An Iterative Method with Fifth-Order Convergence for Nonlinear Equations

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Abstract

In this paper, we suggest and analyze a new four-step iterative method for solving nonlinear equations involving only first derivative of the function using a new decomposition technique which is due to Noor [11] and Noor and Noor [16]. We show that this new iterative method has fifth-order of convergence. Several numerical examples are given to illustrate the efficiency and performance of the new method and a comparison to other results is also presented in this paper.

Mathematics Subject Classification: 65

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1. Introduction

In recent years, many researchers have been paid attention to propose and study several iterative methods for solving nonlinear equations, see [1-16]. Abbasbandy [1] and Chun [5] have proposed and studied several one-step and two-step iterative methods using the Adomian decomposition method [2]. Noor [11] and Noor and Noor [16] have considered another decomposition technique which does not involve the derivative of the Adomian polynomial. Noor and Noor [15] used this alternative decomposition to construct one-step, two-step and three-step iterative methods for solving nonlinear equations. In this paper, we use the alternative decomposition technique proposed in [11] and [16] to construct four-step iterative method for solving nonlinear equations involving only the first derivative of the function.

In Section 2, we outline the main ideas of the alternative decomposition technique and develop the four-step iterative method for solving nonlinear equation. The convergence analysis of this new iterative method will be presented in Section 3. Several numerical examples are given in Section 4 to illustrate the efficiency and the accuracy of the new proposed iterative method. A comparison of our results with other results will be presented also in this section. Some conclusions are pointed in Section 5.

2. Iterative Algorithms

Consider the nonlinear equation

$$f(x) = 0. \tag{2.1}$$

We assume that $f(x)$ has a simple root at α and γ is initial guess sufficiently close to α. Equation (2.1) can be convert into the following coupled system,

$$f(\gamma) + f^{'}(\gamma)(x - \gamma) + g(x) = 0, \tag{2.2}$$

$$g(x) = f(x) - f(\gamma) - f^{'}(\gamma)(x - \gamma), \tag{2.3}$$

where γ is the initial approximation for a zero of (2.1).

Equation (2.2) can be rewritten in the following form:

$$x = c + N(x), \tag{2.4}$$

where

$$c = \gamma - \frac{f(\gamma)}{f^{'}(\gamma)} \tag{2.5}$$

and

$$N(x) = -\frac{g(x)}{f^{'}(\gamma)}. \tag{2.6}$$

We note that if x_0 is the initial guess, then one can observe that

$$f(x_0) = g(x_0). \quad\quad ((2.7))$$

Noor and Noor [15] constructed a sequence of high-order iterative method by using a decomposition method which is due to Noor [11] and Noor and Noor [16]. This decomposition of the nonlinear operator $N(x)$ is quite different than that of the Adomian decomposition.

The main idea of this technique is to look for a solution of equation (2.4) having the series form:

$$x = \sum_{i=0}^{\infty} x_i. \quad\quad ((2.8))$$

The nonlinear operator N can be decomposed as

$$N\left(\sum_{i=0}^{\infty} x_i\right) = N(x_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} x_j\right)\right\}. \quad\quad ((2.9))$$

Upon substituting equations (2.8), and (2.9) into (2.4) yields

$$\sum_{i=0}^{\infty} x_i = c + N(x_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} x_j\right)\right\}. \quad\quad ((2.10))$$

Thus we have the following iterative scheme:

$$x_0 = c$$

$$x_1 = N(x_0)$$

$$x_2 = N(x_0 + x_1)$$

$$\vdots$$

$$x_{n+1} = N \left(x_0 + x_1 + ... + x_n \right), n = 1, 2, ... \qquad ((2.11))$$

It can be shown that the series $\sum\limits_{i=0}^{\infty} x_i$ converges absolutely and uniformly to a unique solution of equation (2.4).

$$x = c + \sum_{i=1}^{\infty} x_i. \qquad ((2.12))$$

Therefore, x is approximated by

$$X_n = x_0 + x_1 + ... + x_{n-1}, \qquad ((2.13))$$

where $\underset{n \to \infty}{Lim} X_n = x$.
For $n = 0$,

$$x \approx X_0 = x_0 = c = \gamma - \frac{f(\gamma)}{f'(\gamma)}, \qquad ((2.14))$$

This allows us to suggest the following one-step iterative method for solving the nonlinear equation (2.1).

Algorithm 1 *2.1. For a given x_0, compute the approximate solution x_{n+1} by the iterative scheme:*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad f'(x_n) \neq 0, \quad n = 0, 1, 2, ...$$

which is known as Newton method and has a second order convergence.
Using (2.13) with $n = 1$, we obtain

$$x \approx X_1 = x_0 + x_1 = c + N(x_0) = \gamma - \frac{f(\gamma)}{f'(\gamma)} - \frac{f(x_0)}{f'(\gamma)}.$$

Using this relation, we suggest the following two-step iterative method for solving nonlinear equation (1.1).

Algorithm 2 *2.2. For a given x_0, compute the approximate solution x_{n+1} by the*

iterative scheme:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0,$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(y_n)}{f'(x_n)}, \quad n = 0, 1, 2, ...$$

It is worth to mention here that Algorithm (2.2) was obtained by Chun [5] using the Adomian decomposition method. It has been shown in [5] that Algorithm (2.2) has cubic convergence.

By using (2.11) with $n = 2$, we can conclude that

$$x_2 = N(x_0 + x_1) = -\frac{g(x_0 + x_1)}{f'(\gamma)} = -\frac{f(x_0 + x_1)}{f'(\gamma)}. \quad ((2.15))$$

Therefore, by using (2.13) and (2.15) with $n = 2$, we get

$$x \approx X_2 = x_0 + x_1 + x_2 = c + N(x_0) + N(x_0 + x_1) = \gamma - \frac{f(\gamma)}{f'(\gamma)} - \frac{f(x_0)}{f'(\gamma)} - \frac{f(x_0 + x_1)}{f'(\gamma)}.$$
$$((2.16))$$

Using this relation, we can suggest the following three-step iterative method for solving nonlinear equation (2.1).

Algorithm 3 *2.3. For a given x_0, compute the approximate solution x_{n+1} by the*

iterative schemes:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0,$$

$$z_n = -\frac{f(y_n)}{f'(x_n)},$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(y_n)}{f'(x_n)} - \frac{f(y_n + z_n)}{f'(x_n)}, \quad n = 0, 1, 2, ...$$

Algorithm 2.3 is called the three-step iterative method for solving nonlinear equation (2.1). This algorithm was obtained by Noor and Noor [15]. It has been shown in [15] that Algorithm (2.3) has fourth order convergence.

Finally use (2.11) with $n = 3$, we conclude that

$$x_3 = N(x_0 + x_1 + x_2) = \gamma - \frac{f(\gamma)}{f'(\gamma)} - \frac{f(x_0)}{f'(\gamma)} - \frac{f(x_0 + x_1)}{f'(\gamma)} - \frac{f(x_0 + x_1 + x_2)}{f'(\gamma)}.$$
$$((2.17))$$

Therefore, by using (2.13) and (2.17) with $n = 3$, we get

$$x \approx X_3 = x_0 + x_1 + x_2 + x_3 = c + N(x_0) + N(x_0 + x_1) + N(x_0 + x_1 + x_2)$$
$$((2.18))$$

$$= \gamma - \frac{f(\gamma)}{f'(\gamma)} - \frac{f(x_0)}{f'(\gamma)} - \frac{f(x_0 + x_1)}{f'(\gamma)} - \frac{f(x_0 + x_1 + x_2)}{f'(\gamma)}.$$

Using this, we can suggest and analyze the following four-step iterative method for solving nonlinear equation (2.1), and this is the main motivation of this paper.

Algorithm 4 *2.4. For a given x_0, compute the approximate solution x_{n+1} by the iterative schemes:*

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad f'(x_n) \neq 0, \tag{((2.19))}$$

$$z_n = -\frac{f(y_n)}{f'(x_n)}, \tag{((2.20))}$$

$$k_n = -\frac{f(y_n + z_n)}{f'(x_n)}, \tag{((2.21))}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(y_n)}{f'(x_n)} - \frac{f(y_n + z_n)}{f'(x_n)} - \frac{f(y_n + z_n + k_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots \tag{((2.22))}$$

This algorithm is called the four-step iterative method for solving the nonlinear equation (2.1). In the next section, we will show that Algorithm (2.4) has fifth order convergence. It is worth to mention here that the new iterative method is free from the second or higher derivatives. Moreover, per any iteration of the new iterative method requires four function and first derivative evaluations. Algorithm (2.4) can be re-considered as a predictor-corrector iterative method as follows:

Algorithm 5 *2.5. For a given x_0, compute the approximate solution x_{n+1} by the*

iterative schemes:
Predictor steps:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad f'(x_n) \neq 0,$$

$$z_n = -\frac{f(y_n)}{f'(x_n)},$$

$$k_n = -\frac{f(y_n + z_n)}{f'(x_n)},$$

Corrector step:

$$x_{n+1} = y_n + z_n + k_n - \frac{f(y_n + z_n + k_n)}{f'(x_n)}, \quad n = 0, 1, 2, ... \qquad ((2.23))$$

3. Convergence Analysis

In this paper, we have proposed a new four-step iterative method (Algorithm 2.4) for solving nonlinear equation (2.1). In this section, we study the convergence analysis of Algorithm 2.4.

Theorem 3.1: Let α be a simple zero of a sufficiently differentiable function $f : I \to R$ for an open interval I. Then, the new four-step iterative method defined in Algorithm 2.4 has the fifth-order convergence and satisfies the following error equation

$$e_{n+1} = 8c_2^4 e_n^5 + O\left(e_n^6\right),$$

where $e_n = x_n - \alpha$ and $c_2 = \frac{f^{(2)}(\alpha)}{2f'(\alpha)}$.

Proof: Let α be a simple zero of f and $e_n = x_n - \alpha$. By expanding $f(x_n)$ and $f'(x_n)$ about α, we obtain

$$f(x_n) = f'(\alpha)\left[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + ...\right] \qquad ((3.1))$$

$$f^{\prime}(x_n) = f^{\prime}(\alpha)\left[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + ...\right] \qquad ((3.2))$$

where $c_k = \frac{f^{(k)}(\alpha)}{k! f^{\prime}(\alpha)}$, for $k = 1, 2, 3, ...$

From (3.1) and (3.2), we have

$$\frac{f(x_n)}{f^{\prime}(x_n)} = e_n - c_2 e_n^2 + 2\left(c_2^2 - c_3\right) e_n^3 + \left(7c_2 c_3 - 3c_4 - 4c_2^3\right) e_n^4 + ... \qquad ((3.3))$$

From (2.19) and (3.3), we have

$$y_n = x_n - \frac{f(x_n)}{f^{\prime}(x_n)} = \alpha + c_2 e_n^2 + 2\left(c_3 - c_2^2\right) e_n^3 + \left(3c_4 + 4c_2^3 - 7c_2 c_3\right) e_n^4 + ...$$
$$((3.4))$$

By expanding $f(y_n)$ about α and using equation (3.4), we have

$$f(y_n) = f^{\prime}(\alpha)\left[c_2 e_n^2 + \left(2c_3 - 2c_2^2\right) e_n^3 + \left(3c_4 + 5c_2^3 - 7c_2 c_3\right) e_n^4 + ...\right] \qquad ((3.5))$$

Now from (2.20) , (3.2) and (3.5), we have

$$z_n = -\frac{f(y_n)}{f^{\prime}(x_n)} = -c_2 e_n^2 - 2\left(c_3 - 2c_2^2\right) e_n^3 - \left(3c_4 + 13c_2^3 - 14c_2 c_3\right) e_n^4 + ... \qquad ((3.6))$$

Now by expanding $f(y_n + z_n)$ about α,we get

$$f(y_n + z_n) = f^{\prime}(\alpha)\left[2c_2^2 e_n^3 + \left(7c_2 c_3 - 9c_2^3\right) e_n^4 + ...\right] \qquad ((3.7))$$

From (2.21), (3.2) and (3.7), we have

$$k_n = -\frac{f(y_n + z_n)}{f^{\prime}(x_n)} = -2c_2^2 e_n^3 + \left(13c_2^3 - 7c_2 c_3\right) e_n^4 + ... \qquad ((3.8))$$

Now by expanding $f(y_n + z_n + k_n)$ about α, we get

$$f(y_n + z_n + k_n) = f^{\prime}(\alpha)\left[\left(17c_2^3 - 13c_2^2\right) e_n^4 + ...\right] \qquad ((3.9))$$

From (3.2) and (3.9), we have

$$-\frac{f(y_n + z_n + k_n)}{f^{\prime}(x_n)} = -4c_2^3 e_n^4 + 8c_2^4 e_n^5 + ... \qquad ((3.10))$$

From (2.22), (3.4), (3.6) and (3.10), we have

$$e_{n+1} = x_{n+1} - \alpha = 8c_2^4 e_n^5 + ... \qquad ((3.11))$$

This shows that Algorithm (2.4) has fifth order convergence.

4. Numerical Examples

In this section, we present some numerical examples to illustrate the efficiency and the accuracy of the new developed iterative method in this paper. These examples are chosen from Chun[5]. We compare our results obtained in this paper (Algorithm 2.4) with Newton's method (NM), the method of Abbasbandy[1] (AM), the method of Homeier [7] (HM), and the methods of Chun [5] (CM2 is referred to method 10 in Chun [5] with fourth-order convergence) and (CM3 is referred to method 11 in Chun [5] with fifth-order convergence), the method of Noor and Noor [15] (NR), the comparisons are given in Tables (1) and (2).

All computations were done using MAPLE using 64 digit floating-point arithmetic. The follwoing stopping criteria is used for computer programs:
(i) $|x_{n+1} - x_n| < \epsilon$.
(ii) $|f(x_{n+1})| < \epsilon$.
We used $\epsilon = 10^{-15}$ and the numerical test examples are:

$$f_1(x) = \sin^2 x - x^2 + 1,$$

$$f_2(x) = x^2 - e^x - 3x + 2,$$

$$f_3(x) = \cos x - x,$$

$$f_4(x) = (x-1)^3 - 1,$$

$$f_5(x) = x^3 - 10,$$

$$f_6(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5,$$

$$f_7(x) = e^{x^2 + 7x - 30} - 1.$$

As for the convergence criteria, it was required that the distance of two consecutive approximations δ for the zero was less than 10^{-15}. Also displayed are the number of iterations to approximate the zero (IT), the approximate zero x_n and the value of $f(x_n)$.

5. Conclusion

In this paper, we have developed a new iterative method for solving nonlinear equations and proved that the method has fifth-order convergence. The derivation of the method is based on the decomposition due to Noor [11] and Noor and Noor [16]. This new iterative method is free from the second or higher derivatives. Moreover, per any iteration of the new iterative method requires only four function and first derivative evaluations. This gives one of the advantages of this new method. From the comparison of the results in Tables (1) and (2) of this new iterative method with other methods reflects that the results we have obtained from this new iterative method are efficient and have high accuracy.

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Table (1)   Comparison of various iterative schemes and the Newton method

	IT	x_n	$f(x_n)$	δ
$f_1,\ x_0 = 1$				
NM	7	1.40449164821534122603508688178	$-1.04e-50$	$7.33e-26$
AM	5	1.40449164821534122603508688178	$-5.81e-55$	$1.39e-18$
HM	4	1.40449164821534122603508688178	$-5.4e-62$	$7.92e-21$
CM2	5	1.40449164821534122603508688178	$-2.0e-63$	$1.31e-17$
CM3	4	1.40449164821534122603508688178	$-2.0e-63$	$1.41e-22$
NR	4	1.40449164821534122603508688178	$-1.3e-40$	$5.4e-41$
Algorithm 2.4	5	1.40449164821534122603508688178	$-5.76e-29$	$2.32e-29$
$f_2,\ x_2 = 2$				
NM	6	0.25753028543986076045536 73094	$2.93e-55$	$9.1e-28$
AM	5	0.25753028543986076045536 73094	$1.0e-63$	$1.45e-26$
HM	5	0.25753028543986076045536 73094	0	$9.33e-43$
CM2	4	0.25753028543986076045536 73094	$1.0e-63$	$9.46e-29$
CM3	4	0.25753028543986076045536 73094	0	$1.61e-53$
NR	4	0.25753028543986076045536 73094	$9.3e-28$	$2.5e-28$
Algorithm 2.4	3	0.25753028543986076045536 73094	$-8.95e-54$	$2.37e-54$
$f_3,\ x_0 = 1.7$				
NM	5	0.73908513321516064165531208767	$-2.03e-32$	$2.34e-16$
AM	4	0.73908513321516064165531208767	$-7.14e-47$	$8.6e-16$
HM	4	0.73908513321516064165531208767	$-5.02e-59$	$9.64e-20$
CM2	4	0.73908513321516064165531208767	0	$1.86e-53$
CM3	3	0.73908513321516064165531208767	0	$5.49e-20$
NR	3	0.73908513321516064165531208767	$-3.7e-54$	$2.4e-54$
Algorithm 2.4	3	0.73908513321516064165531208767	0	0
$f_4,\ x_0 = 3.5$				
NM	8	2	$2.06e-42$	$8.28e-22$
AM	5	2	0	$4.3e-22$
HM	5	2	0	$1.46e-24$
CM2	5	2	0	$2.47e-24$

Table (2)   Comparison of various iterative schemes and the Newton method

	IT	x_n	$f(x_n)$	δ
f_4, $x_0 = 3.5$				
CM3	5	2	0	$4.76e-50$
NR	5	2	$1.75e-24$	$5.8e-24$
Algorithm 2.4	4	2	$6.26e-57$	$2.09e-57$
f_5, $x_0 = 1.5$				
NM	7	2.15443469003188372175929356665	$2.06e-54$	$5.64e-28$
AM	5	2.15443469003188372175929356665	$-5.0e-63$	$1.18e-25$
HM	4	2.15443469003188372175929356665	$-5.0e-63$	$9.8e-23$
CM2	5	2.15443469003188372175929356665	$-5.0e-63$	$1.57e-22$
CM3	5	2.15443469003188372175929356665	$-5.0e-63$	$1.2e-21$
NR	4	2.15443469003188372175929356665	$8.1e-45$	$5.8e-46$
Algorithm 2.4	4	2.15443469003188372175929356665	$-5.19e-47$	$3.73e-48$
f_6, $x_0 = -2$				
NM	9	$-1.20764782713091892700094167584$	$-2.27e-40$	$2.73e-21$
AM	6	$-1.20764782713091892700094167584$	$-4.0e-63$	$4.35e-45$
HM	6	$-1.20764782713091892700094167584$	$-4.0e-63$	$2.57e-32$
CM2	6	$-1.20764782713091892700094167584$	$-4.0e-63$	$2.15e-36$
CM3	5	$-1.20764782713091892700094167584$	$1.3e-62$	$1.1e-18$
NR	5	$-1.20764782713091892700094167584$	$-1.0e-37$	$4.9e-39$
Algorithm 2.4	5	$-1.20764782713091892700094167584$	$-4.0e-63$	0
f_7, $x_0 = 3.5$				
NM	13	3	$1.52e-47$	$4.2e-25$
AM	7	3	$-4.33e-48$	$2.25e-17$
HM	8	3	$2.0e-62$	$2.43e-33$
CM2	8	3	$2.0e-62$	$2.12e-23$
CM3	7	3	$-2.0e-62$	$3.36e-16$
NR	7	3	$5.0e-25$	$3.8e-26$
Algorithm 2.4	7	3	0	0