

On Certain Subclass of Analytic Univalent Functions with Negative Coefficients

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Abstract

For $\lambda \geq 0$, $n \in \mathbb{N}_0$, in [3], the authors introduced the operator D_{λ}^n , which is a generalized Ruscheweyh derivatives operator. In this paper, some results on coefficient inequalities, growth and distortion theorems, closure theorems and extreme points for the class of analytic functions defined by aforementioned operator are obtained.

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1 Introduction

Let $\mathcal{A}(m)$ denote the class of functions of the form

$$f(z) = z + \sum_{k=m+1}^{\infty} a_k z^k, \quad (m \in \mathbb{N}_0 := \{1, 2, 3, \dots\}), \quad (1.1)$$

which are analytic and univalent in the unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also, let $\mathcal{T}(m)$ denote subclasses of \mathcal{A} consisting of functions f of the form

$$f(z) = z - \sum_{k=m+1}^{\infty} |a_k| z^k, \quad (m \in \mathbb{N}_0 := \{1, 2, 3, \dots\}). \quad (1.2)$$

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In [3], the authors introduced the following linear operator D_δ^λ :

$$D_\delta^\lambda f(z) = \frac{z}{(1-z)^{\lambda+1}} * D_\delta f(z)$$

where $D_\lambda f(z) = (1-\lambda)f(z) + \lambda z f'(z)$, $\delta > -1$, $\lambda \geq 0$ and $z \in \mathbb{U}$ which implies that

$$D_\lambda^n f(z) = \frac{z(z^{n-1} D_\lambda f(z))^n}{n!}, \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

It is clear that,

$$\begin{aligned} D_\lambda^0 f(z) &= (1-\lambda)f(z) + \lambda z f'(z) = D_\lambda f(z), \\ D_\delta^1 f(z) &= (1-\lambda)z f'(z) + \lambda z(z f'(z))', \quad \lambda \geq 0. \end{aligned}$$

Note that if f is given by (1.1), then we can write

$$D_\lambda^n f(z) = z + \sum_{k=m+1}^{\infty} [1 + \lambda(k-1)] C(n, k) a_k z^k,$$

where $\lambda \geq 0$, $m, n \in \mathbb{N}_0$ and

$$C(n, k) = \binom{k+n-1}{n} = \frac{\prod_{j=1}^{k-1} (j+n)}{(k-1)!}.$$

Let $K_\lambda^n(m, \alpha)$ the subclass of $\mathcal{A}(m)$ consisting of functions f which satisfy

$$\Re \left\{ \frac{z(D_\lambda^n f(z))'}{D_\lambda^n f(z)} \right\} > \alpha, \quad (z \in \mathbb{U}),$$

for $\lambda \geq 0$, $m, n \in \mathbb{N}_0$ and $0 \leq \alpha < 1$.

Further, we define the class $\mathcal{T}K_\lambda^n(m, \alpha)$ by:

$$\mathcal{T}K_\lambda^n(m, \alpha) = K_\lambda^n(m, \alpha) \cap \mathcal{T}(m)$$

for $\lambda \geq 0$, $m, n \in \mathbb{N}_0$ and $0 \leq \alpha < 1$.

Note that $\mathcal{T}K_\lambda^n(m, \alpha) \subset K_\lambda^n(m, \alpha)$. Also note that various subclasses of $K_\lambda^n(m, \alpha)$ and $\mathcal{T}K_\lambda^n(m, \alpha)$ has been studied by many authors by suitable choices of n , λ and m . For example

$$\begin{aligned} \mathcal{T}K_0^0(1, \alpha) &\equiv \mathcal{T}^*(\alpha), \quad \mathcal{T}K_0^1(1, \alpha) \equiv C(\alpha), \quad \mathcal{T}K_0^0(m, \alpha) \equiv \mathcal{T}_\alpha(m), \\ \mathcal{T}K_0^1(m, \alpha) &\equiv C_\alpha(m), \quad \mathcal{T}K_\lambda^0(m, \alpha) \equiv P(m, \lambda, \alpha) (0 \leq \lambda < 1), \\ \text{and } \mathcal{T}K_\lambda^1(m, \alpha) &\equiv C(m, \lambda, \alpha) (0 \leq \lambda < 1), \end{aligned}$$

etc. The classes $\mathcal{T}^*(\alpha)$ and $C(\alpha)$ were introduced and studied by Silverman [1], and the classes $\mathcal{T}_\alpha(m)$ and $C_\alpha(m)$ were studied by Chatterjea [7](see also Srivastava et al. [2]). Whereas the classes $P(m, \lambda, \alpha)$ and $C(m, \lambda, \alpha)$ were, respectively, studied by Altınatış [6] and Kamali and Aknulut [4]. Finally we note that when $\lambda = 0$ in class $K_\lambda^n(m, \alpha)$ we have the class $R_n(\alpha)$, was introduced and studied by Ahuja [5].

2 Coefficient Inequalities

In this section, we provide a necessary and sufficient condition for a function f analytic in \mathbb{U} to be in $\mathcal{TK}_\lambda^n(m, \alpha)$.

Theorem 2.1 *Let f be defined by (1.1). If $0 \leq \alpha < 1, \lambda > 0,$*

$$\sum_{k=m+1}^{\infty} (k - \alpha)[1 + \lambda(k - 1)]C(n, k)|a_k| \leq (1 - \alpha), \tag{2.1}$$

where $m, n \in \mathbb{N}_0,$ then $f \in K_\lambda^n(m, \alpha)$.

Proof. Assume that (2.1) holds true. It is sufficient to show that

$$\left| \frac{z(D_\lambda^n f(z))'}{D_\lambda^n f(z)} - 1 \right| \leq 1 - \alpha.$$

We have

$$\begin{aligned} \left| \frac{z(D_\lambda^n f(z))'}{D_\lambda^n f(z)} - 1 \right| &= \left| \frac{z(D_\lambda^n f(z))' - D_\lambda^n f(z)}{D_\lambda^n f(z)} \right| \\ &= \left| \frac{\sum_{k=m+1}^{\infty} (k - 1)[1 + \lambda(k - 1)]C(n, k)a_k z^k}{z + \sum_{k=m+1}^{\infty} [1 + \lambda(k - 1)]C(n, k)a_k z^k} \right| \\ &\leq \frac{\sum_{k=m+1}^{\infty} (k - 1)[1 + \lambda(k - 1)]C(n, k)|a_k|}{1 - \sum_{k=m+1}^{\infty} [1 + \lambda(k - 1)]C(n, k)|a_k|}. \end{aligned}$$

This last expression is bounded above by $1 - \alpha.$ We have

$$\sum_{k=m+1}^{\infty} (k - 1)[1 + \lambda(k - 1)]C(n, k)|a_k|$$

$$\leq (1 - \alpha) \left\{ 1 - \sum_{k=m+1}^{\infty} [1 + \lambda(k - 1)] C(n, k) |a_k| \right\},$$

which is equivalent to

$$\sum_{k=m+1}^{\infty} (k - \alpha) [1 + \lambda(k - 1)] C(n, k) |a_k| \leq (1 - \alpha) \quad \square$$

by (2.1). Hence $f \in K_{\lambda}^n(m, \alpha)$.

Theorem 2.2 *Let f be defined by (1.2). Then $f \in \mathcal{TK}_{\lambda}^n(m, \alpha)$ if and only if (2.1) is satisfied.*

Proof. In view of Theorem 2.1, it suffices to show the only *if* part. Assume that

$$\Re \left\{ \frac{z(D_{\lambda}^n f(z))'}{D_{\lambda}^n f(z)} \right\} = \Re \left\{ \frac{z - \sum_{k=m+1}^{\infty} k [1 + \lambda(k - 1)] C(n, k) |a_k| z^k}{z - \sum_{k=m+1}^{\infty} [1 + \lambda(k - 1)] C(n, k) |a_k| z^k} \right\} > \alpha.$$

Choose values of z on real axis so that $\frac{z(D_{\lambda}^n f(z))'}{D_{\lambda}^n f(z)}$ is real. Letting $z \rightarrow 1^-$ through real values, we have

$$1 - \sum_{k=m+1}^{\infty} k [1 + \lambda(k - 1)] C(n, k) |a_k| z^k \geq \alpha - \sum_{k=m+1}^{\infty} \alpha [1 + \lambda(k - 1)] C(n, k) |a_k| z^k.$$

Thus we obtain

$$\sum_{k=m+1}^{\infty} (k - \alpha) [1 + \lambda(k - 1)] C(n, k) |a_k| \leq (1 - \alpha),$$

which is (2.1). Hence the theorem.

Finally the result is sharp with the extremal function f given by

$$f(z) = z^p - \frac{1 - \alpha}{(k - \alpha) [1 + \lambda(k - 1)] C(n, k)} z^k \quad (k \geq m + 1, m \in \mathbb{N}_0). \quad \square \quad (2.2)$$

Corollary 2.1 *Let the function f defined by (1.2) be in the class $\mathcal{TK}_{\lambda}^n(m, \alpha)$. Then we have*

$$|a_k| \leq \frac{1 - \alpha}{(k - \alpha) [1 + \lambda(k - 1)] C(n, k)} \quad (k \geq m + 1, m \in \mathbb{N}_0). \quad (2.3)$$

This equality is attained for the function f given by (2.1).

3 Distortion Theorems

A distortion property for function f to be in the class $\mathcal{TK}_\lambda^n(m, \alpha)$ is given as follows:

Theorem 3.1 *Let the function f defined by (1.2) be in the class $\mathcal{TK}_\lambda^n(m, \alpha)$. Then for $|z| = r$ we have*

$$r - \frac{1 - \alpha}{(m + 1 - \alpha)[1 + \lambda m]C(n, m + 1)}r^{m+1} \leq |f(z)| \leq r + \frac{1 - \alpha}{(m + 1 - \alpha)[1 + \lambda m]C(n, m + 1)}r^{m+1}$$

with equality for

$$f(z) = z - \frac{1 - \alpha}{(m + 1 - \alpha)[1 + \lambda m]C(n, m + 1)}z^{m+1}, \quad (z = \mp r). \quad (3.1)$$

Proof. In view of Theorem 2.2, we have

$$\begin{aligned} & (m + 1 - \alpha)[1 + \lambda m]C(n, m + 1) \sum_{k=m+1}^{\infty} |a_k| \\ & \leq \sum_{k=m+1}^{\infty} (k - \alpha)[1 + \lambda(k - 1)]C(n, k)|a_k| \leq 1 - \alpha, \end{aligned}$$

Hence

$$\begin{aligned} |f(z)| & \leq r + \sum_{k=m+1}^{\infty} |a_k|r^k \leq r + r^{m+1} \sum_{k=m+1}^{\infty} |a_k| \\ & \leq r + \frac{1 - \alpha}{(m + 1 - \alpha)[1 + \lambda m]C(n, m + 1)}r^{m+1} \end{aligned}$$

and

$$\begin{aligned} |f(z)| & \geq r - \sum_{k=m+1}^{\infty} |a_k|r^k \geq r - r^{m+1} \sum_{k=m+1}^{\infty} |a_k| \\ & \geq r - \frac{1 - \alpha}{(m + 1 - \alpha)[1 + \lambda m]C(n, m + 1)}r^{m+1}. \end{aligned}$$

Thus complete the proof. \square

Theorem 3.2 *Let the function f defined by (1.2) be in the class $\mathcal{TK}_\lambda^n(m, \alpha)$. Then for $|z| = r$ we have*

$$1 - \frac{(m+1)(1-\alpha)}{(m+1-\alpha)[1+\lambda m]C(n, m+1)}r^m \leq |f'(z)| \leq 1 + \frac{(m+1)(1-\alpha)}{(m+1-\alpha)[1+\lambda m]C(n, m+1)}r^m$$

with equality for

$$f(z) = z - \frac{(m+1)(1-\alpha)}{(m+1-\alpha)[1+\lambda m]C(n, m+1)}z^{m+1}, \quad (z = \mp r).$$

Proof. we have

$$\begin{aligned} |f'(z)| &\leq 1 + \sum_{k=m+1}^{\infty} k|a_k|r^{k-1} \leq 1 + (m+1)r^m \sum_{k=m+1}^{\infty} |a_k| \\ &\leq 1 + \frac{(m+1)(1-\alpha)}{(m+1-\alpha)[1+\lambda m]C(n, m+1)}r^m \end{aligned}$$

and

$$\begin{aligned} |f'(z)| &\geq 1 - \sum_{k=m+1}^{\infty} k|a_k|r^{k-1} \geq 1 - (m+1)r^m \sum_{k=m+1}^{\infty} |a_k| \\ &\geq 1 - \frac{(m+1)(1-\alpha)}{(m+1-\alpha)[1+\lambda m]C(n, m+1)}r^m. \end{aligned}$$

This complete the proof. \square

Corollary 3.1 *Let the function f defined by (1.2) be in the class $\mathcal{TK}_\lambda^n(m, \alpha)$. Then the disk \mathbb{U} is mapped onto a domain that contains the disk*

$$|w| < 1 - \frac{1-\alpha}{(m+1-\alpha)[1+\lambda m]C(n, m+1)}$$

The result is sharp with extremal function (3.1).

4 Extreme Points

We shall now determine the extreme points of the class $\mathcal{TK}_\lambda^n(m, \alpha)$.

Theorem 4.1 Let $f_n(z) = z$ and

$$f_k(z) = z - \frac{1 - \alpha}{(k - \alpha)[1 + \lambda(k - \lambda)]C(n, k)} z^k \quad (k = m + 1, m + 2, \dots; \lambda \geq 0; n \in \mathbb{N}_0).$$

Then $f \in \mathcal{TK}_\lambda^n(m, \alpha)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=m}^{\infty} \mu_k f_k(z),$$

where $\mu_k \geq 0$ and $\sum_{k=m}^{\infty} \mu_k = 1$.

Proof. Suppose that

$$\begin{aligned} f(z) &= \sum_{k=m}^{\infty} \mu_k f_k(z) = \mu_n f_n(z) + \sum_{k=m+1}^{\infty} \mu_k f_k(z) \\ &= \mu_n z + \sum_{k=m+1}^{\infty} \mu_k \left[z - \frac{1 - \alpha}{(k - \alpha)[1 + \lambda(k - \lambda)]C(n, k)} z^k \right] \\ &= \mu_n z + \sum_{k=m+1}^{\infty} \mu_k z - \sum_{k=m+1}^{\infty} \mu_k \frac{1 - \alpha}{(k - \alpha)[1 + \lambda(k - \lambda)]C(n, k)} z^k \\ &= \left(\sum_{k=m}^{\infty} \mu_k \right) z - \sum_{k=m+1}^{\infty} \mu_k \frac{1 - \alpha}{(k - \alpha)[1 + \lambda(k - \lambda)]C(n, k)} z^k \\ &= z - \sum_{k=m+1}^{\infty} \mu_k \frac{1 - \alpha}{(k - \alpha)[1 + \lambda(k - \lambda)]C(n, k)} z^k. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{k=m+1}^{\infty} \mu_k \left(\frac{1 - \alpha}{(k - \alpha)[1 + \lambda(k - \lambda)]C(n, k)} \right) \left(\frac{(k - \alpha)[1 + \lambda(k - \lambda)]C(n, k)}{1 - \alpha} \right) \\ &= \sum_{k=m+1}^{\infty} \mu_k = \sum_{k=m}^{\infty} \mu_k - \mu_m = 1 - \mu_m \leq 1. \end{aligned}$$

Thus $f \in \mathcal{TK}_\lambda^n(m, \alpha)$ by Theorem 2.2.

Conversely, suppose that $f \in \mathcal{TK}_\lambda^n(m, \alpha)$. By using (2.2) we may set

$$\mu_k = \frac{(k - \alpha)[1 + \lambda(k - \lambda)]C(n, k)}{1 - \alpha} |a_k| \quad \text{and} \quad \mu_m = 1 - \sum_{m+1}^{\infty} \mu_k.$$

Then

$$f(z) = \sum_{k=m}^{\infty} \mu_k f_k(z),$$

and the proof of Theorem 4.1 is complete. \square

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