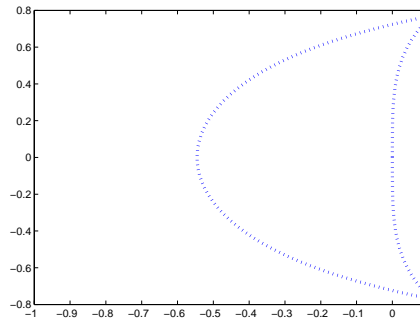


Figure 2: The stability region of the AB3 method.

4 Numerical Results

The aim of this section is to show that the new method created is better compared with third order Adams-Bashforth method. Several scalar autonomous equations are showed and the results are exhibited in Table (1). Relative error was plotted against the step size as it is depicted in Figures (3) and (4).

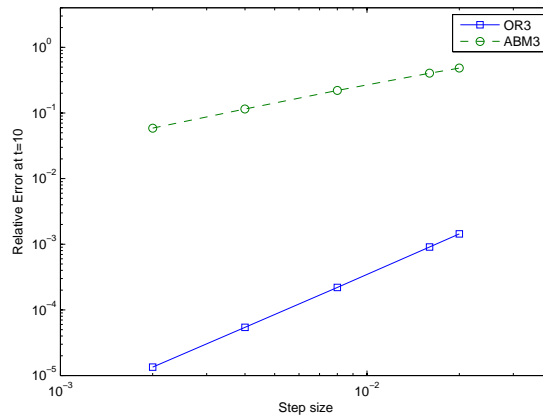


Figure 3: Equation $y' = -y$.

5 Conclusion

New third-order numerical integration technique inspired by the Runge-Kutta method based on harmonic mean have been produced. In particular, a technique utilizing an approximation to f' has been presented resulting in a multi-

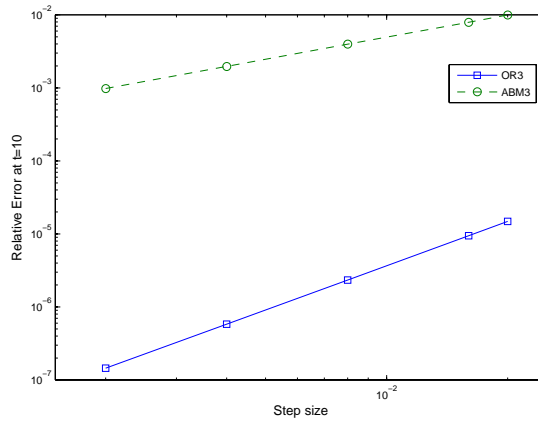


Figure 4: Equation $y' = \frac{-y^3}{2}$.

Table 1: Test problems

Function	Solution	$y(0)$
$y' = -y$	$y = e^{-t}$	1
$y' = \frac{-y^3}{2}$	$y = \frac{1}{\sqrt{t+1}}$	1

step method. The simulation results show an excellent and superior accuracy of the OR3 method. In future research, other means rather than harmonic mean will be used and demonstrated the claim for the autonomous system.

6 Acknowledgements

The authors would like to thank Prof J.C. Butcher for several valuable suggestions, specifically in the stability analysis area.

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Received: February, 2009