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# Generalized Convexity and Invexity in Optimization Theory: Some New Results

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**Abstract**

Convex like properties, without vector space structure are intensively used in the minimax theory Since Ky Fan has proved the first minimax theorem for concave-convexlike functions, several authors have proposed other extensions or generalizations for the convexity.
In this paper we propose a survey of the recent studies concerning convexity and invexity in optimization theory. In fact, in the first part we consider the studies of Hanson (1981), Craven (1986), Giorgi and Mititelu (1993), Jeyakumar (1985), Kaul and Kaur (1985), Martin (1985) and Caristi, Ferrara and Stefanescu (1999, 2001, 2005) considering some new examples and remarks.
In the second part we consider the consistent notation of $(\Phi, \rho)$-invexity establishing new properties in optimization theory.

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## 1 Introduction

A few years later, Hanson (1981) by a seminal paper introduced differentiable function from $R^n$ into $R$ for which there exists a vector function $\eta(x, u) \in R^n$such that

$$f(x) - f(u) \geq [\eta(x, u)]^t \nabla f(u) \tag{1}$$

where $\nabla$ denotes the gradient. The class of functions satisfying (1) were called invex later by Craven (1981) and actually rappresented an important concept in optimization theory because figured as a very broad generalization of convexity. Craven and Glover (1985) showed that the class of invex functions is equivalent to the class of functions whose stationary points are global minima.

In a preceeding paper Caristi, Ferrara and Stefanescu [1] introduced some generalizations of invex functions in a smooth vision of the problem studied. In fact, let $\varphi$ be a differentiable function on a non-empty open set $X \subseteq R^n$, $\varphi : X \to R^n$.

**Definition 1** *$\varphi$ is $\eta$-invex at point $x^0 \in X$ if $\exists\ \eta : X \times X \to R^n$ such that*

$$\varphi(x) - \varphi\left(x^0\right) \geq \eta\left(x, x^0\right), \nabla\varphi\left(x^0\right). \tag{2}$$

$\varphi$ is $\eta$-invex on X if $\exists\ \eta : X \times X \to R^n$ such that $\forall x, y \in X$

$$\varphi(x) - \varphi(y) \geq \eta(x, y), \nabla\varphi(y). \tag{3}$$

$\varphi$ is $\eta$-pseudoinvex at point $x^0 \in X$ if

$$\eta\left(x, x^0\right), \nabla\varphi\left(x^0\right) \geq 0 \tag{4}$$

for some $x \in X \Rightarrow \varphi(x) - \varphi(x^0) \geq 0$.
$\varphi$ is $\eta$-quasinvex at point $x^0 \in X$ if

$$\varphi(x) - \varphi\left(x^0\right) \leq 0 \tag{5}$$

for some $x \in X \Rightarrow \langle \eta(x, x^0), \nabla\varphi(x^0) \rangle \leq 0$.

**Definition 2** *$\varphi$ is $\eta$-infinvex at point $x^0 \in X$ if*

$\inf_{x \in X_0} [\varphi(x) - \varphi(x^0)] \geq \inf_{x \in X_0} \langle \eta(x, x^0), \nabla\varphi(x^0) \rangle$
$\varphi$ is $\eta$-supinvex at point $x^0 \in X$ if

$$\sup_{x \in X_0} \left[\varphi(x) - \varphi\left(x^0\right)\right] \geq \sup_{x \in X_0} \left\langle \eta\left(x, x^0\right), \nabla\varphi\left(x^0\right) \right\rangle. \tag{6}$$

$\varphi$ is $\eta$-infpseudoinvex at point $x^0 \in X$ if

- $\inf_{x \in X_0} \langle \eta(x, x^0), \nabla\varphi(x^0) \rangle \geq 0 \Rightarrow \inf_{x \in X_0} [\varphi(x) - \varphi(x^0)] \geq 0$.

$\varphi$ is $\eta$-supquasiinvex at point $x^0 \in X$ if

- $\sup_{x \in X_0} [\varphi(x) - \varphi(x^0)] \leq 0 \Rightarrow \sup_{x \in X_0} \langle \eta(x, x^0), \nabla\varphi(x^0) \rangle \leq 0$.

where $X_0$ is a subset of $X$.

**Proposition 3** *$\varphi$ $\eta$-invex not implies that $\varphi$ is $\eta-infinvex$ and $\eta-supinvex$. As can be seen from the following example:*

**Example 4** *The function $\theta : [-1,1] \to R$ defined by $\theta(x) = x^3$ is not invex but if we consider $\mu$=0*

$$\sup_x \left(x^3 - \mu^3\right) = 1 - \mu^3 \geq 0 \tag{7}$$

the function $\theta$ is $\eta - supinvex$ and if assume $\mu = -1$

$$\inf_x \left(-\mu^3\right) \geq \inf_x \left(-3\mu^2\right) \tag{8}$$

then function considered is $\eta - infinvex$.

**Remark 5** *Obviously, $\eta - \inf -invexity$ implies $\eta - \inf -peudovexityinvexity$ and $\sup -invexity$ implies $\eta - \sup -quasiivexity$.*

the above properties will be used in the classical framework of the scalar optimization problem:

$$(P) \inf_{x \in X_0} f(x), whereX_0 = \{x \in X \,|g_j(x) \leq 0\,, \quad j = 1, 2, ..., m\} \tag{9}$$

where $f$ and $g_j$ are differentiable.

**Definition 6** *The problem $(P)$ is $\eta - \inf - \sup -invex$ at $y \in X$, if $f$ is $\eta - \inf -invex$ with repect to $X_0$ and $g_j$, $j = 1, 2, ..., m$ are $\eta - \sup -invex$ at $y$, with respect to $X_0$. $(P)$ is $\eta - \inf - \sup -invex$ on $X$ if it is $\eta - \inf -invex$ at every point $y \in X$.*

Let $(D)$ the Wolfe dual in $X$ of $(P)$:

$$(D) \sup_{(x,v) \in V} \left(f(x) + \sum_{j=1}^{m} v_j g_j(x)\right) \tag{10}$$

where,

$$V = \left\{(x,v) \in X \times \mathbf{R}_+^m \,\middle|\, \nabla f(x) + \sum_{j=1}^{m} v_j \nabla g_j(x) = 0\right\}. \tag{11}$$

In [4] authors will show that $(P)$ is $\eta-\inf-\sup-invex$ on $X$ for some $\eta$, then any Kuhn-Tucker point is an optimum solution of $(P)$ and the weak duality property holds.

Assuming $f$ and $g_j$ differentialble on $X$, a Kuhn-Tucker point is a pair $(x_0, v) \in X_0 \times \mathbf{R}_+^m$, satisfying the following two condition:

$$\nabla f(x_0) + \sum_{j=1}^{m} v_j \nabla g_j(x_0) = 0 \tag{12}$$

$$\sum_{j=1}^{m} v_j g_j(x_0) = 0 \tag{13}$$

For $y \in X_0$, let us denote by $J_0(y)$ the set of active constraints at $y$; $J_0(y) = \{j \,| g_j(y) = 0\}$.

**Theorem 7** *Let$(x_0, v)$ be any Kuhn-Tucker point of the problem $(P)$. If there exist some $\eta$, such that $f$ is $\eta-\inf-pseudovex$ at $x_0$ with respect to $X_0$, and for every $j \in J_0(x_0)$, $g_j$ is $\eta-\sup-quasiinvex$ at $y \in X$ with respect to $X_0$, then $x_0$ is an optimum solution of $(P)$.*

**Corollary 8** *Let $(x_0, v)$ be any Kuhn-Tucker point of the problem $(P)$. If there exist some $\eta$ such that $f$ is $\eta-\inf-invex$ at $x_0$ with respect to $X_0$, and for every $j \in J_0(x_0)$, $g_j$ is $\eta-\sup-invex$ at $y \in X$ with respect to $X_0$, then $x_0$ is an optimum solution of $(P)$.*

**Corollary 9** *Assume that there exist some $\eta$ such that $(P)$ is $\eta-\inf-\sup-invex$ on $X$. Then for every Kuhn-Tucker point $(x_0, v)$, $x_0$ is an optimum solution.*

**Remark 10** *An obvious relaxation of tha above theorem can be obtained weakening the requirements for the contraint functions. In fact, it is sufficient to assume that*

$$j \in J(x_0) \Rightarrow \sup_{x \in X_0} \langle \eta(x, x_0), \nabla g_j(x_0) \rangle \leq 0 \tag{14}$$

consequently, one obtains:

**Corollary 11** *Let $(x_0, v)$ be any Kuhn-Tucker point of the problem $(P)$. If there exist some $\eta$ such that the constraint functions satisfy [14] and the objective function satisfies:*

$$\inf_{x\in X_0}\langle\eta(x,x_0),\nabla f(x_0)\rangle\geq 0\Rightarrow\inf_{x\in X_0}[f(x)-f(x_0)]\geq 0 \qquad (15)$$

then $x_0$ is an optimum solution.

**Theorem 12** *Assume that there exist some $\eta$ such that $f$ is $\eta-\inf-$pseudoinvex on $X$ and all $g_j$ are $\eta-\sup-$invex on $X$. Then, for any feasible solution $x$ of $(P)$ $(x\in X_0)$, and for any feasible solution $(y,v)$ of $(D)$ $\quad((y,v)\in V)$, one has:*

$$f(x)\geq f(y)+\sum_{j=1}^{m}v_j g_j(y) \qquad (16)$$

**Corollary 13** *if the problem $(P)$ is $\eta-\inf-\sup-$invex on $X$ for some $\eta$ then it has the weak duality property.*

Always in [4] Caristi, Ferrara and Stefanescu considering that obtain weaker invexity-type conditions which are necessary and succient for the sufficiency of the Kuhn-Tucker conditions.

Martin (1965) showed that if a mathematical programming problem with linear contraints which delimit a bounded feaisible set is invex, than the objective function must be convx.

He called the problem $(P)$ Kuhn-Tucker invex on $X$, if there exist a vector function $\eta: X\times X\to\mathbf{R}^n$, such that:

$$x,y\in X_0\Rightarrow\begin{cases} f(x)-f(y)\geq\langle\eta(x,y),\nabla f(y)\rangle \\ \langle\eta(x,y),\nabla g_j(y)\rangle\leq 0,\text{ whenever } g_j(y)=0,\ \text{for } j=1,2,...,m\end{cases} \qquad (17)$$

**Theorem 14** *(Martin (1985), Theorem 2.1) Every Kuhn-Tucker point of problem $(P)$ is a global minimizer if and only if $(P)$ is Kuhn-Tucker invex.*

**Theorem 15** *A Kuhn-Tucker point $(x_0,v)$ is a global minimizer of the problem $(P)$ if and only if there exist a vector funtion $\eta: X\times X\to\mathbf{R}^n$, such that (14) and (15) hold.*

**Proof.** The sufficiency follows from Corollary 10 of Theorem 6. For the necessity one observe that Kuhn-Tucker invexity implies the conditions of the theorem. ■

It is also obvious that,

**Theorem 16** *Every Kuhn-Tucker point of problem* $(P)$ *is a global minimizer if and only if there exist a vector function* $\eta : X \times X \to \mathbf{R}^n$*, such that* $f$ *is* $\eta-\inf-$*pseudoinvex* ($\eta - \inf -$*invex*) *on* $X_0$ *with respect to* $X_0$*, and, for every* $y \in X_0$*,* $g_j$ *is* $\eta-\sup-$*quasiinvex* ($\eta - \sup -$*invex*) *at* $y$*, whenever* $j \in J_0(y)$*.*

Finally, let us examine an example to show that our conditions actually don't imply convexity.

**Example 17** *consider the problem:*

$$\inf_{x \in X_0} (-x_1 x_2), \text{ where, } X_0 = \left\{ x \in \mathbf{R}^2 \,|x \geq 0, \; x_1 + x_2 \leq 1 \right\}. \tag{18}$$

obviously, $y = (1/2, \; 1/2)$ is Kuhn-Tucker point. One can wasily verify that problem is $\eta - \inf - \sup -$*invex* at $y$, with respect $X_0$, $\eta(x, y) = (x_1^2 - y_1, x_2^2 - y_2)$.

## 2 Convexlike and weakly convex like functions

We begin by a brief survey of some properties emerging from the Fan's convexlike concept. Let $X$ be a nonvoid set, F a family of real-valued functions defined on $X$ and $t \in [0, 1]$.

**Definition 18** *F is* $t-$*convexlike on* $X$ *if*

$$\forall x_1, x_2 \in X, \exists x_0 \in X, \forall f \in \mathcal{F}, \quad f(x_0) \leq t f(x_1) + (1-t) f(x_2) \tag{19}$$

**Definition 19** *F is* $t-$*subconvexlike on* $X$ *if*

$$\forall x_1, x_2 \in X, \forall \varepsilon > 0, \exists x_\varepsilon \in X, \forall f \in \mathcal{F}, f(x_\varepsilon) \leq t f(x_1) + (1-t) f(x_2) + \varepsilon \tag{20}$$

The assertions of the following proposition are obvious:

**Proposition 20** *If F is* $t-$*convexlike (* $t-$*subconvexlike) on* $X$*, so is:*

- a) Any subfamily $\mathcal{F}'$ of F
- b) $\mathcal{F} \cup \{c\}$, for any constant function $c$

- c) The convex hull $co\mathcal{F}$ of F

It is also known (see Paeck [6]) that:

**Proposition 21** *If F is $t-convexlike$ ($t-subconvexlike$) on $X$for some$t\in$ $(0,1)$, then there exists a dense subset $D(t)$ of the interval $[0,1]$, such that F is $s-convexlike$ ( $s-subconvexlike$) on$X$, for every $s\in D(t)$.*

**Definition 22** *F is weakly convexlike on X if*

$$\forall x_1,\ x_2\in X|,\forall t\in[0,1],\inf_{x\in X}\sup_{f\in\mathcal{F}}f(x)\leq\sup_{f\in\mathcal{F}}\left[tf(x_1)+(1-t)f(x_2)\right] \quad (21)$$

The following implications are obvious:

F is $t-convexlike$ on $X$, for some $t\in(0,1)\Rightarrow\mathcal{F}$ is $t-subconvexlike$ on $X$, for the same $t\Rightarrow\mathcal{F}$ is weakly convexlike on $X$

The counterpart of a) in Proposition 1 doesn't hold for the weakly convexlike families, but the other two assertions are true, as it is shown below.

**Proposition 23** *If the family F is weakly convexlike on $X$, so is $\mathcal{F}\cup\{c\}$, for every constant function c .*

**Proposition 24** *If the family F is weakly convexlike on $X$, so is $co\mathcal{F}$ . introduced by a specialized concept.*

**Definition 25** *The ordered finite family $\mathcal{F}=\{f1,f2,...,fn\}$ is sequential weakly convexlike (s.w.c.) on $X$ if, for every $i=1,2,...,n-1$, the family $\{f1,f2,...,f\}$ is weakly convexlike on $X$, for each $f\in co\{fi+1,...,fn\}$.*

**Remark 26** *If $\{f1,f2,...,fn\}$ is s.w.c. on $X$, so is $\{f1,f2,...,fi,f\}$, for every $f\in co\{fi+1,...,fn\}$ and $i=1,2,...,n-1$.*

**Remark 27** *If $\mathcal{F}=\{f1,f2,...,fn\}$ is $t-subconvexlike$ on $X$, for some $t\in(0,1)$, then every subfamily of $co\mathcal{F}$ is weakly convexlike on $X$, and hence, F is s.w.c.*

Stefanescu in [18] proved that:

**Theorem 28** *Let F be any family of functions on $X$. If each finite subfamily of $co\mathcal{F}$ is weakly convexlike on $X$, then, for every finite subfamily $\mathcal{F}'$ of F exactly one of the following situations occurs:*

$$\exists \overline{x} \in X, \forall f \in \mathcal{F}', \quad f(\overline{x}) < 0, \tag{22}$$

$$\exists f \in co\mathcal{F}', \forall x \in X, \ \overline{f}(x) \geq 0 \tag{23}$$

We establish below, a slight generalization of the above theorem, that will allow us to prove a saddle-point theorem for constrained optimization. The proof of the theorem basically follows the line in the above cited paper. For the sake of completeness, we will prove first a lemma.

For any family F of real-valued functions on $X$, denote by $X(\mathcal{F}) = \{x \in X | f(x) < 0, \forall f \in F\}(X(\emptyset) = X)$.

**Lemma 29** *(see also [18].) Let F be a finite family of real-valued functions defined on $X$ and $h_1, h_2 : X \to \mathbb{R}$. Assume that*

$$X(\mathcal{F} \cup \{h_1, h_2\}) = \emptyset \tag{24}$$

If $\mathcal{F} \cup \{h_1, h_2\}$ is weakly convexlike on $X$, then there exists $t \in [0, 1]$, such that

$$\forall x \in X(\mathcal{F}), th_1(x) + (1 - t)h_2(x) \geq 0 \tag{25}$$

Now, let $T_i = \{s \in [0,1] \ \mid sh_1(x) + (1-s)h_2(x) < 0$, for some $x \in X(\mathcal{F} \cup \{h_i\}\}, i = 1, 2$. One can easily verify that both $T_1$ and $T_2$ are open in $[0, 1]$, and $0 \in T_2, 1 \in T_1$. Since, by (24), $T_1 \cap T_2 = \emptyset$, it follows that $[0, 1]\backslash(T_1 \cup T_2) = \emptyset$. Thus, (25) holds for at least one $t$ in $[0, 1]$.

**Theorem 30** *Assume $H = \{h_1, h_2, ..., h_n\}$ be s.w.c. on $X$. Then exactly one of the following two situations occurs:*

$$\exists \overline{x} \in X, h_i(\overline{x}) < 0, i = 1, 2, ..., n \tag{26}$$

$$\exists \overline{h} \in coH, \forall x \in X, \ \overline{h}(x) \geq 0 \tag{27}$$

**Proof.** We prove, by induction on n, that (27) holds whenever (26) doesn't hold.

For $n = 2, H = \{h_1, h_2\}$ is weakly convexlike. If $max_{i=1,2}h_i(x) \geq 0, \forall x \in X$, apply Lemma 16 for $\mathcal{F} = \emptyset$ and (27) results.

Assume now that the theorem is true for every s.w.c. family of at most $n \geq 2$ functions, and consider $H = \{h_1, h_2, ..., h_n + 1\}$ be s.w.c. such that $max_{1\leq i\leq n+1}h_i(x) \geq 0, \forall x \in X$. Denote by $X' = X(\{h_1, h_2, ..., h_{n-1}\})$. If $X' = \emptyset$, pick an $h_0 \in co\{h_n, h_{n+1}\}$.

If $X' = \emptyset$, pick an $h_0 \in co\{h_n, h_{n+1}\}$.

If $X' \neq \emptyset$, then $max_{i=n,n+1}h_i(x) \geq 0, \forall x \in X'$, and by Lemma 16 (for $\mathcal{F} = \{h_1, h_2, ..., h_{n-1}\}$) it follows the existence of $h_0 \in co\{h_n, h_{n+1}\}$ such that $h_0(x) \geq 0, \forall x \in X'$.
In both cases, $max_{0\leq i\leq n-1}h_i(x) \geq 0, \forall x \in X$, and since $\{h_1, h_2, ..., h_{n-1}, h_0\}$ is s.w.c., it follows by induction that $h(x) \geq 0, \forall x \in X$, for some $h \in co\{h_1, h_2, ..., h_{n-1}, h_0\} \subseteq coH$. ■

The following corollary results from Theorem (17) and Remark (4).

**Corollary 31** *If $H = \{h_1, h_2, ..., h_n\}$ is $t-$ subconvexlike on $X$ for some $t \in (0, 1)$, then the alternative (26)-(27) holds*

# 3 Saddle-point theorem for constrained optimization.

In [3] Caristi and Ferrara showed that the Lagrange's Saddle-point inequalities should be verified by any optimum solution of an optimization problem if the objective and the constraints functions form a s.w.c. family. The supporting space could be any set, not endowed with topology or vector space structure.

Particularly, our result holds for discrete optimization.

Let X be any nonvoid set. Consider the constrained optimization problem:

$$(P) : inf\{f(x)|x \in X, g_j(x) \leq 0, \qquad j = 1, 2, ..., m\} \tag{28}$$

Denoting by g the vector function $(g_1, g_2, ..., g_m)$, the associated Lagrange function $L : X \times \mathbb{R}^m_+ \to \mathbb{R}$, is defined by:

$$L(x, u) = f(x) + \langle u, g(x)\rangle \tag{29}$$

**Theorem 32** *Assume that the family $\{g_1, g_2, ..., g_m,\ f,\ c\}$ is s.w.c. for every constant function c. If the Slater's constraints qualification condition*

$$(c-q) : \exists x_0 \in X,\ \ g_j(x_0) < 0, j = 1, 2, ..., m \tag{30}$$

is satisfied, then for every optimum solution $\overline{x}$ of $(P)$ there exists $u \in \mathbb{R}^m_+$ such that $(\overline{x}, \overline{u})$ is a saddle-point of $L$, i.e.

$$L(x, u) \leq L(x, u) \leq L(x, u), \forall x \in X, \forall u \in \mathbb{R}^m_+ \tag{31}$$

**Corollary 33** *If $\{g_1, g_2, ..., g_m, f\}$ is $t-$ subconvexlike on $X$ for some $t \in (0, 1)$, then for every optimum solution $\overline{x}$ of $(P)$, there exists $u \in \mathbb{R}^m_+$ satisfying (31).*

# 4 $(\Phi, \rho)$ invexity

We begin by introducing a consistent notation for vector inequalities and for derivative operators.

In the following, $\mathbb{R}^n$ denotes the $n-$dimensional Euclidean space. If $x, y \in \mathbb{R}^n$; then $x \geq y$ means $x_i \geq y_i$ for all $i = 1, 2, ..., n$, while $x > y$ means $x_i > y_i$ for all $i = 1, 2, ..., n$. An element of $\mathbb{R}^{n+1}$ may be regarded as $(t, r)$ with $t \in \mathbb{R}^n$ and $r \in \mathbb{R}$.

Let $\varphi : D \subseteq \mathbb{R}^n \to \mathbb{R}$ be a differentiable (twice dierentiable) function of the independent variable $x$; and $a \in D$.

We will denote by $\nabla_x \varphi_{x=a}$ the gradient of $\varphi$ at the point $a$, and $\nabla^2_{xx} \varphi_{x=a}$ stands for the matrix formed by the second order derivatives of $\varphi$. When any confusion is avoided, we will omit the subscript, writing simply $\nabla \varphi(a)$ respectively, $\nabla^2 \varphi(a)$. In the next definitions, $\varphi$ is a real number and $\Phi$ is a real-valued function defined on $D \times D \times \mathbb{R}^{n+1}$; such that $\Phi(x, a, .)$ is convex on $\mathbb{R}^{n+1}$ and $\Phi(x, a, )0, r)) \geq 0$ for every $(x, a) \in D \times D$ and $r \in \mathbb{R}_+$.

**Definition 34** *We say that $\varphi$ is $(\Phi, \rho)$ - invex at $a$ with respect to $X \subseteq D$, if*

$$\varphi(x) - \varphi(a) \geq \Phi(x, a, (\nabla \varphi(a), \rho)) \forall x \in X \tag{32}$$

$\varphi$ is $(\Phi, \rho)$ - invex on $D$ if it is $(\Phi, \rho)$ - invex at $a$, for every $a \in D$.

**Remark 35** *If $\varphi_1$ is $(\Phi, \rho_1)$ - invex and $\varphi_2$ is $(\Phi, \rho_2)$ - invex then $\lambda\varphi_1 + (1-\lambda)\varphi_2$ is $(\Phi, \lambda\rho_1 + (1-\lambda)\rho_2$-invex, whenever $\lambda \in [1, 0]$. in particular if $\varphi_1$ and $\varphi_2$ is $(\Phi, \rho)$ - invex with repsect to same $\Phi$ and $\rho$, then so is $\lambda\varphi_1 + (1-\lambda)\varphi_2$.*

The following two de ... nitions generalizes $(\Phi, \rho)$ - invexity.

**Definition 36** *We say that $\varphi$ is pseudo $(\Phi, \rho)$ - invex at $a$ with repsect to $X$, if whennever $\Phi(x, a, (\nabla\varphi(a), \rho)) \geq 0$ for some $x \in X$, then $\varphi(x) - \varphi(a) \geq 0$.*

**Definition 37** *We say that $\varphi$ is quasi $(\Phi, \rho)$ - invex at $a$ with repsect to $X$, if whennever $\varphi(x) - \varphi(a) \leq 0$ for some $x \in X$ then $\Phi(x, a, (\nabla\varphi(a), \rho)) \leq 0$*

**Remark 38** *For $\Phi(x, a, (y, a)) = F(x, a, y) + rd^2(xa)$, where $F(x, a, .)$ is sublinear on $\mathbb{R}^n$, the definition of $(\Phi, \rho)$ - invexity reduces to the definition of $(F, \rho)$-convexity introduced by Preda [13], which in turn generalizes the concepts of $F$-convexity ([9]) and $\rho$-invexity ([19]).*

More comments on the relationships between $(\Phi, \rho)$ - invexity and invexity and their earlier extensions are in the next two sections.

# 5 Optimality conditions

The typical mathematical programming problem

$$(P) : \inf \{f(x) | x \in X_0, g_i(x) \leq 0, j = 1, 2, ..., m\} \tag{33}$$

where $X_0$ is a nonvoid open subset of $\mathbb{R}^n$, $f : X_0 \to \mathbb{R}$, $g_j : X_0 \to \mathbb{R}$, $j = 1, 2, ..., m$.

Let $X$ be the set of all feasible solutions of (P);

$$X = \{x \in X_0, \ g_j(x) \leq 0, \quad j = 1, 2, ..., m\} \tag{34}$$

Everywhere in this paper $f$ and $g_j$, $j = 1, 2, ..., m$ are assumed to be differentiable on $X_0$; and we will refer to a Kuhn-Tucker point of (P) according to the usual definition.

**Definition 39** *$(a, v) \in X \times \mathbb{R}^m$ is said to be a Kuhn-Tucker point of the problem $(P)$ if:*

$$\nabla f(a) + \sum_{j=1}^{m} v_j \nabla g_j(a) = 0 \tag{35}$$

$$\sum_{j=1}^{m} \upsilon_j g_j (a) = 0 \tag{36}$$

Denoting by $J(a) = \{j \in \{1, 2, ..., m\} \quad |g_j (a) = 0\}$, then summation in (35) and (36) is over $J(a)$.

Now we estabilish the necessity and sufficiency of Kuhn-Tucker conditions for the optimality in $(P)$ under $(\Phi, \rho)$ -invexity.

Everywhere in the following, we will assume invexity with respect to the set $X$ of the feasible solutions of $(P)$, but for the sake of simplicity we will omit to mention $X$.

**Theorem 40** *Let $(a, v)$ be a Kuhn-Tucker point of $(P)$: If $f$ is pseudo $(\Phi, \rho)$ -invex at $a$, and for each $j \in J(a)$, $g_j$ is quasi $(\Phi, \rho)$-invex at $a$, for some $\rho_0, \rho_1,\ j \in J(a)$ such that $\rho_0 + \sum_{j \in J(a)} \upsilon_j \rho_j \geq 0$, then $a$ is an optimum solution of $(P)$.*

**Proof.** Set $\lambda_0 = 1/\left(1 + \sum_{j=1}^{m} \upsilon_j\right),\ \lambda_j = \lambda_0 \upsilon_j,\ j = 1, 2, ..., m$. Obviously.

$$\sum_{j=0}^{m} \lambda_j \rho_j = \lambda_0 \rho_0 + \sum_{j \in J(a)} \lambda_j \rho_j \geq 0 \tag{37}$$

and

$$\lambda_0 \nabla f (a) + \sum_{j \in J(a)} \lambda_j \nabla g_j (a) = 0 \tag{38}$$

Then, it follows from the definition of $\Phi$ that

$$\begin{aligned} 0 \ \leq \ \Phi\left(x, a, \left(\lambda_0 \nabla f (a) + \sum_{j \in J(a)} \lambda_j \nabla g_j (a), \lambda_0 \rho_0 + \sum_{j \in J(a)} \lambda_j \rho_j\right)\right) \leq \\ \lambda_0 \Phi (x, a, (\nabla f (a), \rho_0)) + \sum_{j \in J(a)} \lambda_j \Phi (x, a, (\nabla g_j (a), \rho_2)) \end{aligned} \tag{39}$$

for every $x \in \mathbb{R}^{|n}$.

Now, let $x \in X$ be a feasible solution. Since $g_j(x) - g_j(a) \leq 0$ and $g_j$ is quasi $(\Phi, \rho_j)$-invex, it results that $\Phi (x, a, (\nabla g_j (a), \rho_j)) \leq 0$, for each $j \in J(a)$. Hence, the above inequalities imply $\Phi (x; a, (\nabla f (a), \rho_0)) \geq 0$ and the pseudo $(\Phi, \rho)$-invexity of $f$ implies $f (x) - f (a) \geq 0$. ■

**Theorem 41** *Let $a$ be an optimum solution of $(P)$. Suppose that Slater's constraint qualification holds for restrictions in $J(a)$ (i.e. there exists $x^* \in X_0$ such that $g_j(x^*) < 0$, for all $j \in J(a)$). If, for each $j \in J(a)$, $g_j$ is $(\Phi, \rho_j)$-invex at $a$ for some $\rho_j \geq 0$, then there exists $\upsilon \in \mathbb{R}^m_+$ such that $(a; \upsilon)$ is a Kuhn-Tucker point of $(P)$.*

**Proof.** Since $f$ and $g_j$ are differentiable, then there exist Fritz-John multipliers $\mu \in \mathbb{R}^m_+$ such that:

$$\mu \nabla f(a) + \sum_{j=1}^{m} \lambda_j \nabla g_j(a) = 0 \tag{40}$$

$$\sum_{j=1}^{m} \lambda_j \,| g_j(a) = 0 \tag{41}$$

$$\mu + \sum_{j=1}^{m} \lambda_j > 0 \tag{42}$$

All that we need is to prove that $\mu > 0$.

Suppose, by way of contradiction, that $\mu = 0$. Then $\sum_{j \in J(a)} \lambda_j > 0$ from (42), and we can define $\mu_j = \lambda_j / \sum_{j \in J(a)} \lambda_j$. Obviously $\sum_{j \in J(a)} \mu_j g_j(a) = 0$ and $\sum_{j \in J(a)} \mu_j \rho_j \geq 0$.

Hence, since each $g_j$ is $(\Phi, \rho_j)$-invex,

$$\begin{gathered} 0 \leq \Phi\left(x^*, a, \left(\sum_{j \in J(a)} \mu_j \rho_j\right)\right) \leq \\ \sum_{j \in J(a)} \mu_j \Phi\left(x^*, a, \left(\nabla g_j(a), \rho_j\right)\right) \leq \\ \sum_{j \in J(a)} \mu_j \left(g_j(x^*) - g_j(a)\right) \end{gathered} \tag{43}$$

but$\sum_{j \in J(a)} \mu_j g_j(a) = 0$ by (41), so that $\sum_{j \in J(a)} \mu_j \left(g_j(x^*) - g_j(a)\right) < 0$, contradicting the above inequalities. ■

**Remark 42** *Unlike Martin's conditions, where the properties of all functions involved in the problem, ( $f$ and $g_j$), are defined in respect to the same scale* *function $\eta$, our conditions are allowed to be satisfied for different scale functions. In fact, considering different values of $\rho$, $f$ and each $g_j$ should satisfy different invexity conditions. Thus in the definitions of Section 4, $\rho$ should be interpreted as a parameter, and $\Phi$ generates a family of functions, one for each value of $\rho$. Similar situation appears in the case of $(F,\ \rho)$-convexity (or $\rho-$invexity), but in that case the sign of $\rho$ determines explicitly the properties of the function subjected to such condition. As we can observe in the proof of Theorem 21 (and in all results bellow), all that we need is that $\Phi(.,.,(0,r))$ is non-negative for some values of $r$. We have asked this condition to be satisfied whenever $r \geq 0$, but this is a convention which can be replaced by any other one.*

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