Applied Mathematical Sciences, Vol. 3, 2009, no. 46, 2271 - 2293

Stabilization of Solutions to Unidimensional Nonlinear Parabolic Problems

Asmaâ ABASSI, Abderrahmane EL HACHIMI and Ahmed JAMEA

UFR: Applied and Industrial Mathematics
University of Chouaib Doukkali, El Jadida, Morocco
elhachimi@ucd.ac.ma
abassi_asmaa@yahoo.fr
a.jamea@yahoo.fr

Abstract

In this paper, we consider the following unidimensional nonlinear parabolic problem

$$(P)\begin{cases} \frac{\partial u}{\partial t} = (|u_x|^{p-2}u_x)_x + f(u) & on\ (-L,L)\times\mathbb{R}^+, \\ u(\pm L,t) = 0 & on\ \mathbb{R}^+, \\ u(x,0) = u_0(x) & on\ ]-L,L[. \end{cases}$$

We begin by describing the set $E(L)$ of nonnegative equilibrium solutions to the motivating example, which consists of problem (P) with the special choice $f(u) = u(1-u)(u-a)$ and $0 < a < \frac{1}{2}$. This will be followed by the study of existence, uniqueness and stabilization of solutions to problem (P) when f is a general function satisfying suitable assumptions. Finally, we show, in part of application, the stability of the trivial solution and of a large positive equilibrium solution.

1 Introduction

The aim of this paper is the study of the large time behaviour of nonnegative solutions to the initial boundary value problem

$$(P)\begin{cases} \frac{\partial u}{\partial t} = (|u_x|^{p-2}u_x)_x + f(u) & on\ (-L,L)\times\mathbb{R}^+, \\ u(\pm L,t) = 0 & on\ \mathbb{R}^+, \\ u(x,0) = u_0(x) & on\ ]-L,L[. \end{cases}$$

where $p > 1$, f is locally lipschitz continuous with $f(0) = 0$ and u_0 is bounded. This kind of problems arise in many fields of science: Non-newtonian fluid

mechanics, gas or fluid flow in porous media, spread of certain biological populations,...
We primarily focus our study on a motivating example, which consists of problem (P) with the special choice

$$f(u) = u(1-u)(u-a) \qquad \text{where} \quad 0 < a < \frac{1}{2}.$$

We base our analysis on properties of the time-map related to the elliptic problem associated with problem (P) in this case, in order to obtain characterization of nonnegative equilibrium solutions and thus describe in detail their set $E = E(L)$ that we can write as

$$E(L) = E^*(L) \cup \{0\},$$

since $v \equiv 0$ is a trivial solution. According, we shall show here the following results.

If $p \in]1,2]$, there is one critical parameter value $L_p > 0$, such that

i) $E^*(L) = \emptyset$ for all $0 < L < L_p$,

ii) $E^*(L_p)$ consists of one isolated positive solution,

iii) for all $L > L_p$, $E^*(L)$ consists of two isolated positive solutions noted respectively $s \equiv s(L)$ and $q \equiv q(L)$ with $s < q$ on $(-L, L)$.

If $p \in]2, +\infty[$, there exist tree critical values of L: $0 < L_p < L_p^1 < L_p^0$ and such that

i) $E^*(L) = \emptyset$ for all $0 < L < L_p$,

ii) $E^*(L_p)$ consists of one isolated positive solution,

iii) $E^*(L)$ consists of tow isolated positive solutions noted respectively s and q with $s < q$ on $(-L, L)$ for $L_p < L < L_p^1$,

iv) for $L > L_p^1$, N a positive integer, and $NL_p^1 < L < (N+1)L_p^1$, $E^*(L)$ consists of one isolated positive solution q and N j-parameter families $S_j(L)$, $j = 1, ..., N$ of nonnegative solutions for $L_p^1 < L < L_p^0$, however, for $L > L_p^0$ it contains only Nj-parameter families $S_j(L)$, $j = 1, ..., N$ of nonnegative solutions.

Our work extends interesting results obtained by D.Aronson, M.G.Crandall and L.Peletier in [2], where the study of the set of equilibrium solutions $E(L)$ extends the one done by Smoller and Wasserman in [13], and determinate it for problem (P) with a cubic nonlinearity f when the elliptic term is of the form $(u^m)_{xx}$ with $m > 1$, instead of $(|u_x|^{p-2}u_x)_x$.
In our study of $E(L)$, we distinguish two cases according to $p > 2$ or $1 < p \leq 2$.

In the case where $p > 2$, we show that $L_p^0 < +\infty$ and that the set of equilibrium solutions is the same as in the study done in [2] even if their parameter L_p^0 is infinite. In contrast, when $1 < p \leq 2$, Our set $E(L)$ is characterized by similar elements to those found by smoller and wasserman in [13] for the operator u_{xx}.
On the other hand, the detailed description of $E(L)$ allows us to prove that $u(t, u_0)$ converges, as t tends to $+\infty$, to a limit in $E(L)$. More precisely, we establish the stability of the trivial solution and of the large positive solution q, obtained in the first part, of the elliptic problem associated with problem (P) by exhibiting suitable invariant set $K \subset X$, where X is a complete metric space of functions, and $K \cap E(L)$ is either $\{0\}$ or $\{q(L)\}$.
These last stabilization results are obtained thanks to a general stabilization theorem that we establish for the general problem (P), after proving various basic existence, uniqueness, comparison and regularity theorems of problem (P), and defining a complete metric space of functions in which orbits of problem (P) are precompact. Moreover, if $0 \leq u_0 \leq 1$ and $u(t, u_0)$ is solution of (P), then we show, by means of a Lyapunov function associated with (P) that the w-limit set

$$w(u_0) = \{w \in X, u(t_n, u_0) \to w \text{ in} X, \text{ for some sequence } (t_n) \text{ with } t_n \underset{n\to\infty}{\longrightarrow} \infty\}$$

is contained in $E(L)$.
To this end, we shall follow the same approach used by Aronson, Grandall and Pelletier in [2] for problem (P) when the elliptic term is of the form $(u^m)_{xx}$.
Let us mention works [5] and [6] of A.El hachimi and F.De Thelin, where the authors showed stabilization results for problem (P) when $\Omega \subset \mathbb{R}^N,\ N > 1$; their approach was based on the use of supersolutions of problem (P), they also obtained that $w(u_0) \subset E(L)$ by using regularizing effects that they established through their analysis.
This paper is organized as follows: we devote the second section to determinate the set of equilibrium solutions of the motivating example. In section III, we return our attention to the general case of (P) and establish existence, uniqueness, comparison and stabilization theorems. Finally, section IV, contains applications of precedent general results: we prove the stability of some equilibrium solutions in the case of our motivating example.

2 Equilibrium solutions

We begin our analysis by establishing a characterization of equilibrium solutions to problem (P) in the case where f is defined by

$$f(u) = u(1-u)(u-a) \qquad \textit{with} \quad 0 < a < \frac{1}{2}.$$

Definition 1. *A function $u : [-L, L] \longrightarrow \mathbb{R}^+$ is called an equilibrium solution of problem (P) when it is a classical solution of the following problem*

$$(P_e)\begin{cases}(|u_x|^{p-2}u_x)_x + f(u) = 0 & \text{on } (-L, L),\\ u(\pm L) = 0.\end{cases}$$

It is clear that $u \equiv 0$ is a trivial solution of problem (P_e). We shall show below that, in this case, problem (P) possesses nontrivial solutions obtained under some conditions on $L > 0$.

2.1 A characterization of equilibrium solutions

We set

$$F(s) = \int_0^s f(t)dt \quad \text{and} \quad \lambda_p(\mu) = \left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_0^{\mu} \frac{dv}{(F(\mu) - F(v))^{\frac{1}{p}}},$$

we have the following

Proposition 1. *u is a positive solution of problem (P_e) if and only if*

$$\left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_{u(x)}^{\mu} \frac{dv}{(F(\mu) - F(v))^{\frac{1}{p}}} = |x| \qquad \text{for} \quad |x| \leq L,$$

where $\mu \in (\alpha, 1)$ and $L \in \mathbb{R}^+$ are related by $\lambda_p(\mu) = L$ and α is the unique root of F in $(a, 1)$.

Proof. Let us consider the following problem

$$(P_e^*)\begin{cases}(|u'|^{p-2}u')' + f(u) = 0,\\ u(\xi) = \mu,\ u'(\xi) = 0,\end{cases}$$

with $\xi \in (-L, L)$ and $\mu \in \mathbb{R}^+$.

We shall seek conditions on ξ that allow problem (P_e^*) to be equivalent to (P_e) in the sens that a solution of (P_e^*) is also a solution of (P_e); since, for a positive solution u of problem (P_e), there exists $\xi \in (-L, L)$ such that $u'(\xi) = 0$ and $0 < u(x) \leqslant u(\xi)$, for all $x \in (-L, L)$.ie. there exist ξ and μ for which u is a solution of (P_e^*).

Conversely, let u be a solution of (P_e^*).

In the case where $\mu = 1$, the unique solution of (P_e^*) is $u \equiv 1$, since f is a locally lipschitzian function satisfying $f(1) = 0$.

For $\mu > 1$, it is clear that solution of (P_e^*) is convex on its domain of definition since we have $f(u) < 0$ for $u > 1$.

Consequently, there is no solution of problem (P_e^*) satisfying the boundary condition $u(\pm L)=0$, when $\mu \geqslant 1$.
Hence, we consider $\mu \in (0,1)$.
Next, multiplying the equation of problem (P_e^*) by u' gives

$$\frac{p-1}{p}(|u'|^p)' + f(u)u' = 0.$$

So, for $u \leq \mu$, we get

$$\frac{p-1}{p}|u'(x)|^p = F(\mu) - F(u).$$

This last equation has a sense provided that $F(\mu)-F(u) \geq 0$.
First, it is easy to see that F is nonincreasing on $(0,a)$ and that there exists a unique $\alpha \in (a,1)$ such that $F(\alpha)=0$, $F(x)>0$ on $(\alpha,1)$ and $F(x)<0$ on (a,α).
Arguing as in [2], page 1004, we obtain

$$\left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_u^\mu \frac{dv}{(F(\mu)-F(v))^{\frac{1}{p}}} = |x-\xi|, \qquad for \quad \alpha < \mu < 1. \tag{2.1}$$

Remark 1. *The singularity at $v=\mu$ in (2.1) is integrable for $p>1$ since*

$$\lim_{v\to\mu} \frac{F(\mu)-F(v)}{\mu - v} = f(\mu) > 0,$$

which implies that $F(\mu)-F(v) > M(\mu-v)$ for some $M>0$ and v near μ.

Remark 2. *The integrand in (2.1) can be extended down to $u=0$ as follows*

$$\lambda_p(\mu) = \left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_0^\mu \frac{dv}{(F(\mu)-F(v))^{\frac{1}{p}}}, \qquad for \quad \alpha < \mu < 1. \tag{2.2}$$

Indeed, for any $v \in (0,\mu)$ we have $F(\mu)-F(v) > 0$, and so (2.2) is well defined.

Now, if u is a solution of (P_e), we have $u(\pm L)=0$, then

$$\lambda_p(\mu) = |L-\xi| = |L+\xi|,$$

which implies that $\xi = 0$, and so,

$$\left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_u^\mu \frac{dv}{(F(\mu)-F(v))^{\frac{1}{p}}} = |x|, \qquad for \quad |x| \leq L.$$

This ends the proof of proposition 2. □

Lemma 1. *$\lambda_p(\alpha) < +\infty$ if and only if $p > 2$.*

Proof. Since
$$\lim_{v\to\alpha}\frac{F(v)-F(\alpha)}{v-\alpha} = f(\alpha) > 0,$$
there exists $\delta > 0$ and $M > 0$ such that
$$F(\alpha) - F(v) > M(\alpha - v), \qquad \forall v \in (\alpha-\delta, \alpha).$$
Thus,
$$\int_{\alpha-\delta}^{\alpha}\frac{dv}{(F(\alpha)-F(v))^{\frac{1}{p}}} < +\infty \qquad for \quad p > 1.$$
On the other hand, since
$$\lim_{v\to 0^+}\frac{F(0)-F(v)}{-v^2} = -\frac{f'(0)}{2} < 0,$$
there exist $\epsilon > 0$, $m_1 < 0$ and $m_2 < 0$ such that
$$m_1 \le \frac{F(0)-F(v)}{-v^2} \le m_2, \qquad \forall v \in (0,\epsilon).$$
Hence,
$$(-m_1)^{-\frac{1}{p}}\int_0^{\epsilon}\frac{dv}{v^{\frac{2}{p}}} \le \int_0^{\epsilon}\frac{dv}{(F(0)-F(v))^{\frac{1}{p}}} \le (-m_2)^{-\frac{1}{p}}\int_0^{\epsilon}\frac{dv}{v^{\frac{2}{p}}}.$$
So,
$$\int_0^{\epsilon}\frac{dv}{(F(0)-F(v))^{\frac{1}{p}}} < +\infty \qquad \text{if and only if} \quad p > 2.$$
Consequently
$$\lambda_p(\alpha) < +\infty \qquad \text{if and only if} \quad p > 2.$$
□

In the next, we shall give conditions on L in order to obtain the existence of a positive solution for problem (P_e).
We shall begin by proving some properties of the associated time-map.

2.2 properties of the time-map

Proposition 2. *We have the following properties of λ_p*

(i) $\lambda_p \in \mathcal{C}^1((\alpha, 1))$, for any $p > 1$,

(ii) λ_p is continuous at α for any $p > 2$,

(iii) $\lim\limits_{\mu\to 1}\lambda_p(\mu) < +\infty$ *if and only if* $p > 2$,

(iv) $\lim\limits_{\mu\to\alpha}\lambda_p^{'}(\mu) = -\infty$, *for any* $p > 1$,

(v) $\lim\limits_{\mu\to 1}\lambda_p^{'}(\mu) = +\infty$ *for any* $p > 1$.

Proof. **(i)**Define

$$\Lambda_p(\mu) = \int_0^{\mu} \frac{dv}{(F(\mu) - F(v))^{\frac{1}{p}}},$$

which becomes by the change of variables $\tau = \frac{v}{\mu}$, as $\Lambda_p(\mu) = \mu G_p(\mu)$, where

$$G_p(\mu) = \int_0^{1} \frac{dv}{(F(\mu) - F(\tau\mu))^{\frac{1}{p}}}.$$

One can easily verify that the function G_p is derivable on $(\alpha, 1)$ and that

$$G_p^{'}(\mu) = -\frac{1}{p}\int_0^{1} \frac{f(\mu) - \tau f(\tau\mu)}{(F(\mu) - F(\tau\mu))^{1+\frac{1}{p}}} d\tau.$$

Hence, it is straightforward that $\lambda_p \in \mathcal{C}^1((\alpha, 1))$.
(ii) By lemma 1, it suffices to show that $\lim\limits_{\mu\to\alpha}\lambda_p(\mu) = \lambda_p(\alpha)$ for $p > 2$, to obtain (ii).
Indeed, we can write

$$\Lambda_p(\mu) = I_1(\mu) + I_2(\mu),$$

where

$$I_1(\mu) = \int_0^{\alpha} \frac{dv}{(F(\mu) - F(v))^{\frac{1}{p}}} \quad \text{and} \quad I_2(\mu) = \int_{\alpha}^{\mu} \frac{dv}{(F(\mu) - F(v))^{\frac{1}{p}}}.$$

It is straightforward that

$$\lim_{\mu\to\alpha} I_2(\mu) = 0.$$

On the other hand, for $v \in [0, \alpha]$ we have,

$$\frac{1}{(F(\mu) - F(v))^{\frac{1}{p}}} < \frac{1}{(-F(v))^{\frac{1}{p}}}.$$

By lemma 1, the second part function of the inequality is integrable only for $p > 2$. Hence

$$\lim_{\mu\to\alpha} I_1(\mu) = \Lambda_p(\alpha) \quad \text{for any} \quad p > 2.$$

(iii) We have

$$\lim_{\mu\to 1}\frac{1}{(F(\mu) - F(\tau\mu))^{\frac{1}{p}}} = \frac{1}{(F(1) - F(\tau))^{\frac{1}{p}}},$$

and
$$\int_{1-\epsilon}^{1} \frac{d\tau}{(F(1)-F(\tau))^{\frac{1}{p}}} < +\infty \qquad \text{if and only if} \quad p > 2,$$
then, we deduce (iii).
(iv) From
$$\Lambda_p(\mu) = \mu G_p(\mu),$$
we have
$$\Lambda_p^{'}(\mu) = \frac{1}{\mu p} \int_0^{\mu} \frac{\theta_p(\mu)-\theta_p(v)}{(F(\mu)-F(v))^{1+\frac{1}{p}}} dv,$$
where
$$\theta_p(\mu) = pF(\mu) - \mu f(\mu).$$
Since $\theta_p(\alpha) = -\alpha f(\alpha) < 0$, there exists $\delta_p > 0$ such that
$$\theta_p(\mu) < \frac{\theta_p(\alpha)}{2} < 0, \qquad \forall \mu \in [\alpha, \alpha+\delta_p).$$
On the other hand, $\theta_p(0) = 0$. Then there exists $\gamma_p \in (0, \alpha)$ such that
$$|\theta_p(v) - \theta_p(0)| < -\frac{\theta_p(\alpha)}{4} \qquad \forall v \in [0, \gamma_p].$$
So, for $\mu \in [\alpha, \alpha+\delta_p)$ and $v \in [0, \gamma_p]$ we get
$$\theta_p(\mu) - \theta_p(v) < \frac{\theta_p(\alpha)}{4} < 0. \tag{2.3}$$
Let now
$$\Lambda_p^{'}(\mu) = J_1(\mu) + J_2(\mu),$$
where
$$J_1(\mu) = \frac{1}{p\mu} \int_0^{\gamma_p} \frac{\theta_p(\mu)-\theta_p(v)}{(F(\mu)-F(v))^{1+\frac{1}{p}}} dv$$
and
$$J_2(\mu) = \frac{1}{p\mu} \int_{\gamma_p}^{\mu} \frac{\theta_p(\mu)-\theta_p(v)}{(F(\mu)-F(v))^{1+\frac{1}{p}}} dv.$$
Arguing as above, we can show that near μ we have
$$\left| \frac{\theta_p(\mu)-\theta_p(v)}{(F(\mu)-F(v))^{1+\frac{1}{p}}} \right| \leq \frac{c}{(\mu-v)^{\frac{1}{p}}}$$
and so $J_2(\mu)$ remains bounded as $\mu \to \alpha$.
Now, using (2.3), we obtain that
$$J_1(\mu) < \frac{\theta_p(\alpha)}{4p\mu} \int_0^{\gamma_p} \frac{dv}{(F(\mu)-F(v))^{1+\frac{1}{p}}}.$$

But for $v < \gamma_p < \alpha$, we have

$$\lim_{\mu\to\alpha}\frac{1}{(F(\mu)-F(v))^{1+\frac{1}{p}}} = \frac{1}{(-F(v))^{1+\frac{1}{p}}},$$

hence, in order to obtain (iv) it suffices to study the singularity of the function $v \mapsto \frac{1}{(-F(v))^{1+\frac{1}{p}}}$ near zero.

Near zero we have

$$m_1 < \frac{F(0)-F(v)}{-v^2} < m_2.$$

So, by application of Fatou's lemma we deduce that

$$\lim_{\mu\to\alpha}\int_0^{\gamma_p}\frac{dv}{(F(\mu)-F(v))^{1+\frac{1}{p}}} = +\infty.$$

Therefore

$$\lim_{\mu\to\alpha}\lambda_p^{'}(\mu) = -\infty.$$

(v) Using the change of variables $\tau = \frac{v}{\mu}$, we can write

$$\Lambda_p^{'}(\mu) = \frac{1}{p}\int_0^1\frac{\theta_p(\mu)-\theta_p(\tau\mu)}{(F(\mu)-F(\tau\mu))^{1+\frac{1}{p}}}d\tau.$$

Moreover,

$$\lim_{\mu\to 1}\frac{\theta_p(\mu)-\theta_p(\tau\mu)}{(F(\mu)-F(\tau\mu))^{1+\frac{1}{p}}} = \frac{\theta_p(1)-\theta_p(\tau)}{(F(1)-F(\tau))^{1+\frac{1}{p}}}$$

and

$$\lim_{\tau\to 1}\frac{\theta_p(1)-\theta_p(\tau)}{1-\tau} = \theta_p^{'}(1) > 0.$$

Hence, we deduce that, for some positive constant c, we have

$$\frac{c}{(1-\tau)^{1+\frac{2}{p}}} < \frac{\theta_p(1)-\theta_p(\tau)}{(F(1)-F(\tau))^{1+\frac{1}{p}}} < \frac{c}{(1-\tau)^{1+\frac{2}{p}}}.$$

Now, since

$$\int_{1-\epsilon}^1\frac{\theta_p(1)-\theta_p(\tau)}{(F(1)-F(\tau))^{1+\frac{1}{p}}}d\tau = +\infty,$$

we have

$$\int_{1-\epsilon}^1\frac{\theta_p(1)-\theta_p(\tau)}{(F(1)-F(\tau))^{1+\frac{1}{p}}}d\tau = +\infty,$$

and so, by application of Fatou's lemma, we get

$$\lim_{\mu\to 1}\Lambda_p^{'}(\mu) = +\infty.$$

□

I.ADDOU obtained in [7] more properties of λ_p. We shall recall the following one, which will be used near.

Proposition 3. *For any $p > 1$, we assume that*

$$\theta_p''(\mu) \le 0 \qquad \text{for all } \mu \in (0, x_p]$$

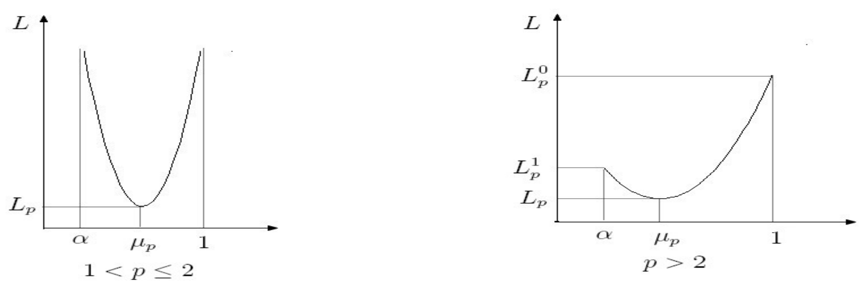
with strict inequality in an open interval $I_p \subset (0, x_p]$, and

$$\theta_p''(\mu) \ge 0 \qquad \text{for all } \mu \in [x_p, 1).$$

Where x_p is some point in $(0,1)$ for which θ_p'' changes sign.
Then, the time map λ_p admits a unique critical point; which is a minimum.

Proof. (See [5]). □

Remark 3. *In order to interpret the results of proposition 2 and proposition 3 we translate them to the following graphs of λ_p as follows:*



This interpretation takes form in the following

Lemma 2. *Let μ_p be the unique root of the equation $\lambda_p(\mu) = L$ stated in proposition 3 and $L_p = \lambda_p(\mu_p)$. Then for all $p \in (1,2]$, we have $\lambda_p((\alpha,1)) = [L_p, +\infty)$ and*

$$\lambda_p(\mu) = L \text{ has } \begin{cases} \text{no solution}, & if\ 0 < L < L_p, \\ \text{one solution}, & if\ L = L_p, \\ \text{two solutions noted } \mu_p^+(L) \text{ and } \mu_p^-(L), & if\ L > L_p, \end{cases}$$

where μ_p^+ and μ_p^- are respectively the largest and the smallest solutions of $L = \lambda_p(\mu)$.

Lemma 3. *For* $p \in]2, +\infty)$, *set* $L_p^0 = \lim_{\mu \to 1} \lambda_p(\mu)$, $L_p^1 = \lambda_p(\alpha)$ *and suppose that* $L_p^0 > L_p^1$. *Then we have* $\lambda_p([\alpha, 1)) = [L_p, L_p^0)$ *and*

$$\lambda_p(\mu) = L \text{ has } \begin{cases} \text{no solution}, \quad if\ 0 < L < L_p, \\ \text{one solution}, \quad if\ L = L_p \text{ or } L_p^1 < L < L_p^0, \\ \text{two solutions noted } \mu_p^+(L) \text{ and } \mu_p^-(L), \quad if\ L_p < L < L_p^1. \end{cases}$$

Theorem 2.1. *The set* $E^*(L)$ *of positive solutions of problem* (P_e) *can be characterized as follows: for any* $p > 1$

- *If* $0 < L < L_p$ *there is no positive solution for problem* (P_e).
- *If* $L = L_p$ *(resp* $L = L_p$ *and* $L_p^1 < L < L_p^0$*), for* $p \in (1, 2]$ *(resp* $p > 2$*), problem* (P_e) *admits one positive solution denoted by* $u(., \mu_p)$.
- *If* $L > L_p$ *(resp* $L_p < L < L_p^1$*), for* $p \in (1, 2]$*(resp* $p > 2$*), problem* (P_e) *admits two positive solutions denoted by* $s(., L) = u(., \mu_p^-)$ *and* $q(., L) = u(., \mu_p^+)$

Remark 4. *By proposition 2,* λ_p *is a continuous function of* μ*'s if and only if* $p > 2$; *so* $u(., \alpha) = u(., \mu_p^-(L_p^1))$ *generates families of nonnegative solutions of problem* (P_e) *on intervals* $(-L, L)$ *with* $L > L_p^1$ *for* $p > 2$.
So that $u(., \alpha)$ *extended by 0 for* $L_p^1 \leqslant |x| \leqslant L$ *is also solution of* (P_e) *for* $L > L_p^1$ *and so does the function defined by*

$$v(x, h) = \begin{cases} u(x - h, \alpha) & if\ |x - h| \leq L_p^1, \\ 0 & if\ |x - h| > L_p^1; \end{cases}$$

provided that $|h| \leqslant L - L_p^1$ *and* $p > 2$.

More generally, let N *be a positive integer and* L *satisfying* $L \geq N L_p^1$. *For each vector* $\xi = (\xi_1, ..., \xi_N)$ *such that*

$$-L \leq \xi_1 - L_p^1,\ \xi_i + L_p^1 \leq \xi_{i+1} - L_p^1,\ i = 1, ..., N-1 \text{ and } \xi_N + L_p^1 \leq L; \quad (2.4)$$

it is straightforward that the function

$$v(x, \xi) = \begin{cases} u(x - \xi_i, \alpha) & if\ |x - \xi_i| \leq L_p^1, \\ 0 & if\ |x - \xi_i| > L_p^1,\ for\ i = 1, ..., N, \end{cases}$$

is a nonnegative solution of problem (P_e).

Let $S_N(L)$ denotes the collection of functions $v(., \xi)$ where $\xi \in \mathbb{R}^N$ satisfies (2.4). We have the following

Proposition 4. *For $p > 2$ and $L > L_p^1$, we set*

$$S(L) = \bigcup_{j=1}^{N} S_j(L)$$

where N is the integral part of $\frac{L}{L_p^1}$.
Then we have

$$E^*(L) = \{q(.,L)\} \cup S(L).$$

Theorem 2.2. *The set $E(L)$ of positive solutions of (P_e) is given by the following*

- *For $p \in]1,2]$ we have*

$$E(L) = \begin{cases} \{0\} & for\ 0 < L < L_p, \\ \{0, q(.,L_p)\} & for\ L = L_p, \\ \{0, s(.,L), q(.,L)\} & for\ L_p < L. \end{cases}$$

- *For $p \in]2,+\infty)$ we have*

$$E(L) = \begin{cases} \{0\} & for\ 0 < L < L_p, \\ \{0, q(.,L_p)\} & for\ L = L_p, \\ \{0, s(.,L), q(.,L)\} & for\ L_p < L < L_p^1, \\ \{0, q(.,L)\} \cup S(L) & for\ L > L_p^1. \end{cases}$$

Remark 5. *In our study of $E(L)$, we distinguish two cases according to $p > 2$ or $1 < p \leq 2$.*
In the first case where $p > 2$, we find that $L_p^0 < +\infty$ and prove that the set of equilibrium solutions is the same as the one obtained by D.Aronson, M.G.Crandall and L.Peletier in [2] for $p = 2$. While in the case where $1 < p \leq 2$, we show that the set $E(L)$ is characterized by similar elements to those found by smoller and Wasserman in [13] for the operator u_{xx}.

3 general case

This part is devoted to the various basic, existence uniqueness continuous dependence on initial data comparison and stabilization results concerning problem (P).

Throughout this section, we set $\Omega = (-L, L)$ and $Q_t = \Omega \times [0, t]$.
Let us consider problem (P) defined by

$$(P)\begin{cases}\frac{\partial u}{\partial t} = (|u_x|^{p-2}u_x)_x + f(u) & on\ (-L, L) \times \mathbb{R}^+,\\ u(\pm L, t) = 0 & on\ \mathbb{R}^+,\\ u(x, 0) = u_0(x) & on\ ]-L, L[,\end{cases}$$

and assume that the data f and u_0 satisfy the following assumptions

(H_1) $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a locally lipschitzian function satisfying $f(0) = f(1) = 0$.

(H_2) $u_0 \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$ and $0 \leq u_0 \leq 1$.

Definition 2.

- *By a solution u of problem (P) on $[0, T]$ we mean a function satisfying the following properties:*

  (i) $u \in C([0, T], L^1(\Omega)) \cap L^\infty(Q_T) \cap L^\infty(0, T, W_0^{1,p}(\Omega))$,

  (ii) $\int_\Omega u(t)\varphi(t) - \int\int_{Q_t} u\varphi_t - |u_x|^{p-2}u_x\varphi_x = \int_\Omega u_0\varphi(0) + \int\int_{Q_t} f\varphi$, *for all* $\varphi \in \mathcal{C}^1(\bar{Q}_T)$ *such that* $\varphi \geq 0$ *and* $\varphi(\pm L, t) = 0\ \forall\ t \in [0, T]$.

- *A solution on $[0, \infty)$ means a solution on each $[0, T]$, $\forall\ T > 0$.*

- *A subsolution (supersolution) is defined by (i) and (ii) with equality replaced by $\leq$ $(\geq)$.*

3.1 Existence, uniqueness and continuous dependence for problem (P)

Theorem 3.1. *Assume that assumptions (H_1) and (H_2) are satisfied, then, problem (P) admits a unique solution such that $0 \leq u \leq 1$*

Proof. The solution of problem (P) is obtained as a limit, as $\varepsilon \to 0$, of a sequence u_ε whose terms are solutions to a regularized problem associated with problem (P).
Since $u_0 \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$, then there exist a sequence $u_{0\varepsilon}$ in $\mathcal{C}_0^\infty(\Omega)$ such that $0 \leqslant u_{0\varepsilon} \leqslant 1$ and $\| u_0 - u_{0\varepsilon} \|_{W_0^{1,p}(\Omega)} \underset{\varepsilon\to 0}{\longrightarrow} 0$.
Consequently, the regularized problem associated with problem (P) defined by

$$(P_\epsilon)\begin{cases}\frac{\partial u}{\partial t} = \triangle_p^\epsilon u + f_\epsilon(u) & on\ Q_T,\\ u(\pm L, t) = \epsilon & on\ (0, T],\\ u(x, 0) = u_{0\varepsilon}(x) & sur\ \bar{\Omega},\end{cases}$$

where $\triangle_p^\epsilon u = \nabla.\phi_\epsilon(\nabla u)$, $\phi_\epsilon(\nabla u) = (|\nabla u|^2 + \epsilon)^{\frac{p-2}{2}}\nabla u$ and $(f_\varepsilon) \subset \mathcal{C}^1(\mathbb{R}^+)$, such that f_ε converges uniformly, as $\varepsilon \to 0$, to f, $\frac{\partial f_\varepsilon}{\partial u}(u(t, x)) \leqslant K$, for some constant $K > 0$, and $f_\varepsilon(0) \geqslant 0$, possesses a unique solution $u_\epsilon \in \mathcal{C}^{2,1}(\bar{Q}_T)$ satisfying $0 \leq u_\epsilon \leq 1$, and we have the following estimates.

Lemma 4. *For $\epsilon > 0$, we have*

(i) $\| \frac{\partial u_\epsilon}{\partial t} \|_{L^2(0,T,L^2(\Omega))} \leq C$,

(ii) $\| u_\epsilon \|_{L^\infty(0,T,W^{1,p}(\Omega))} \leq C$,

(iii) $\| u_\epsilon \|_{L^p(0,T.W^{1,p}(\Omega))} \leq C$,

(iv) $\| \phi_\epsilon(\nabla u_\varepsilon) \|_{L^\infty(0,T,L^{p'}(\Omega))} \leq C$.

Remark 6. *These estimates are proved in [5] and [6] in the case where problem (P) is defined on a bounded subset Ω of $\mathbb{R}^N$ with $N \geq 1$. The solution u of problem (P) is showed to belong to $L^\infty(0,T,W^{1,p}(\Omega) \cap L^\infty(\Omega))$. It remains to show that $u \in \mathcal{C}([0,T], L^1(\Omega))$ to conclude that u is a solution of problem (P) in the sens of definition 2.*

On the one hand, from estimates (i) and (ii) we have

$$\frac{\partial u_\varepsilon}{\partial t} \text{ is bounded in } L^2(0,T,L^2(\Omega)),$$

and

$$u_\varepsilon \quad \text{is bounded in } L^\infty(0,T,W^{1,p}(\Omega)).$$

On the other hand, according to the *Rellich-Kondrakov* theorem (see [1] in page 144), the space $W^{1,p}(\Omega)$ is compactly imbedded in $L^q(\Omega)$, $\forall 1 \leqslant q \leqslant +\infty$ and $p > 1$ in the case of a unidimensional space. This allow us to conclude, by application of corollary 4 of [12], that there exists a subsequence u_{ε_n} such that $\varepsilon_n \underset{n\to+\infty}{\longrightarrow} +\infty$ and $u_{\varepsilon_n} \longrightarrow u$ in $\mathcal{C}([0,T], L^2(\Omega))$. Consequently $u_{\varepsilon_n} \longrightarrow u$ in $\mathcal{C}([0,T], L^1(\Omega))$, since $\mathcal{C}([0,T], L^2(\Omega)) \subset \mathcal{C}([0,T], L^1(\Omega))$. □

Remark 7. *The space $W^{1,p}(\Omega)$ is compactly imbedded in $L^q(\Omega)$, for $1 \leqslant q \leqslant +\infty$, only if $N = 1$. But, it's compactly imbedded in $L^p(\Omega)$ for any $N \in \mathbb{N}^*$. So, when $N > 1$ and $p > 2$, we have $L^p \hookrightarrow L^2$, then, we can apply corollary 4 of [12] again to obtain that $u \in C([0,T], L^1(\Omega))$.*

Proposition 5. *Let u_1 and u_2 be two solutions of problem (P) on $[0,T]$ associated respectively with u_{01}, f_1 and u_{02}, f_2. Then*

$$\| u_1(t) - u_2(t) \|_{L^1(\Omega)} \leq \| u_{01} - u_{02} \|_{L^1(\Omega)} + \| f_1 - f_2 \|_{L^1(Q_t)} . \tag{3.1}$$

Proof. If u_1 and u_2 are two solutions of problem (P) associated respectively with u_{01} and u_{02}, then for any test function $\varphi \in C^1(Q_T)$ with $\varphi \geq 0$ and $\varphi(\pm L, t) = 0$, we have

$$\int_\Omega (u_1(t) - u_2(t))\varphi(t) - \int\int_{Q_t} (u_1 - u_2)(\varphi_t - \frac{|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2}{u_1 - u_2}\varphi_x)dxdt = \int_\Omega (u_{01} - u_{02})\varphi(0) + \int\int_{Q_t} (f_1 - f_2)\varphi dxdt. \tag{3.2}$$

Let

$$\eta = \frac{|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2}{u_1 - u_2}.$$

From the monotonicity of the function $\xi \longrightarrow |\xi|^{p-2}\xi$ and the fact that $\| u_1 - u_2 \|_{L^\infty(\Omega)} < M$, we can deduce that the function η verifies the following assertions

(i) $\eta \geq 0$

(ii) $\eta \in L^{p'}(\Omega)$.

In the following step we construct an appropriate function to use in (3.2) as a test function and which enables us to conclude to inequality (3.1).
To this end, choose a sequence (η_n) in $\mathcal{C}_0^\infty(\Omega)$ such that (η_n) converges to η in $L^{p'}(\Omega)$ and let $\chi \in C_0^\infty(\Omega)$ such that $0 \leq \chi \leq 1$. Then the following parabolic problem

$$\begin{cases} \varphi_{nt} - \eta_n \varphi_{nx} = \lambda \varphi_n & on\ \Omega \times (0,T), \\ \varphi_n(\pm L, t) = 0 & on\ [0,T), \\ \varphi_n(x,T) = \chi(x) & on\ \Omega, \end{cases}$$

admits a unique solution in $\mathcal{C}^\infty(\bar{Q}_T)$: This result is allowed by the classical theory developed in [8]. Moreover, we have the following assertions

(1) $0 \leq \varphi_n \leq e^{\lambda(t-T)} \quad on\ \Omega \times (0,T)$,

(2) $\sup\limits_{\bar{Q}_T} |\varphi_{nx}| \leq M$.

Set $t = T$ and $\varphi = \varphi_n$ in (3.2), to obtain

$$\begin{aligned} \int_\Omega (u_1 - u_2)\chi + \int\int_{\bar{Q}_T} (u_1 - u_2)(\eta - \eta_n)\varphi_{nx} = \int_\Omega (u_{01} - u_{02})\varphi_n(0) \\ + \int\int_{\bar{Q}_T} ((f_1 - f_2) + \lambda(u_1 - u_2))\varphi_n. \end{aligned} \tag{3.3}$$

But

$$|\int\int_{Q_T} (u_1 - u_2)(\eta - \eta_n)\varphi_{nx}| \leq C \sup_{Q_T} |\varphi_{nx}| \, \| \eta - \eta_n \|_{L^{p'}(\Omega)} \| u_1 - u_2 \|_{L^p(\Omega)} .$$

Then, by passage to the limit, as $n \longrightarrow +\infty$, in (3.3), we get

$$\int_\Omega (u_1(T) - u_2(T))\chi \leq \int_\Omega (u_{01} - u_{02})^+ e^{-\lambda T} + \int\int_{\bar{Q}_T} ((f_1 - f_2) + \lambda(u_1 - u_2))^+ e^{\lambda(s-T)},$$

for all $\chi \in \mathcal{C}_0^\infty(\Omega)$ with $0 \leq \chi \leq 1$.
Set, $\chi(x) = 1$ on $\{x, u_1(T) > u_2(T)\}$ and $\chi = 0$ otherwise. We have

$$\int_\Omega (u_1(t) - u_2(t))^+ \leq \int_\Omega (u_{01} - u_{02})^+ + \int\int_{Q_t} e^{\lambda(s-T)}(f_1 - f_2) + \lambda(u_1 - u_2))^+.$$

Thus, for $\lambda = 0$, we deduce

$$\int_\Omega (u_1(t) - u_2(t))^+ \leq \int_\Omega (u_{01} - u_{02})^+ + \int\int_{Q_t} (f_1 - f_2)^+.$$

Hence

$$\| u_1(t) - u_2(t) \|_{L^1(\Omega)} \leq \| u_{01} - u_{02} \|_{L^1(\Omega)} + \| f_1 - f_2 \|_{L^1(Q_t)} .$$

□

Remark 8. *In the proof of proposition 5 we obtained*

$$\int_\Omega (u_1(T) - u_2(T))\chi \leq \int_\Omega (u_{01} - u_{02})^+ e^{-\lambda T} + \int\int_{\bar{Q}_T} ((f_1 - f_2) + \lambda(u_1 - u_2))^+ e^{\lambda(s-T)},$$

which is the equation that leads to estimation (3.1) *and will also lead to the point* (i) *and also to the comparison principle* (ii) *in the following theorem.*

Theorem 3.2. *(i) Let u_1 and u_2 be two solutions of problem (P) on $[0, T]$, associated respectively with initial data u_{01} and u_{02}. Let K be a lipschitz constant for f on $[-M, M]$, with $M = \max(\| u_1 \|_{L^\infty(Q_T)}, \| u_2 \|_{L^\infty(Q_T)})$. Then*

$$\| u_1(t) - u_2(t) \|_{L^1(\Omega)} \leq e^{Kt} \| u_{01} - u_{02} \|_{L^1(\Omega)} . \tag{3.4}$$

(ii) Let u be a subsolution and $\hat{u}$ a supersolution of problem (P) with initial data u_0 and $\hat{u}_0$. If $u_0 \leq \hat{u}_0$ then we have

$$u \leq \hat{u}.$$

3.2 Regularization

We begin this paragraph by proving the lipschitz property of the solution operator of problem (P), by using the regularizing effects results concerning evolution equations given in [4], which is an important ingredient used to prove the main regularizing theorem.

Lemma 5. *Under hypotheses (H_1) and (H_2), we have*

i) The function $t \mapsto u(t, u_0)$ is lipschitz continuous from $[\tau, \infty)$ into $L^1(\Omega)$ with constant K_τ independent of u_0.

ii) In the case where $p > 2$, the function $t \mapsto \phi(\nabla u(t))$ is continuous from $[\tau, \infty)$ into $L^1(\Omega)$, where $\phi(x) = |x|^{p-2}x$.

Proof. **(i)**Let $S(t, u_0, f)$ denotes the solution of problem (P) at time t. So, by inequality (3.1) the operator S satisfies

$$\| S(t, u_{01}, f_1) - S(t, u_{02}, f_2) \|_{L^1(\Omega)} \leq \| u_{01} - u_{02} \|_{L^1(\Omega)} + \| f_1 - f_2 \|_{L^1(Q_t)} . \quad (3.5)$$

On the other hand, it suffices to verify that $\lambda^{\frac{1}{m-1}} S(\lambda t, u_0, f)$ is a solution of problem (P) associated with $\lambda^{\frac{1}{m-1}} u_0$ and $\lambda^{\frac{m}{m-1}} f_\lambda$, to conclude, thanks to the uniqueness of the solution of problem (P), that

$$\lambda^{\frac{1}{m-1}} S(\lambda t, u_0, f) = S(t, \lambda^{\frac{1}{m-1}} u_0, \lambda^{\frac{m}{m-1}} f_\lambda), \quad \lambda \geq 0, \quad (3.6)$$

where $f_\lambda(t)(.) = f(\lambda t)(.)$ and $m = p - 1$.
Now by properties (3.5), (3.6) of S and the Lipschitz continuity of f, we get, by applying theorem 7 of [4], that the solution u of problem (P) verify, the following regularizing effect of the solution u: for $\tau > 0$, $0 < h \leq \tau$ and $t \geq 0$, we have

$$\begin{aligned} \frac{1}{h} \| u(t+\tau+h, u_0) - u(t+\tau, u_0) \|_{L^1(\Omega)} &= \frac{1}{\tau} \left(\frac{\tau}{h} \| u(\tau+h, u(t, u_0)) - u(\tau, u(t, u_0)) \|_{L^1(\Omega)} \right) \\ &\leq \frac{1}{\tau} H(\tau, \| u(t, u_0) \|_{L^1(\Omega)}), \end{aligned}$$

where H is a nondecreasing function of its arguments. Moreover, since we have $0 \leq u \leq 1$, then $\| u(t, u_0) \|_{L^1(\Omega)} \leq meas\Omega = 2L$. So, it follows that $\tau^{-1} H(\tau, 2L)$ is a Lipschitz constant for $t \longrightarrow u(t, u_0)$ on $[\tau, \infty)$.
(ii) Following [5](in page 1392,1393), one can obtain that $\frac{\partial}{\partial t} \phi_\varepsilon(\nabla u_\varepsilon)$ is bounded in $L^2(t_0, \infty, L^{p'}(\Omega))$ and from (iv) of lemma 6 we have that $\phi_\varepsilon(\nabla u_\varepsilon)$ is bounded in $L^\infty(0, \infty, L^{p'}(\Omega))$. Hence by application of corollary 4 of [12] we get the continuity of the function $t \rightarrow \phi(\nabla u(t))$ from $[\tau, \infty)$ into $L^1(\Omega)$.

□

Now, our main regularizing theorem is the following:

Theorem 3.3. *Assume that assumptions (H_1) and (H_2) hold, $p > 2$ and let u be the solution of problem (P). Then for each $\tau > 0$ there exists a constant M_τ, independent of u_0, such that*

(i) $\phi(\nabla u) \in L^\infty(\Omega)$ for $t > \tau$.

(ii) $\| \phi(\nabla u) \|_{L^\infty(\Omega)} \leq M_\tau$ and ess var$\phi(\nabla u) \leq M_\tau$ for $t \geq \tau$.

To prove this result we shall use the following lemma.

Lemma 6. *Let $v(t)$ be Lipschitz continuous function with constant K, and $w(t),\ z(t)$ be continuous functions from $[0,\infty)$ into $L^1(\Omega)$ with*

$$v_t = w_x + z \qquad in\ D'(\Omega).$$

Then $w(t) \in L^\infty(\Omega)$ for all t and

$$ess\ var\ w(t) \leq Kmeas(\Omega) + \| z(t) \|_{L^1(\Omega)}\ .$$

The proof is similar to that of lemma 15 in [2] and we avoid it.
By virtue of lemma 5, we can apply lemma 6 to the equation

$$u_t = (\phi(\nabla u))_x + f(u)$$

which hold in the sense of distributions.

Thus, $\phi(\nabla u(t)) \in L^\infty(\Omega)$ for $t \geq \tau > 0$ and the variation of $\phi(\nabla u(t))$ is bounded by $K_\tau + \| f(u(t)) \|_{L^1(\Omega)}$, which is bounded.
Using corollary 2.4 of [11] we get

$$\| \phi(\nabla(u)) \|_{L^\infty(\Omega)} \leq ess\ var \phi(\nabla u).$$

So assertions of theorem 3.3 are hence proved.

3.3 Stabilization

Let $p > 2$, $0 \leq u_0 \leq 1$ and $u = u(t, u_0)$ the solution of problem (P) associated with u_0. For each $\tau > 0$ define the semiorbit

$$\gamma_\tau = \{u(t, u_0), t \geq \tau\}.$$

According to theorem 3.3, we have $\gamma_\tau(u_0) \subset X_\tau$, where X_τ is the metric space whose elements $w \in L^\infty(\Omega)$ satisfy

$0 \leq w \leq 1,\ w_x \in L^\infty(\Omega),\ \| w_x \|_{L^\infty(\Omega)} \leq M_\tau$ and essential variation$\phi(w_x) \leq M_\tau$.

Where M_τ is as in theorem 3.3.

Lemma 7. *i) The space X_τ equipped with the metric*

$$d(u, v) = \| u - v \|_{L^1(\Omega)} + \| (u - v)_x \|_{L^p(\Omega)}$$

is compact.

ii) The semi-orbit γ_τ is precompact.

Proof. **i)** It is clear that X_τ is complete. Moreover X_τ is bounded in $W^{1,\infty}(\Omega)$, and is thus precompact in $L^1(\Omega)$.
On the other hand the subset $\{\phi(w_x), w \in X_\tau\}$ is bounded in $L^\infty(\Omega)$ and in variation. Thus, it is precompact in $L^1(\Omega)$ and then by the L^∞-boundedness, the set $\{w_x, w \in X_\tau\}$ is precompact in $L^p(\Omega)$ for every $1 \leq p < \infty$. Consequently X_τ is compact.
ii) Since $\gamma_\tau \subset X_\tau$ which is compact, then (ii) follows. □

The following statement is an immediate result of lemma 7.

Corollary 1. *i) If $(u_n) \subset X_\tau$ and $\| u_n - u \|_{L^1(\Omega)} \rightarrow 0$, then $u \in X_\tau$ and $d(u_n, u) \rightarrow 0$.*

ii) The solution $u(., u_0) \in C((0,\infty), X)$, where X is the space defined by

$$X = \{u \in L^\infty(\Omega), \quad 0 \leq u \leq 1, \quad u_x \in L^p(\Omega)\}.$$

Define the w-limit set as

$$w(u_0) = \{w \in X, u(t_n, u_0) \rightarrow w \text{ in} X, \text{ for some sequence } (t_n) \text{ with } t_n \underset{n\rightarrow\infty}{\longrightarrow} \infty\}.$$

We have the following

Proposition 6. *Assume that hypothesis (H_1) and (H_2) are satisfied and that $p > 2$, then*

i) $w(u_0)$ is nonempty and connected in X,

ii) if $w \in w(u_0)$ then $u(t, w) \in w(u_0)$ for $t > 0$.

Proof. **i)** Since γ_τ is precompact, then $w(u_0)$ is nonempty.
ii) As $u(t_n, u_0) \rightarrow w$ in X and so in $L^1(\Omega)$ we get, by assertion (3.4) of theorem 3.2, that $u(t + t_n, u_0) \rightarrow u(t, w)$ in $L^1(\Omega)$ and thus in X.
Hence, as $u(t + t_n, u_0) = u(t, u(t_n, u_0))$, we get $u(t, w) \in w(u_0)$. □

Now, the main result of this section is the following.

Theorem 3.4. *Let hypothesis (H_1) and (H_2) hold and $p > 2$. Then $w(u_0) \subset E$.*

To prove this theorem, we will introduce the function $V : X \longrightarrow \mathbb{R}$ defined by

$$V(\varphi) = \int_\Omega \left(\frac{1}{p} |\varphi^{'}|^p - F(\varphi) \right) dx,$$

where $F(r) = \int_0^r f(s) ds$.
One can easily check that V is continuous and satisfies the following statements

Lemma 8. *under assumptions* (H_1) *and* (H_2) *we have*

$$u_t \in L^2(0, \infty, L^2(\Omega))$$

and

$$\int_s^t \int_\Omega (u_t)^2 + V(u(t, u_0)) \leq V(u(s, u_0)) \quad \textit{for } t > s > 0. \tag{3.7}$$

Proof. For $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, we choose $u_{0n} \subset C_0^\infty(\Omega)$ such that $u_{0n} \underset{n\to\infty}{\longrightarrow} u_0$. Let u_n the sequence of solutions of (P) associated with u_{0n}, we so get that

$$\int_s^t \int_\Omega (u_{nt})^2 + V(u_n(t, u_{0n})) \leq V(u_n(s, u_{0n})),$$

and by letting $n \to +\infty$ we get

$$\int_s^t \int_\Omega (u_t)^2 + V(u(t, u_0)) \leq V(u(s, u_0)).$$

□

Now, we are ready to prove theorem 3.4

Proof. (of theorem 3.4).
By lemma 8, the function $t \longrightarrow V(u(t, u_0))$ is nonincreasing for $t > 0$. Moreover V is continuous on X, thus we have

$$V(w) = \inf_{t>0} V(u(t, u_0)) \quad \text{for } w \in w(u_0).$$

On the other hand, form The assertion (ii) of proposition 6, $w(u_0)$ is an invariant subset of X, so

$$V(u(t, w)) = V(w) \quad \forall w \in w(u_0),\ \forall t > 0. \tag{3.8}$$

Then, from (3.8) and (3.7), we can deduce that $(u(t, w))_t \equiv 0$ and thus $u(t, w) = w$. This means that w is a solution to problem (P) and thus satisfies the following relation

$$\int_\Omega (|w_x|^{p-2} w_x \varphi_x + f(w)\varphi) = 0,$$

with $\varphi \in C^1(\bar{\Omega})$, $\varphi \geq 0$ and $\varphi(\pm L) = 0$.
But this implies that $\Delta_p w + f(w) = 0$ only in $D'(\Omega)$. Now, since $w \in L^\infty(\Omega)$ and f is lipschitz, then $|w_x|^{p-2} w_x \in C^1(\bar{\Omega})$. Moreover, $w = 0$ at $\pm L$. Consequently $w \in E$. □

4 Applications

This part is devoted to the study of the stability of some equilibrium solutions of the motivating example, namely $u = 0$ and $u = q$, using the stabilization result proved above. To this end we begin by defining the notion of subsolutions and supersolutions of problem (P^*)

Definition 3. *A weak subsolution of problem (P^*) is a function $u \in C([-L, L])$ for which $\int_\Omega (|u'|^{p-2}u'\varphi' + f(u)\varphi)dx \geq 0$ for all $\varphi \in C^1(\bar{\Omega})$, $\varphi \geq 0$ and $\varphi(\pm L) = 0$ and $u(\pm L) \leq 0$.*
A weak supersolution is defined by reversing the inequality and $u(\pm L) \geq 0$.

Next, let $\underline{u}$ and $\bar{u}$ be respectively a subsolution and a supersolution of problem (P^*) and define

$$[\underline{u}, \bar{u}] = \{w \in L^\infty(\Omega) \cap W^{1,p}(\Omega), \underline{u} \leq w \leq \bar{u} \ a.e \text{ on } \Omega\}$$

Proposition 7. *If $u_0 \in [\underline{u}, \bar{u}]$ such that (H_2) is satisfied, then*

i) $u(t, u_0) \in [\underline{u}, \bar{u}]$ *for all* $t \geq 0$.
and

ii) $w(u_0) \subset [\underline{u}, \bar{u}] \cap E$.

Proof. To prove (i), we use theorem 3.2 and the definition of $\underline{u}$ and $\bar{u}$ which are time-independent.
The statement (ii) follows immediately from (i) and proposition 6. □

Corollary 2. *If $u_0 \in [\underline{u}, \bar{u}]$ such that hypotheses (H_2) is satisfied and $[\underline{u}, \bar{u}] \cap E = \{g\}$ is a singleton, then $u(t, u_0) \to g$ in X as $t \to \infty$.*

Now, as examples of application of corollary 2 to problem (P) where $f(u) = u(1-u)(a-u)$, we will determinate some domains of attraction for some isolated elements of $E(L)$.

Example 1

For $p > 2$, let $L \in [L_p^1, L_p^0)$ and choose l such that

$$\max_A(|\xi_1 - L_p^1|, |\xi_N + L_p^1|) \leq l < L,$$

where

$$A = \{\xi = (\xi_1, \xi_2, ..., \xi_N) \in \mathbb{R}^N, -L \leq \xi_1 - L_p^1,\ \xi_i + L_p^1 \leq \xi_{i+1} - L_p^1,\ i = 1, ..., N-1,\ \xi_N + L_p^1 \leq L\}$$

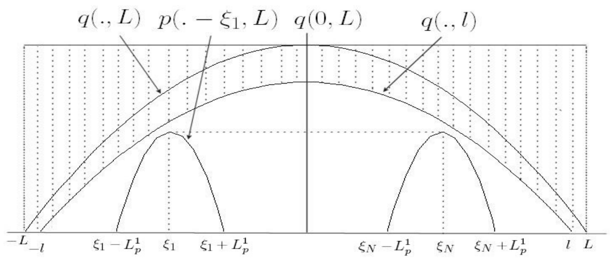
Then

$$\underline{u}(x) = \begin{cases} q(x, l) & if \quad x \in [-l, l], \\ 0 & if \quad x \notin [-l, l]. \end{cases}$$

is a subsolution of (P_e). On the other hand $\bar{u} \equiv q(0,L)$ is a supersolution of (P_e).
So,

$$\lim_{t\to+\infty} u(t,u_0) = q(L) \text{ for } u_0 \in [\underline{u},\bar{u}].$$

The domain of attration for $q(.,L)$ is

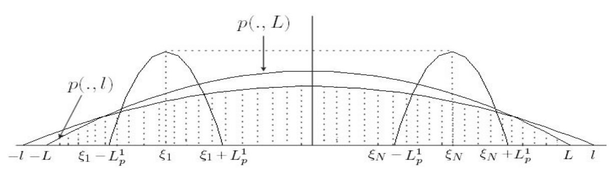


Example 2
For $p > 2$, let $L \in [L_p^1, L_p^0)$ and choose $l \in [L, L_p^0]$. Set

$$\bar{u}(x) = \begin{cases} p(x,l) & if \quad x \in [-l,l], \\ 0 & if \quad x \notin [-l,l]. \end{cases}$$

Then $\bar{u}$ is a supersolution of (P_e). Also $\underline{u} \equiv 0$ is a subsolution of (P_e).
In this case, we have

$$\lim_{t\to+\infty} u(t,u_0) = 0 \text{ for } u_0 \in [\underline{u},\bar{u}].$$



Domain of attraction for O where $p > 2$ and $L \in [L_p^1, L_p^0)$

References

[1] R.A.Adams Sobolev spaces. *Academic Press* (1975).

[2] D.Aronson, M.G.Crandall and L.Peletier. Stabilization of solutions of a degenerate nonlinear diffusion problem. *Nonlinear Analysis, Theory, Methods& Applications.* , **vol 6**, No 10, pp.1001-1022(1982).

[3] H.Brezis ANALYSE FONCTIONNELLE Theorie et applications.*Masson, Paris.* (1983).

[4] M.G.Crandall and Philippe Benilan. Regularizing effects of homogeneous evolution equations. *Mathematics Research center TSR ♯ 2076.* University of wisconsin at Madison (1980).

[5] A.El Hachimi and F.De Thelin Supersolution and stabilization of the solutions of the equation $\frac{\partial u}{\partial t} - \nabla(|\nabla u|^{p-2}\nabla u) = f(x,u)$. PartI. *Nonliear Analysis, Theory, Methods and Applications.* **Vol 12**.No 12. 1385-1398(1988).

[6] A.El Hachimi and F.De Thelin Supersolution and stabilization of the solutions of the equation $\frac{\partial u}{\partial t} - \nabla(|\nabla u|^{p-2}\nabla u) = f(x,u)$. PartII. *Publicacions Matematiques.* **Vol 35** (1991), 347-362.

[7] IDRISS ADDOU. Exact multiplicity results for quasilinear boundary-value problems with cubic-like nonlinearities, *e.j.d.e* **vol 2000** No 01. pp 1-26, (2002).

[8] O.A.Ladyzenskaja, V.A.Solonnikov and N.N.Ural'ceva: Linear and quasiliear equations parabolic type(1968), AMS Transl.Monogr *vol 23* Providence, RI.

[9] M.Langlais and D.Phillips Stabilization of solutions of nonlinear and degenerate evolution equations *Nonlinear analysis T.M.A* **9**. pp. 321-333(1985).

[10] J.L.Lions. Quelques methodes de resolution des problemes aux limites non lineaires. *Dunod, Paris*, (1969).

[11] S.Osher and O.Scherzer. G-Norm properties of bounded variation regularization. *Comm. Math. Sci*, **Vol 2**, No 2, pp.237-254 (2004).

[12] J.SIMON. Compact Sets in the Space $L^p(0,T,B)$, *Ann.Mat. Pura Appl.*(4)**146**, 65-96 (1987)

[13] J.Smoller and A.Wasserman. Global bifurcation of steady-state solutions,*Journal of differential equations.* **39**, 269-290 (1981).

Received: December, 2008