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On Reduction of Banerjee Shandil Semi Circle

Riyaz Ahmad Khan, Abdul Wadood Khan and Gulam Muhiuddin

Department of Mathematics
Integral University, Lucknow-226 026, India
riyazakhan68@yahoo.co.in
khanawadood71@gmail.com
chishtygm@gmail.com

Abstract. In this paper, we stablish a more accurate discription of the region of confinement for the point (μ, σ) in the $\mu\sigma$- plane by applying the Banerjee Shandil semi circle theorem on the square of the random variate x which gives another necessary condition for the point (μ, σ) in the $\mu\sigma$-plane. In this condition, we obtain a fourth degree curve involving fourth central moment μ_4. On applying Kendal's inequality, this fourth central moment μ_4 converts into the third central moment μ_3 about mean which leads the reduction in the Banerjee-Shandil semi circle.

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1. Introduction.

Let x be a real and continous random statistical variate defined over a finite domain $a \leq x \leq b$ and let $\phi(x)$ be a continous, non-negative real valued function of x on $a \leq x \leq b$ satisfying the condition $\int_a^b \phi(x)dx = 1 \quad \ldots (1)$

Then the ordered pair $(x, \phi(x))$ is said to be the probability distribution of x and $\phi(x)$ is said to be the probability density function of variate x over $a \leq x \leq b$. The mean μ and the variance σ^2 of the variate x are given as

$$\mu = \int_a^b x\phi(x)dx \qquad \ldots (2)$$

and

$$\sigma^2 = \int_a^b (x-\mu)^2\phi(x)dx \quad \ldots (3)$$

In [2], Banerjee-Shandil have shown that the point (μ, σ) in $\mu\sigma$-plane must satisfy the condition (known as Banerjee Shandil semi circle theorem) given as

$$[\mu - \frac{a+b}{2}]^2 + \sigma^2 \leq [\frac{b-a}{2}]^2 \qquad \ldots (4)$$

This is uniformally valid for all probability density functions defined over a finite domain $a \leq x \leq b$. Kapur [4] have shown that for two points distribution i.e., the variate takes only two values say x_1 and x_2 with the probability p and $1-p$, the Banerjee-Shandil semi circle inequality becomes equality.

Thus to obtain the reduction in Banarjee Shandil semi circle theorem, some more information are needed about the variate except the prescription of the domain of the variate as taken by Banerjee-Shandil [2].

3. Main Result.

Theorem 3.1. Let μ and $\sigma(> 0)$ be the mean and the standard deviation respectively of a continous random statistical variate x define over a finite domain $a \leq x \leq b$ and let $\phi(x)$ be the probability density function. Then the point (μ, σ) lies inside the region of the first quadrant bounded by the curve

$$[\mu^2 + \sigma^2 - (a^2+b^2)/2]^2 + \sigma^4[\frac{\mu_3}{\sigma^3} + 2\frac{\mu}{\sigma}]^2 \leq [(b^2-a^2)/2]^2$$

provided $a > 0$ and the skewness factor $\frac{\mu_3}{\sigma^3} \geq 1$.

Proof. The mean and the variance of variate x^2 are given by

$$\text{mean} = \int_a^b x^2\phi(x)dx \qquad \ldots (5)$$

and

$$\text{variance} = \int_a^b (x^2 - \text{mean})^2\phi(x)dx \quad \ldots (6)$$

From eqn. (3), we have

$$\begin{aligned}\sigma^2 &= \int_a^b (x-\mu)^2\phi(x)dx \\ &= \int_a^b [x^2+\mu^2-2\mu x]\phi(x)dx \\ &= \int_a^b x^2\phi(x)dx + \mu^2\int_a^b \phi(x)dx - 2\mu\int_a^b x\phi(x)dx \\ &= \int_a^b x^2\phi(x)dx - \mu^2\end{aligned}$$

$$\Longrightarrow \quad \int_a^b x^2\phi(x)dx = \mu^2+\sigma^2$$

Henceforth, the mean and the variance of the variate x^2 are given as

$$\text{mean} = \int_a^b x^2\phi(x)d(x) = \mu^2+\sigma^2 \qquad \ldots (7)$$

and

$$\text{variance} = \int_a^b [x^2-(\mu^2+\sigma^2)]^2\phi(x)dx \qquad \ldots (8)$$

By applying Banerjee-Shandil semi circle theorem on the variate x^2, we have

$$[\mu^2+\sigma^2-(a^2+b^2)/2]^2 + \int_a^b [x^2-(\mu^2+\sigma^2)]^2\phi(x)dx \leq [(b^2-a^2)/2]^2 \qquad \ldots (9)$$

Now, we first calculate

$$\begin{aligned}\int_a^b [x^2-(\mu^2+\sigma^2)]^2\phi(x)dx &= \int_a^b x^4\phi(x)dx - (\mu^2+\sigma^2)^2 \quad (\text{ By Eqn. (7)}) \\ &= \mu_4' - (\mu^2+\sigma^2)^2 \\ &= \mu_4 + 6\mu^2\sigma^2 + 4\mu\mu_3 + \mu^4 - (\mu^2+\sigma^2)^2\end{aligned}$$

Therefore, Eqn. (9) becomes

$$[\mu^2+\sigma^2-(a^2+b^2)/2]^2 + \mu_4 + 6\mu^2\sigma^2 + 4\mu\mu_3 + \mu^4 - (\mu^2+\sigma^2)^2 \leq [(b^2-a^2)/2]^2$$

Using Kandel's inequality i.e., $\mu_4 \geq \frac{{\mu_3}^2}{\sigma^2} + \sigma^4$, we have

$$[\mu^2+\sigma^2-(a^2+b^2)/2]^2 + \sigma^4(\frac{\mu_3}{\sigma^3} + 2\frac{\mu}{\sigma})^2 \leq [(b^2-a^2)/2]^2 \qquad \ldots (10)$$

For symmetrical distribution $\mu_3 = 0$, therefore eqn. (10) becomes

$$[\mu^2 + \sigma^2 - (a^2 + b^2)/2]^2 + 4\mu^2\sigma^2 \leq [(b^2 - a^2)/2]^2 \quad \ldots (11)$$

We now show that by using skewness factor, there is a reduction in Banerjee-Shandil semi circle (4).

Setting $\mu = \frac{a+b}{2}$ and $\sigma = \frac{b-a}{2}$ for a point (μ, σ) on the semi circle (4). It can be easily see that the point $(\frac{a+b}{2}, \frac{b-a}{2})$ satisfies the inequality (11) which amounts to say that the point $(\frac{a+b}{2}, \frac{b-a}{2})$ lies on (11).

Therefore, inequality (10) does not hold good for $\frac{\mu_3}{\sigma^3} \geq 1$. Hence the point $(\frac{a+b}{2}, \frac{b-a}{2})$ lies on the cicle (4) must lie out side the curve (10). Thus, for at least one point, there is a reduction in the Banerjee-Shandil semi circle.

Further, since $\frac{\mu_3}{\sigma^3} \geq 1$, we have from eqn. (10)

$$[\mu^2 + \sigma^2 - (a^2 + b^2)/2]^2 + \sigma^4(1 + 2\frac{\mu}{\sigma})^2 \leq [(b^2 - a^2)/2]^2 \quad \ldots (12)$$

Thus, the point (μ, σ) must lie within or on the region of the first quadrant bounded by the curve given by the inequality (12).

Therefore, the inequality (12) together with inequality (4) exclude a portion of Banarjee-Shandil semi circle and hence the point (μ, σ) must lie in the intersection region of the two curves given by the inequalities (12) and (4) respectively. This completes the proof.

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