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On Comparison of the Tail Index of Heavy Tail Distributions Using Pitman's Measure of Closeness

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Abstract

The objective of this paper is to compare estimators which are a function of sample averages (a modification of Fan's estimators) based on different sample sizes for all symmetric stable distributions with exponent α $(0 < \alpha \leq 2)$, according to the Pitman's measure of closeness criterion.

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1 Introduction

Statistical modeling is usually based on special assumptions on the distribution of the parent population. The well known assumption is normality. But in many situations we come across in the real world, the normality is not an appropriate model. It has been found that in many fields of telecommunications, hydrology, biology and sociology; the normal distributions can not be used to describe the data set. Since, empirical studies show so many outliers. It is also known that the fluctuation of Stock returns, were originally supposed to follow a normal distribution or mixtures of normal distributions can not be explained by a normal white noise and the underlying distribution must posses a much fatter tail than a normal distribution.

Mandelbrot and Taylor (1967) and Fama (1965) suggested to use stable distributions with tail index $0 < \alpha < 2$ to model stock prices. The α -stable distributions are a special type of heavy tailed distributions with tail index α, where $0 < \alpha < 2$. The stable distributed samples are only rarely observed in practice and it is more suitable to model the observations with distributions in the domain of attraction of some stable law. Tail indices persist under convolutions and stable distributions appear limit laws of properly normalized means. For this reason, estimation of tail indices of heavy tail distributions has been an important field of research. Fama and Roll (1968), Press (1972), Zolotarev (1986), Hill (1975), Pickands (1975), De Haan and Resnick (1980), Csorgo et al. (1985), De Haan and Pereira (1999), Meerschaert and Scheffler (1998) and Fan (2004) have proposed different methods of estimating the tail index. None of these estimators is perfect. From different point of views, each one has some disadvantages. Indeed, they can hardly be used simply and effectively in practice, see Fan (2004). An easy-to-calculate estimate is given by Fan (2004).

In this paper we introduce a modification of the Fan's estimators and then will compare this estimator based on different sample sizes for all symmetric stable distributions with exponent $\alpha(0 < \alpha < 2)$ according to the Pitman's measure of closeness criterion. Our findings indicate that for $0 < \alpha < 1$, $\frac{\log|\overline{X}_{n+m}|}{\log n} + 1$ is not necessarily Pitman-closer than $\frac{\log|\overline{X}_n|}{\log n} + 1$ for all $n > 1$ and $m > 1$, but for $1 < \alpha < 2$, $\frac{\log|\overline{X}_{n+m}|}{\log n} + 1$ is Pitman-closer than $\frac{\log|\overline{X}_n|}{\log n} + 1$ for all $n > 1$ and $m > 1$.

2 Main Results

First, we begin with the notion of Pitman's measure of closeness (PMC in short). The PMC criterion (Pitman, 1937) has received a great deal of attention in recent years and has existed a criterion for more than sixty years. It is an alternative to mean square error in comparison of estimators. According to the PMC criterion, rival estimators are usually compared two at a time.

Let $\theta(\in \Theta)$ be the parameter of interest. For estimating θ consider two rival estimators $\hat{\theta}_1$ and $\hat{\theta}_2$. The PMC of $\hat{\theta}_1$ relative to $\hat{\theta}_2$ is the probability that the estimator $\hat{\theta}_1$ is closer than the estimator $\hat{\theta}_2$ to the parameter θ . That is,

$$P(\hat{\theta}_1, \hat{\theta}_2|\theta) = \Pr(|\hat{\theta}_1 - \theta| < |\hat{\theta}_2 - \theta|).$$

The estimator $\hat{\theta}_1$ is Pitman - closer than $\hat{\theta}_2$ whenever

$$P(\hat{\theta}_1, \hat{\theta}_2|\theta) \geq 1/2, \forall \theta \in \Theta,$$

with strict inequality holding for at least one θ .

Let us start with the sequence of i.i.d. strictly α - stable random variables

$X_1, X_2, ..., X_n, ...$ The sum-preserving property of stable laws says

$$\frac{X_1 + X_2 + ... + X_n}{n^{1/\alpha}} \stackrel{d}{=} X_1$$

This Leads to

$$\log n \left[\frac{\log |\sum_{i=1}^{n} X_i|}{\log n} - \frac{1}{\alpha} \right] \stackrel{d}{=} \log |X_1| \, .$$

Let

$$\hat{\alpha}_n^{-1} := \frac{\log |\sum_{i=1}^{n} X_i|}{\log n} = \frac{\log |\overline{X}_n|}{\log n} + 1, \tag{2.1}$$

then $\hat{\alpha}_n^{-1} \longrightarrow \alpha^{-1}$ in probability as $n \to \infty$. This estimator is introduced by Fan (2004). In the following we compare sample averages of a modification of this estimator based on different sample sizes for all symmetric stable distributions with tail index α ($0 < \alpha < 2$) according to PMC. Our approach is based on the work of Bose, Datta and Ghosh (1993) and heavily based the following results. They showed that if $X_1, ..., X_n$ are independent and identically distributed, each symmetric about μ, X_1 having a stable density with exponent α, then

$$P_\theta(|\overline{X}_{n+m} - \mu| \ \leq \left|\overline{X}_n - \mu\right|) = 2 \int_{\circ}^{\infty} [F_{X_1}(u^{\frac{\alpha-1}{\alpha}} x) - F_{X_1}(-\frac{u+2}{u} u^{\frac{\alpha-1}{\alpha}} x] dF_{X_1}(x), \tag{2.2}$$

where $u = \frac{m}{n}$.

We have used this result for expressing our main result. First we introduce a modification of Fan's estimators by $\tilde{\alpha}_{n+m}^{-1} := \frac{\log |\overline{X}_{n+m}|}{\log n} + 1$.

Theorem 2.1 Let $X_1, ..., X_n$ be independent and identically distributed, each symmetric about 0, X_1 having a stable density with exponent α . Then
(i) If $0 < \alpha < 1$, then $\tilde{\alpha}_{n+m}^{-1} := \frac{\log |\overline{X}_{n+m}|}{\log n} + 1$ is Pitman-closer to $\frac{1}{\alpha}$ than $\hat{\alpha}_n^{-1} = \frac{\log |\overline{X}_n|}{\log n} + 1$ for $m \leq n$.
(ii) If $1 < \alpha < 2$, then $\tilde{\alpha}_{n+m}^{-1} = \frac{\log |\overline{X}_{n+m}|}{\log n} + 1$ is pitman closer to $\frac{1}{\alpha}$ than $\hat{\alpha}_n^{-1} = \frac{\log |\overline{X}_n|}{\log n} + 1$ for all $n > 1$ and $m > 1$.

Proof.

(i) First consider the case $0 < \alpha < 1$. It follows now from (2.2) that if $m \leq n$ so that $u \leq 1$, using the symmetry of F_{X_1} ,

$$P(|\tilde{\alpha}_{n+m}^{-1} - \frac{1}{\alpha}| \quad \leq \quad |\hat{\alpha}_n^{-1} - \frac{1}{\alpha}|) = P(|\frac{\log |\overline{X}_{n+m}|}{\log n} + 1 - \frac{1}{\alpha}| \leq |\frac{\log |\overline{X}_n|}{\log n} + 1 - \frac{1}{\alpha}|)$$

$$
\begin{aligned}
&\geq P\left(\frac{\log|\overline{X}_{n+m}|}{\log n} \leq \frac{\log|\overline{X}_n|}{\log n}\right)\\
&\geq P(\log_n^{|\overline{X}_{n+m}|} \leq \log_n^{|\overline{X}_n|})\\
&= P(|\overline{X_{n+m}}| \leq |\overline{X}_n|)\\
&\geq 2\int_\circ^\infty [2F_{x_1}(u^{1-\frac{1}{\alpha}}x) - 1]dF_{x_1}(x)\\
&\geq 2\int_\circ^\infty [2F_{x_1}(x) - 1]dF_{x_1}(x)\\
&= \frac{1}{2}.
\end{aligned}
$$

So that $\tilde{\alpha}_{n+m}^{-1} = \frac{\log|\overline{X}_{n+m}|}{\log n} + 1$ $(m \leq n)$ is Pitman-closer than $\hat{\alpha}_n^{-1} = \frac{\log|\overline{X}_n|}{\log n} +$ 1 .

(ii) Next consider $1 < \alpha < 2$. If $m \geq n$, i.e. if $u \geq 1$.

$$
\begin{aligned}
P(|\tilde{\alpha}_{n+m}^{-1} - \frac{1}{\alpha}| &\leq |\hat{\alpha}_n^{-1} - \frac{1}{\alpha}|) = P(|\frac{\log|\overline{X}_{n+m}|}{\log n} + 1 - \frac{1}{\alpha}| \leq |\frac{\log|\overline{X}_n|}{\log n} + 1 - \frac{1}{\alpha}|)\\
&\geq P\left(\frac{\log|\overline{X}_{n+m}|}{\log n} \leq \frac{\log|\overline{X}_n|}{\log n}\right)\\
&= P(\log_n^{|\overline{X}_{n+m}|} \leq \log_n^{|\overline{X}_n|})\\
&= P(|\overline{X}_{n+m}| \leq |\overline{X_n}|)\\
&\geq 2\int_0^\infty [2F_{X_1}(u^{1-\frac{1}{\alpha}}x) - 1]dF_{X_1}(x)\\
&\geq 2\int_0^\infty [2F_{X_1}(x) - 1]dF_{X_1}(x)\\
&= \frac{1}{2}
\end{aligned}
$$

i.e. $\tilde{\alpha}_{n+m}^{-1} = \frac{\log|\overline{X}_{n+m}|}{\log n} + 1$ is pitman closer than $\hat{\alpha}_n^{-1} = \frac{\log|\overline{X}_n|}{\log n} + 1$. If $m < n$, i.e. $u < 1$, using the symmetry of F_{X_1}, it follows form 2.2 that

$$
\begin{aligned}
P(|\tilde{\alpha}_{n+m}^{-1} - \frac{1}{\alpha}| &\leq |\hat{\alpha}_n^{-1} - \frac{1}{\alpha}) = P(|\frac{\log|\overline{X}_{n+m}|}{\log n} + 1 - \frac{1}{\alpha}| \leq |\frac{\log|\overline{X}_n|}{\log n} + 1 - \frac{1}{\alpha}|)\\
&\geq P(|\overline{X_{n+m}}| \leq |\overline{X}_n|)\\
&= 2\int_\circ^\infty [F_{X_1}(u^{1-\frac{1}{\alpha}}x) + F_{X_1}(\frac{u+2}{u}u^{1-\frac{1}{\alpha}}x) - 1]dF_{X_1}(x)\\
&= 2\int_\circ^\infty [2F_{X_1}(x) - 1]dF_{X_1}(x) + 2\int_0^\infty [F_{X_1}(\frac{u+2}{u}u^{1-\frac{1}{\alpha}}x)\\
&\quad -F_{X_1}(x)]dF_{X_1}(x) - 2\int_\circ^\infty [F_{X_1}(x) - F_{X_1}(u^{1-\frac{1}{\alpha}}x)]dF_{X_1}(x)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2} + \frac{1}{2}\int_{\circ}^{\infty}[G(\frac{u+2}{u}u^{1-\frac{1}{\alpha}}x) - G(x)]\ dG(x) \\
&\quad -\frac{1}{2}\int_{0}^{\infty}[G(x) - G(u^{1-\frac{1}{\alpha}}x)]\ dG(x) \\
&= \frac{1}{2} + \frac{1}{2}P(W_2 \leq W_1 \leq \frac{u+2}{u}u^{1-\frac{1}{\alpha}}W_2) \\
&\quad -\frac{1}{2}P(W_1 u^{1-\frac{1}{\alpha}} \leq W_2 \leq W_1) \\
&\geq \frac{1}{2} + \frac{1}{2}P(W_2 \leq W_1 \leq W_2 u^{-1/\alpha}) - \frac{1}{2}P(W_2 \leq W_1 \leq W_2 u^{-1+1/\alpha}) \\
&= \frac{1}{2} + \frac{1}{2}P(W_2 u^{-1+1/\alpha} \leq W_1 \leq W_2 u^{-1/\alpha}) \\
&\geq \frac{1}{2},
\end{aligned}
$$

where W_1, W_2 are two i.i.d non-negative random variables having common distribution function $G(x) = 2F_{X_1}(x) - 1, \quad 0 \leq x < \infty$.

Thus, for $1 < \alpha < 2$, $\tilde{\alpha}_{n+m}^{-1} = \frac{\log|\overline{X}_{n+m}|}{\log n} + 1$ is Pitman-closer to $\frac{1}{\alpha}$ than $\hat{\alpha}_n^{-1} = \frac{\log|\overline{X}_n|}{\log n} + 1$ for all $n > 1$ and $m > 1$.

Remark 2.1 In case $0 < \alpha < 1$, it is not necessary true that $\tilde{\alpha}_{n+m}^{-1} = \frac{\log|\overline{X}_{n+m}|}{\log n} + 1$ is Pitman closer than $\hat{\alpha}_n^{-1} = \frac{\log|\overline{X}_n|}{\log n} + 1$ for all $m \geq 1$. For example, as $m \to \infty$, n fixed, $u \to \infty$ and so $u^{1-\frac{1}{\alpha}} \to 0$ since $0 < \alpha < 1$. Then

$$
P(|\frac{\log|\overline{X}_{n+m}|}{\log n} + 1 - \frac{1}{\alpha}| \leq |\frac{\log|\overline{X}_n|}{\log n} + 1 - \frac{1}{\alpha}|) \to 0\,, \text{as}\ \ m \to \infty,
$$

for fixed n .

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