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A Gradient Recovery-based a Posteriori Error Estimators for the Ciarlet - Raviart Formulation of the Second Biharmonic Equations

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Abstract

This paper proposes a posteriori error estimators of gradient recovery type for the Ciarlet-Raviart formulation of the second biharmonic equations. By the appropriate modification of weighted $Cle'ment$-type interpolation, we give the proper scaling of the gradient recovery leading to both lower and upper estimation on the non-uniform meshes. Moreover, it is proved that a posteriori error estimators is also asymptotically exact on the uniform meshes if the solution is smooth enough.

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1 Introduction

Adaptive finite element appromixation via a posteriori error estimators is not only among most important means of boosting the accuracy and efficiency of finite element discretization but also widely used in engineering numerical simulation. The literature in this area is huge. Most of the known a posteriori error estimators are either of residual type or only for the second order elliptic problems, see, for example, [1-5], however, there is not a paper both for the fourth order problem and by gradient recovery type of a posteriori error estimators.

In this paper, we discuss a posteriori error estimators of gradient recovery type for the second biharmonic problem. Using the Ciarlet-Raviart mixed FEM, we can obtain and prove both lower and upper bounds for the discretization error on the non-uniform meshes. Moreover, it is proved that a posteriori error estimators is also asymptotically exact on the uniform meshes if the solution is smooth enough.

2 Preliminaries and the main results

In this paper, let $\Omega \subset R^2$ be an open bounded domain with a Lipschitz boundary $\partial\Omega$, we adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with the norm $\|\bullet\|_{W^{m,q}(\Omega)}$ (or $\|\bullet\|_{m,q,\Omega}$ as a simplification) and semi-norm $|\bullet|_{W^{m,q}(\Omega)}$ (or $|\bullet|_{m,q,\Omega}$). We set $W_0^{m,q}(\Omega) \equiv \{\omega \in W^{m,q}(\Omega) : \omega|_{\partial\Omega} = 0\}$. We denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ with the norm $\|\bullet\|_{m,\Omega}$ and semi-norm $|\bullet|_{m,\Omega}$. We denote the measure of the domain D by $|D|$. In addition, C denotes a general positive constant independent of the mesh size h, which may take different values in different occurrences.

2.1 Review of the Ciarlet-Raviart mixed element scheme for the second biharmonic equation.

Consider the second biharmonic problem (the model for the deformation of simply supported thin elastic plates)

$$\begin{cases} \Delta^2\phi = f \ , \quad \text{in } \Omega, \\ \phi = \Delta\phi = 0 \ , \quad \text{on } \Gamma = \partial\Omega, \end{cases} \qquad (2.1)$$

where the domain Ω is a convex polygon in R^2.

It was proved that for any $f \in L^2(\Omega)$ the weak solution of the second biharmonic problem on a rectangle satisfies $\phi \in H^4(\Omega)$ (see [2]).

With $u = -\Delta\phi$, the second biharmonic equation can be decomposed into two poisson equations with Dirichlet boundary conditions.

$$\begin{cases} -\Delta u = f, \ \text{in } \Omega, \\ u + \Delta\phi = 0 \ , \ \text{in } \Omega, \\ \phi = \Delta\phi = 0 \ , \ \text{on } \Gamma = \partial\Omega, \end{cases} \qquad (2.2)$$

From the analysis as the above, we can see that the second biharmonic equation have not only the better regularity of the solution but also the simpler structual characteristic. That makes its studying much easier.

Let $(u, v) = \int uv$, and $a(u,v) = \int_\Omega \nabla u \nabla v$, the following variational problem corresponding to (2.1):

Find $(u, \phi) \in H_0^1(\Omega) \times H_0^1(\Omega)$, such that

$$\begin{cases} (u,v) - a(v,\phi) = 0, \forall v \in H_0^1(\Omega); \\ a(u,\varphi) + (f,\varphi) = 0, \forall \varphi \in H_0^1(\Omega). \end{cases} \qquad (2.3)$$

Let $T^h = \{K\}$ be a quasi-uniform triangular or rectangluar partition of $\Omega \subset R^2$ with $h = \max_{K \in T^h} h_K$ the maximum diameter of the partition. Set $X = \left\{ v \in C^0(\overline{\Omega}) : v|_K \in P_m, \forall K \in T^h \right\}, \ (m \geq 1)$, and $M^h = X^h = X \cap H_0^1(\Omega)$.

Condsider the discerete variational problem used to approximate (2.3):

Find $(u_h, \phi_h) \in X^h \times M^h$, such that

$$\begin{cases}(u_h, v) - a(v, \phi_h) = 0, \forall \nabla v \in X^h, \\ a(u_h, \varphi) + (f, \varphi) = 0, \forall \varphi \in M^h.\end{cases} \quad (2.4)$$

2.2 Weighted $Cle'ment - type$ interpolation

A posteriori error estimators are based on the so-called weighted $Cle'ment$ type interpolation on the finite space M^h (or X^h), First of all, we need introduce this interpolation.

Let $\partial^2 T^h$ be the set of nodes, σ_z be the basic function of M^h (or X^h) on $z \in \partial^2 T^h$, $\omega_z = \text{supp}\, \sigma_z(x)$, where $\Lambda = \partial^2 T^h \setminus \partial\Omega$.

The weighted $Cle'ment - type$ interpolation is defined by:

$$\pi v = \sum_{z \in (\partial^2 T^h)} v_z \sigma_z, \forall v \in H_0^1(\Omega), \quad \text{and} \quad v_z = \frac{(\sigma_z, v)}{(\sigma_z, 1)}, \forall z \in \Lambda .$$

2.3 Gradient recovery on non-uniform meshes

In order to construct a posteriori error estimate on irregular meshes, we need a gradient recovery operator Gv_h on M^h (or X^h) which stafies

$$Gv_h = \sum_{z \in \partial^2 T^h} Gv_h(z)\phi_z , \quad Gv_h(z) = \sum_{j=1}^{J_z} \alpha_z^j (\nabla v_h)_{K_z^j}, \forall v_h \in M^h(X^h).;$$

where $$\bigcup_{j=1}^{J_z} \overline{K_z^j} = \overline{\omega_z}, \sum_{j=1}^{J_z} \alpha_z^j = 1, 0 \le \alpha_z^j \le 1, j = 1, \cdots, J_z .$$

Using the gradient recovery operator, we can then construct a posteriori estimators as follows:

$$\eta^2 = \sum_K (\|Gu_h - \nabla u_h\|_{0,K}^2 + \|G\phi_h - \nabla \phi_h\|_{0,K}^2) .$$

2.4 The main results

For the above a posteriori error estimators of gradient recovery type, we can derive both efficiency (upper bound) and reliability (lower bound) of a posteriori error estimators on general meshes for the second biharmonic equation, that is as follow:

Theorem 2.1 Suppose (u,ϕ) and (u_h,ϕ_h) are solutions (2.3) and (2.4), respectively. Then：

$$c\eta^2 - C_2\varepsilon_2^2 \le \left|u-u_h\right|_{1,\Omega}^2 + \left|\phi-\phi_h\right|_{1,\Omega}^2 \le C\eta^2 + C_1\varepsilon_1^2 ,$$

where

$$\eta^2 = \sum_K (\left\|Gu_h - \nabla u_h\right\|_{0,K}^2 + \left\|G\phi_h - \nabla\phi_h\right\|_{0,K}^2) ,$$

$$\varepsilon_1^2 = \sum_{z\in\Lambda} h_z^2 \int_{\omega_z} [(f-\overline{f_z})^2 + (u_h - \overline{(u_h)_z})^2 + (\Delta u_h - \overline{(\Delta u_h)_z})^2 + (\Delta\phi_h - \overline{(\Delta\phi_h)_z})^2] ,$$

$$\varepsilon_2^2 = \sum_K h_K^2 \int_K [(f-\overline{f})^2 + (u_h - \overline{u_h})^2] ,$$

With $$\overline{v_z} = \frac{\int_{\omega_z} v}{\int_{\omega_z} 1}, \quad \overline{v_K} = \frac{\int_K v}{\int_K 1}, \quad h_z = \max_{K\subset\omega_z} h_K .$$

3 Proof of the theorm

Lemma 3.1 Assume that is the weighted $Cle'ment-type$ interpolation of defined as the above. then, for all, there holds

$$\sum_{K\in T^h} \left\|h_K^{-1}(v-\pi v)\right\|_{0,K}^2 \le c\left|v\right|_{1,\Omega}^2 , \forall v\in H_0^1(\Omega)$$

$$\left|\pi v\right|_{1,\Omega}^2 \le c\left|v\right|_{1,\Omega}^2 , \forall v\in H_0^1(\Omega)$$

Furthermore, if $f\in L^2(\Omega), \forall v\in H_0^1(\Omega)$, then

$$\int_\Omega f(v-\pi v) \le C\left|v\right|_{1,\Omega} (\sum_{z\in\Lambda}\int_{\omega_z} h_z^2 \left|f-\overline{f_z}\right|^2)^{1/2} ,$$

Where $\overline{f_z} = \dfrac{\int_{\omega_z} f}{\int_{\omega_z} 1}$, $h_z = \max_{K \in \omega_z} h_K$.

The proof of this lemma can be found in [5].

Lemma 3.2 *Suppose* (u, ϕ) and (u_h, ϕ_h) are solutions of (2.3) and (2.4), respectively, Then:

$$c\eta_j^2 - C_2\varepsilon_2^2 \le \left|u - u_h\right|_{1,\Omega}^2 + \left|\phi - \phi_h\right|_{1,\Omega}^2 \le C\eta_j^2 + C_1\varepsilon_1^2 , \qquad (3.1)$$

Where l is the edge of the element, h_l is the size of l ,$[v]$ denote the jump of v on the edge, and

$$\eta_j^2 = \sum_{l \cap \partial\Omega = \varnothing} h_l \int_l [\frac{\partial u_h}{\partial n}]^2 + \sum_{l \cap \partial\Omega = \varnothing} h_l \int_l [\frac{\partial \phi_h}{\partial n}]^2$$

$$\varepsilon_1^2 = \sum_{z \in \Lambda} h_z^2 \int_{\omega_z} [(f - \overline{f_z})^2 + (u_h - \overline{(u_h)_z})^2 + (\Delta u_h - \overline{(\Delta u_h)_z})^2 + (\Delta\phi_h - \overline{(\Delta\phi_h)_z})^2],$$

$$\varepsilon_2^2 = \sum_K h_K^2 \int_K [(f - \overline{f})^2 + (u_h - \overline{u_h})^2],$$

with $\overline{v_z} = \dfrac{\int_{\omega_z} v}{\int_{\omega_z} 1}$, $\overline{v_k} = \dfrac{\int_K v}{\int_K 1}$, $h_z = \max_{K \subset \omega_z} h_K$, $h = \max_{K \in T^h} h_K$.

Proof（1）Proof of the right inequation.

Let $e_u = u - u_h$, $e_\phi = \phi - \phi_h$. It follows form（2.3）and (2.4) that

$$\begin{aligned}
&\left|u - u_h\right|_{1,\Omega}^2 + \left|\phi - \phi_h\right|_{1,\Omega}^2 \\
&= a(u - u_h, u - u_h) + a(\phi - \phi_h, \phi - \phi_h) \\
&= a(u - u_h, e_u) + a(e_\phi, \phi - \phi_h) \\
&= a(u - u_h, e_u - \pi e_u) + a(e_\phi - \pi e_\phi, \phi - \phi_h)
\end{aligned}$$

$$= (-f, e_u - \pi e_u) + (\Delta u_h, e_u - \pi e_u) - \sum_{l \cap \partial\Omega = \varnothing} \int_l [\frac{\partial u_h}{\partial n}](e_u - \pi e_u)$$

$$+(u, e_\phi - \pi e_\phi) + (\Delta \phi_h, e_\phi - \pi e_\phi) - \sum_{l \cap \partial\Omega = \varnothing} \int_l [\frac{\partial \phi_h}{\partial n}](e_\phi - \pi e_\phi), \quad (3.2)$$

Where $\pi e_u \in X^h$ and $\pi e_\phi \in M^h$ are the weighted $Cle'ment-type$ interpolations of e_u and e_ϕ defined as the above.

By lemma3.1 and the Young inequation $|ab| \le \lambda a^2 + \frac{b^2}{\lambda}, \forall \lambda > 0$,then

$$\left|(\Delta u_h, e_u - \pi e_u)\right| \le C \sum_{z \in \Lambda} h_z^2 \int_{\omega_z} (\Delta u_h - \overline{(\Delta u_h)_z})^2 + \frac{1}{8} |e_u|_{1,\Omega}^2, \forall u \in H_0^1(\Omega), \quad (3.3)$$

$$\left|(\Delta \phi_h, e_\phi - \pi e_\phi)\right| \le C \sum_{z \in \Lambda} h_z^2 \int_{\omega_z} (\Delta \phi_h - \overline{(\Delta \phi_h)_z})^2 + \frac{1}{8} |e_\phi|_{1,\Omega}^2, \forall \phi \in H_0^1(\Omega), \quad (3.4)$$

$$\left|(-f, e_u - \pi e_u)\right| \le C \sum_{z \in \Lambda} h_z^2 \int_{\omega_z} (f - \overline{f_z})^2 + \frac{1}{16} |e_u|_{1,\Omega}^2, \forall u \in H_0^1(\Omega), \quad (3.5)$$

$$\left|(u, e_\phi - \pi e_\phi)\right| \le C \sum_{z \in \Lambda} h_z^2 \int_{\omega_z} (u - \overline{u_z})^2 + \frac{1}{8} |e_\phi|_{1,\Omega}^2, \forall \phi \in H_0^1(\Omega), \quad (3.6)$$

Note that the functions of u and $\overline{u_z}$ is not computable in (3.6) , it can not be the right terms of the a posteriori error estimators.
It is well known that

$$|a-d| \le |a-b| + |b-c| + |c-d|, \forall a,b,c,d \in R,$$

and $\qquad (|a| + |b| + |c|)^2 \le 3(a^2 + b^2 + c^2), \forall a,b,c \in R$.

it can be proved that

$$\left|u - \overline{u_z}\right|^2 \le (|u - u_h| + \left|u_h - \overline{(u_h)_z}\right| + \left|\overline{(u_h)_z} - \overline{u_z}\right|)^2$$

$$\le 3(|u - u_h|^2 + \left|u_h - \overline{(u_h)_z}\right|^2 + \left|\overline{(u_h)_z} - \overline{u_z}\right|^2)$$

$$\sum_{z\in\Lambda}\int_{\omega_z} h_z^2 \left|u-\overline{u_z}\right|^2 \le 3\sum_{z\in\Lambda}\int_{\omega_z} h_z^2(\left|u-u_h\right|^2+\left|u_h-\overline{(u_h)_z}\right|^2+\left|\overline{(u_h)_z}-\overline{u_z}\right|^2) \tag{3.7}$$

Note that $\overline{u_z}=\frac{\int_{\omega_z} u}{|\omega_z|}$ and $\overline{(u_h)_z}=\frac{\int_{\omega_z} u_h}{|\omega_z|}$, it follows from swchwarz inequality that

$$\left|\overline{u_z}-\overline{(u_h)_z}\right|^2=\left|\frac{\int_{\omega_z}(u-u_h)}{|\omega_z|}\right|^2 \le \frac{(\int_{\omega_z}|u-u_h|)^2}{|\omega_z|^2} \le \frac{\int_{\omega_z}|u-u_h|^2}{|\omega_z|},$$

Hence $\int_{\omega_z} h_z^2\left|\overline{u_z}-\overline{(u_h)_z}\right|^2 \le \int_{\omega_z} h_z^2\frac{\int_{\omega_z}|u-u_h|^2}{|\omega_z|}=h_z^2\int_{\omega_z}|u-u_h|^2=\int_{\omega_z} h_z^2|u-u_h|^2$,

$$\sum_{z\in\Lambda}\int_{\omega_z} h_z^2(\left|u-u_h\right|^2+\left|\overline{(u_h)_z}-\overline{u_z}\right|^2) \le 2\sum_{z\in\Lambda}\int_{\omega_z} h_z^2\left|u-u_h\right|^2 \le 2h^2\left\|u-u_h\right\|_{0,\Omega}^2, \tag{3.8}$$

Using the priori error estimate of the second elliptic problem, then,

$$\left\|u-u_h\right\|_{0,\Omega}^2 \le Ch^2\left|u-u_h\right|_{1,\Omega}^2, \tag{3.9}$$

From （3.6）－ (3.9)，then,

$$\left|(u,e_\phi-\pi e_\phi)\right| \le C\sum_{z\in\Lambda} h_z^2\int_{\omega_z}(u_h-\overline{(u_h)_z})^2+Ch^4\left|e_u\right|_{1,\Omega}^2+\frac{1}{8}\left|e_\phi\right|_{1,\Omega}^2$$

$$\le C\sum_{z\in\Lambda} h_z^2\int_{\omega_z}(u_h-\overline{(u_h)_z})^2+\frac{1}{16}\left|e_u\right|_{1,\Omega}^2+\frac{1}{8}\left|e_\phi\right|_{1,\Omega}^2, \forall\phi\in H_0^1(\Omega) \tag{3.10}$$

$$\left|\sum_{l\cap\partial\Omega=\varnothing}\int_l[\frac{\partial u_h}{\partial n}](e_u-\pi e_u)\right| \le C\sum_{l\cap\partial\Omega=\varnothing}(h_l\int_l[\frac{\partial u_h}{\partial n}]^2)^{1/2}h_l^{-1/2}\left\|e_u-\pi e_u\right\|_{0,l}$$

$$\le C\sum_{l\cap\partial\Omega=\varnothing}(h_l\int_l[\frac{\partial u_h}{\partial n}]^2)^{1/2}(h_l^{-1}\left\|e_u-\pi e_u\right\|_{0,K_l}+\left|e_u-\pi e_u\right|_{1,K_l})$$

$$\le C\sum_{l\cap\partial\Omega=\varnothing}(h_l\int_l[\frac{\partial u_h}{\partial n}]^2)^{1/2}\left|e_u\right|_{1,K_l}$$

$$\le C\sum_{l\cap\partial\Omega=\varnothing}h_l\int_l[\frac{\partial u_h}{\partial n}]^2+\frac{1}{4}\left|e_u\right|_{1,\Omega}^2, \tag{3.11}$$

Where K_l is an element such that $l \subset \overline{K_l}$. Similarly,

$$\left|\sum_{l \cap \partial\Omega = \varnothing} \int_l [\frac{\partial \phi_h}{\partial n}](e_\phi - \pi e_\phi)\right| \le C \sum_{l \cap \partial\Omega = \varnothing} h_l \int_l [\frac{\partial \phi_h}{\partial n}]^2 + \frac{1}{4}|e_\phi|_{1,\Omega}^2 \quad , \qquad (3.12)$$

Therefore, it follows from （3.2）－（3.5）and (3.10)－(3.12) that：

$$|u - u_h|_{1,\Omega}^2 + |\phi - \phi_h|_{1,\Omega}^2 \le C\eta_j^2 + C_1\varepsilon_1^2 ,$$

（2）Proof of the left inequation.

Using the Bubble function technique (see,e.g.,[1]), it can be proved that

$$\left|\sum_{l \cap \partial\Omega = \varnothing} h_l \int_l [\frac{\partial u_h}{\partial n}]^2\right| \le C(|u - u_h|_{1,\Omega}^2 + C\sum_K h_K^2 \int_K |f + \Delta u_h|^2 , \qquad (3.13)$$

$$\left|\sum_{l \cap \partial\Omega = \varnothing} h_l \int_l [\frac{\partial \phi_h}{\partial n}]^2\right| \le C|\phi - \phi_h|_{1,\Omega}^2 + C\sum_K h_K^2 \int_K |u + \Delta \phi_h|^2 , \qquad (3.14)$$

$$\sum_K h_K^2 \int_K |f + \Delta u_h|^2 \le C|u - u_h|_{1,\Omega}^2 + C\sum_K h_K^2 \int_K (f - \overline{f})^2 , \qquad (3.15)$$

$$\sum_K h_K^2 \int_K |u + \Delta \phi_h|^2 \le C|\phi - \phi_h|_{1,\Omega}^2 + C\sum_K h_K^2 \int_K (u - \overline{u})^2 , \qquad (3.16)$$

Note that the functions of u and $\overline{u}$ is not computable in (3.16) , it can not be the right term of the a posteriori error estimators, thus,

Similarly,

$$|u - \overline{u}|^2 \le (|u - u_h| + |u_h - \overline{u_h}| + |\overline{u_h} - \overline{u}|)^2$$

$$\le 3(|u - u_h|^2 + |u_h - \overline{u_h}|^2 + |\overline{u_h} - \overline{u}|^2) ,$$

So $$\sum_K \int_K h_K^2 |u - \overline{u}|^2 \le 3\sum_K \int_K h_K^2 (|u - u_h|^2 + |u_h - \overline{u_h}|^2 + |\overline{u_h} - \overline{u}|^2) , \qquad (3.17)$$

Note that $\overline{u} = \frac{\int_K u}{|K|}$ and $\overline{u_h} = \frac{\int_K u_h}{|K|}$， it follows from schwarz inequality that

$$|\overline{u} - \overline{u_h}|^2 = \left|\frac{\int_K (u - u_h)}{|K|}\right|^2 \le \frac{(\int_K |u - u_h|)^2}{|K|^2} \le \frac{\int_K |u - u_h|^2}{|K|} ,$$

$$\int_K h_K^2 \left|\overline{u}-\overline{u_h}\right|^2 \le \int_K h_K^2 \frac{\int_K |u-u_h|^2}{|K|} = h_K^2 \int_K |u-u_h|^2 = \int_K h_K^2 |u-u_h|^2 ,$$

$$\sum_K \int_K h_K^2 (|u-u_h|^2 + |\overline{u_h}-\overline{u}|^2) \le 2\sum_K \int_K h_K^2 |u-u_h|^2 \le 2h^2 \|u-u_h\|_{0,\Omega}^2 , \qquad (3.18)$$

Using the priori error estimators of the second elliptic problem, then,

$$\|u-u_h\|_{0,\Omega}^2 \le Ch^2 |u-u_h|_{1,\Omega}^2 , \qquad (3.19)$$

From（3.16）—(3.19) , then,

$$\sum_K h_K^2 \int_K |u+\Delta\phi_h|^2 \le C|\phi-\phi_h|_{1,\Omega}^2 + C\sum_K h_K^2 \int_K (u_h-\overline{u_h})^2 . \qquad (3.20)$$

From （3.13）—（3.15）and （3.20）, thus,

$$\eta_j^2 \le C(|u-u_h|_{1,\Omega}^2 + |\phi-\phi_h|_{1,\Omega}^2) + C_2\varepsilon_2^2 ,$$

From the step (1) and (2), the proof of lemma 3.2 is completed.

Lemma 3.3 *Under the conditions and the definitions of the lemma 3.2,* then

$$c\eta_j \le \eta \le C\eta_j .$$

The proof is similarly of the lemma 4.3 in [4]

The proof of the theorem 2.1 is the direct result of Lemma 3.2 and 3.3.

4 Discussions

In the proof of theorem in this paper, the end of the definition of $X^h = M^h$ is only to be easier for the proof. In fact, the same result can be obtained under the condition of $X^h \ne M^h$. In general, X^h and M^h are taken as the quadratic piecewise polynomial spaces. Similar as [4], using the result in this paper, we can prove that the a posteriori error estimators are asymptotically exact which the mesh is uniform and the solution is smooth enough. That is to say :

$$\eta^2 = |u-u_h|_{1,\Omega}^2 + |\phi-\phi_h|_{1,\Omega}^2 + o(h^2) .$$

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