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Stability of a Four Body Relative Equilibrium

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Abstract

In the present work we study the lineal stability of a relative equilibrium for the problem of the gyrostat in newtonian interaction with three spherical rigid bodies or punctual masses. Geometrically the relative equilibrium is characterized by a particular symmetry, i.e., the rigid bodies have all the same mass m and form an equilateral triangle. On the other hand, the gyrostat of mass m_0 with revolution symmetry around the third axis of inertia is located in the center of this triangle rotating with an angular velocity ω_e, that will be determined, perpendicular to the plane formed by the previous spherical masses.

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1 Introduction

In the last few years some papers about the problem of roto-translational motion of celestial bodies have appeared. They show a new interest in the study of configurations of relative equilibria and new methods have been proposed (see [1], [2] for details about these new methods).

Let us remember that a gyrostat is a mechanical system S, composed of a rigid body S', and other bodies S'' (deformable or rigid) connected to it, in such a way that their relative motion with respect to its rigid part do not change the distribution of mass of the total system S [3].

In our previous works [4,5,6], the non-canonical Hamiltonian dynamics of $n+1$ bodies in Newtonian attraction, where n of them are rigid bodies with spherical distribution of mass (or material points) and the other one is a triaxial gyrostat, is considered. Using the symmetries we obtain the equations of motion of the reduced problem, the Casimir function of the system, the equations that determine the relative equilibria and global conditions for their existence of them. Besides, the variational characterization of these equilibria and three invariant manifolds of the problem were obtained. The equations of motion in these manifolds are described by means of a canonical Hamiltonian system.

In a first approach to the qualitative study of this system, we will describe the approximate dynamics that arises in a natural way when we take the multipolar development of the potential and truncate it until first order.

In [4,6] we have obtained a family of relative equilibria for the problem of the gyrostat in newtonian interaction with three spherical rigid bodies or punctual masses. Geometrically the relative equilibrium is characterized by a particular symmetry, i.e., the rigid bodies have all the same mass m and form an equilateral triangle. On the other hand, the gyrostat of mass m_0 with revolution symmetry around the third axis of inertia is located in the center of this triangle rotating with an angular velocity ω_e, that will be determined, perpendicular to the plane formed by the previous spherical masses.

The purpose of the present work is the study of the lineal stability of this family of relative equilibria as a natural continuation to the previously mentioned works.

Equations of motion

Let us consider m_0, m_1, m_2, m_3 the masses of the gyrostat S_0 and the spherical rigid bodies (or material points) S_1, S_2 and S_3. According with [6] we use the following notation

$$M_3 = m_2 + m_3, \quad M_2 = m_1 + m_2 + m_3, \quad M_1 = m_0 + m_1 + m_2 + m_3$$

$$g_1 = \frac{m_2 m_3}{M_3}, \qquad g_2 = \frac{m_0 M_3}{M_2}, \qquad g_3 = \frac{m_0 M_2}{M_1}$$

For $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^3$, $\mathbf{u}\cdot\mathbf{v}$ is the dot product, $|\,\mathbf{u}\,|$ is the Euclidean norm of the vector $\mathbf{u}$ and $\mathbf{u}\times\mathbf{v}$ is the cross product. $\mathbf{I}_{\mathbb{R}^3}$ is the identity matrix and $\mathbf{0}$ is the zero matrix of order three. $\mathbf{z} = (\Pi,\ \mathbf{u}_1,\ \mathbf{p}_1,\ \mathbf{u}_2,\ \mathbf{p}_2, \mathbf{u}_3, \mathbf{p}_3) \in \mathbb{R}^{21}$ to be a generic element of the twice reduced problem, where $\Pi = \mathbb{I}\Omega + \mathbf{l}_r$ is the total rotational angular momentum vector of the gyrostat in the body frame, which is attached to its rigid part $\mathfrak{J}$ and whose axes have the direction of the principal axes of inertia of S_0 and $\mathbf{l}_r = (0, 0, l)$ is the constant gyrostatic momentum. $\mathbb{I} = diag(I_1, I_1, I_3)$ is the diagonal tensor of inertia of the gyrostat, $\mathbf{u}_1$, $\mathbf{u}_2$,

$\mathbf{u}_3$, $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$ are respectively the barycentric coordinates and the linear momenta expressed in the body frame $\mathfrak{J}$.

The non-canonical dynamics of a gyrostat in Newtonian attraction with three spherical rigid bodies (or mass points), is described in [6] by means of the following Lie-Poisson system $(\mathbb{R}^{21}, \mathbf{B}, \mathcal{H})$, being

$$\mathcal{H}(\mathbf{z}) = \sum_{i=1}^{3} \frac{|\ \mathbf{p}_i\ |^2}{2g_i} + \frac{1}{2}\Pi\mathbb{I}^{-1}\Pi - \mathbf{l}_r \cdot \mathbb{I}^{-1}\Pi + \mathcal{V}$$

the Hamiltonian of the system and

$$\mathbf{B}(\mathbf{z}) = \begin{pmatrix} \widehat{\Pi} & \widehat{\mathbf{u}_1} & \widehat{\mathbf{p}_1} & \widehat{\mathbf{u}_2} & \widehat{\mathbf{p}_2} & \widehat{\mathbf{u}_3} & \widehat{\mathbf{p}_3} \\ \widehat{\mathbf{u}_1} & \mathbf{0} & \mathbf{I}_{\mathbb{R}^3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \widehat{\mathbf{p}_1} & -\mathbf{I}_{\mathbb{R}^3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \widehat{\mathbf{u}_2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{\mathbb{R}^3} & \mathbf{0} & \mathbf{0} \\ \widehat{\mathbf{p}_2} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_{\mathbb{R}^3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \widehat{\mathbf{u}_3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{\mathbb{R}^3} \\ \widehat{\mathbf{p}_3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_{\mathbb{R}^3} & \mathbf{0} \end{pmatrix}$$

the Poisson tensor.

In $\mathbf{B}(\mathbf{z})$, $\widehat{\mathbf{v}}$ is considered to be the image of the vector $\mathbf{v} \in \mathbb{R}^3$ by the standard isomorphism between the Lie Algebras $\mathbb{R}^3$ and $\mathfrak{so}(3)$, i.e.

$$\widehat{\mathbf{v}} = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}$$

The equations of motion can be written in the following form

$$\frac{d\mathbf{z}}{dt} = \mathbf{B}(\mathbf{z})\nabla_{\mathbf{z}}\mathcal{H}(\mathbf{z}). \tag{1}$$

From the previous formula, we obtain the differential equations of motion $(i = 1, 2, 3)$

$$\begin{aligned} \frac{d\Pi}{dt} &= \Pi\times\Omega + \mathbf{u}_1\times\nabla_{\mathbf{u}_1}\mathcal{V} + \mathbf{u}_2\times\nabla_{\mathbf{u}_2}\mathcal{V} + \mathbf{u}_3\times\nabla_{\mathbf{u}_3}\mathcal{V} \\ \frac{d\mathbf{u}_i}{dt} &= \mathbf{u}_i\times\Omega + \frac{1}{g_i}\mathbf{p}_i, \qquad \frac{d\mathbf{p}_i}{dt} = \mathbf{p}_i\times\Omega - \nabla_{\mathbf{u}_i}\mathcal{V} \end{aligned} \tag{2}$$

where $\Omega = \mathbb{I}^{-1}(\Pi - \mathbf{l}_r)$ is the angular velocity of the gyrostat S_0 and $\nabla_{\mathbf{u}_i}\mathcal{V}$ the gradient of $\mathcal{V}$ with respect to the variable $\mathbf{u}_i$.

Important elements of $\mathbf{B}(\mathbf{z})$ are the associate Casimir functions. We consider the total angular momentum $\mathbf{L}$ given by

$$\mathbf{L} = \Pi + \sum_{i=1}^{3}\mathbf{u}_i\times\mathbf{p}_i$$

Then the following result is obtained (see [6] for details).

Proposition 1. *If φ is a real smooth function no constant, then $\varphi(\frac{|\mathbf{L}|^2}{2})$ is a Casimir function of the Poisson tensor $\mathbf{B}(\mathbf{z})$. Moreover $Ker\mathbf{B}(\mathbf{z}) =< \nabla_{\mathbf{z}}\varphi >$. We also have $\frac{d\mathbf{L}}{dt} = \mathbf{0}$, which means that the total angular momentum vector remains constant.*

Potential energy of the system

The potential of the system is

$$V(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = -\left(\frac{Gm_2m_3}{|\mathbf{u}_1|} + \frac{Gm_1m_2}{|\mathbf{u}_2 - \frac{m_3}{M_3}\mathbf{u}_1|} + \frac{Gm_1m_3}{|\mathbf{u}_2 + \frac{m_2}{M_3}\mathbf{u}_1|} + \right.$$
$$Gm_1\int_{S_0}\frac{dm(\mathbf{Q})}{|\mathbf{Q} + \mathbf{u_3} - \frac{M_3}{M_2}\mathbf{u}_2|} + Gm_2\int_{S_0}\frac{dm(\mathbf{Q})}{|\mathbf{Q} + \mathbf{u_3} + \frac{m_1}{M_2}\mathbf{u}_2 - \frac{m_3}{M_3}\mathbf{u}_1|}$$
$$\left. + Gm_3\int_{S_0}\frac{dm(\mathbf{Q})}{|\mathbf{Q} + \mathbf{u_3} + \frac{m_1}{M_2}\mathbf{u}_2 + \frac{m_2}{M_3}\mathbf{u}_1|}\right)$$

Considering the multipolar development of the potential, supposing that the involved bodies are at much more mutual distances than their individual dimensions of the same ones, we can develop the potential in quickly convergent series.

Under the previous considerations the potential, until the first order, comes given by

$$\mathcal{V}^{(1)} = -\left(\frac{Gm_2m_3}{|\mathbf{u}_1|} + \frac{Gm_1m_2}{|\mathbf{u}_2 + \frac{m_3}{M_3}\mathbf{u}_1|} + \frac{Gm_1m_3}{|\mathbf{u}_2 - \frac{m_2}{M_3}\mathbf{u}_1|}\right)$$
$$-\frac{1}{2}\left(\frac{Gm_1\alpha}{|\mathbf{u}_3 - \frac{M_3}{M_2}\mathbf{u}_2|^3} + \frac{Gm_2\alpha}{|\mathbf{u}_3 + \frac{m_1}{M_2}\mathbf{u}_2 + \frac{m_3}{M_3}\mathbf{u}_1|^3} + \frac{Gm_3\alpha}{|\mathbf{u}_3 + \frac{m_1}{M_2}\mathbf{u}_2 - \frac{m_2}{M_3}\mathbf{u}_1|^3}\right)$$
$$+\frac{1}{2}\left(\frac{3Gm_1f_1}{|\mathbf{u}_3 - \frac{M_3}{M_2}\mathbf{u}_2|^5} + \frac{Gm_2f_2}{|\mathbf{u}_3 + \frac{m_1}{M_2}\mathbf{u}_2 + \frac{m_3}{M_3}\mathbf{u}_1|^5} + \frac{Gm_3f_3}{|\mathbf{u}_3 + \frac{m_1}{M_2}\mathbf{u}_2 - \frac{m_2}{M_3}\mathbf{u}_1|^5}\right)$$

with $\alpha = 2I_1 + I_3$ and

$$f_1(\mathbf{u}_2, \mathbf{u}_3) = \left(\mathbf{u}_3 - \frac{M_3}{M_2}\mathbf{u}_2\right)^t \cdot \mathbb{I}\left(\mathbf{u}_3 - \frac{M_3}{M_2}\mathbf{u}_2\right)$$
$$f_2(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = \left(\mathbf{u}_3 + \frac{m_1}{M_2}\mathbf{u}_2 + \frac{m_3}{M_3}\mathbf{u}\right)^t \cdot \mathbb{I}\left(\mathbf{u}_3 + \frac{m_1}{M_2}\mathbf{u}_2 + \frac{m_3}{M_3}\mathbf{u}_1\right)$$
$$f_3(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = \left(\mathbf{u}_3 + \frac{m_1}{M_2}\mathbf{u}_2 - \frac{m_2}{M_3}\mathbf{u}_1\right)^t \cdot \mathbb{I}\left(\mathbf{u}_3 + \frac{m_1}{M_2}\mathbf{u}_2 - \frac{m_2}{M_3}\mathbf{u}_1\right)$$

1.1 Approximate dynamics of order one

We will call *approximate dynamics of order one* to the differential equations given by the following expression

$$\frac{d\mathbf{z}}{dt} = \{\mathbf{z}, \mathcal{H}(\mathbf{z})\}(\mathbf{z}) = \mathbf{B}(\mathbf{z})\nabla_{\mathbf{z}}\mathcal{H}(\mathbf{z}) \tag{3}$$

with

$$\mathcal{H}(\mathbf{z}) = \sum_{i=1}^{3} \frac{|\ \mathbf{p}_i\ |^2}{2g_i} + \frac{1}{2}\Pi\mathbb{I}^{-1}\Pi - \mathbf{l}_r \cdot \mathbb{I}^{-1}\Pi + \mathcal{V}^{(1)}$$

Of the equations of motion, it is easy to verify the following result.

Proposition 2. *π_3 is an integral of the motion.*

2 Relative equilibria

The relative equilibria are the equilibria of the doubly reduced problem. Let us consider $\mathbf{z}_e = (\Pi_e, \mathbf{u}_i^e, \mathbf{p}_i^e)$ the equilibrium in an approximate dynamics of first order, this verifies the following equations for $i = 1, 2, 3$

$$\begin{array}{l} \Pi_e \times \Omega_e + \mathbf{u}_1^e \times (\nabla_{\mathbf{u}_1}\mathcal{V})_e + \mathbf{u}_2^e \times (\nabla_{\mathbf{u}_2}\mathcal{V})_e + \mathbf{u}_3^e \times (\nabla_{\mathbf{u}_3}\mathcal{V})_e = \mathbf{0} \\ \dfrac{\mathbf{p}_i^e}{g_i} + \mathbf{u}_i^e \times \Omega_e = \mathbf{0}, \quad \mathbf{p}_i^e \times \Omega_e = (\nabla_{\mathbf{u}_i}\mathcal{V})_e \end{array} \tag{4}$$

with $(\nabla_{\mathbf{u}_i}\mathcal{V})_e$ is the evaluation of $\nabla_{\mathbf{u}_i}\mathcal{V}$ in $\mathbf{z}_e$.

Of the previous conditions, by means of algebraic manipulations, the following equations are deduced

$$|\ \Omega_e\ |^2 |\ \mathbf{u}_i^e\ |^2 - (\mathbf{u}_i^e \cdot \Omega_e)^2 = \tfrac{1}{g_i}(\mathbf{u}_i^e \cdot (\nabla_{\mathbf{u}_i}\mathcal{V})_e), \ (i = 1, 2, 3) \tag{5}$$

The last equation allows us to obtain $|\ \Omega_e\ |$ in the equilibria.

2.1 Lagrangian equilibria in a approximate dynamics of order one

Definition 3. Will we say that $\mathbf{z}_e$ is a *Lagrangian relative equilibrium* when Ω_e be orthogonal to $U = span(\mathbf{u}_1^e, \mathbf{u}_2^e, \mathbf{u}_3^e)$ and $dim(U) = 2$.

It is easy to verify the following result.

Proposition 4. *If $\mathbf{z}_e$ is a Lagrangian relative equilibrium then*

$$\mathbf{u}_1^e \times (\nabla_{\mathbf{u}_1}\mathcal{V})_e + \mathbf{u}_2^e \times (\nabla_{\mathbf{u}_2}\mathcal{V})_e + \mathbf{u}_3^e \times (\nabla_{\mathbf{u}_3}\mathcal{V})_e = \mathbf{0}$$

that is to say moments are not exercised on the gyrostat.

After some algebraic calculations with (4) and (5) the following result is obtained (see [6] for details).

Proposition 5. $\mathbf{z}_e = (\Pi_e, \mathbf{u}_1^e, \mathbf{p}_1^e, \mathbf{u}_2^e, \mathbf{p}_2^e, \mathbf{u}_3^e, \mathbf{p}_3^e)$ *given by*

$$\begin{array}{ll} \mathbf{u}_1^e = (\sqrt{3}Z, 0, 0), & \mathbf{p}_1^e = (0, \sqrt{3}g_1\omega_e Z, 0) \\ \mathbf{u}_2^e = (0, \frac{3}{2}Z, 0), & \mathbf{p}_2^e = (-\frac{3}{2}g_2\omega_e Z, 0, 0) \\ \mathbf{u}_3^e = (0, 0, 0), & \mathbf{p}_3^e = (0, 0, 0) \\ \Omega_e = (0, 0, \omega_e), & \Pi_e = (0, 0, C\omega_e + l) \end{array}$$

where

$$\omega_e^2 = \frac{G(m\sqrt{3} + 3m_0)}{3Z^3} + \frac{G\beta}{Z^5}$$

with $m_1 = m_2 = m_3 = m$, the masses of the S_i $(i = 1, 2, 3)$, m_0 the mass of S_0, $\sqrt{3}Z$ is the distance from S_i to S_j, $(i \neq j$ and $i, j \in \{1, 2, 3\})$ and $\beta = 3(I_3 - I_1)/2$ is a relative equilibrium in approximate dynamics of order one.

2.2 Stability of $\mathbf{z}_e$

The tangent flow of the equations (3) in the equilibrium $\mathbf{z}_e$, comes given by

$$\frac{d\delta\mathbf{z}}{dt} = \mathfrak{U}(\mathbf{z}_e)\delta\mathbf{z}$$

with $\delta\mathbf{z} = \mathbf{z} - \mathbf{z}_e$ and $\mathfrak{U}(\mathbf{z}_e)$ the jacobian matrix of (3) in $\mathbf{z}_e$.

Carrying out an appropriate election of the units, in order to have the minimum number of parameters, the characteristic polynomial of $\mathfrak{U}(\mathbf{z}_e)$ is determined by the following expression

$$P(\lambda) = \lambda^3(\lambda^2 + \Phi^2)(\lambda^2 + \widetilde{\omega}_e^2)^2(\lambda^4 + m\lambda^2 + n)(\lambda^8 + p\lambda^6 + q\lambda^4 + r\lambda^2 + s)$$

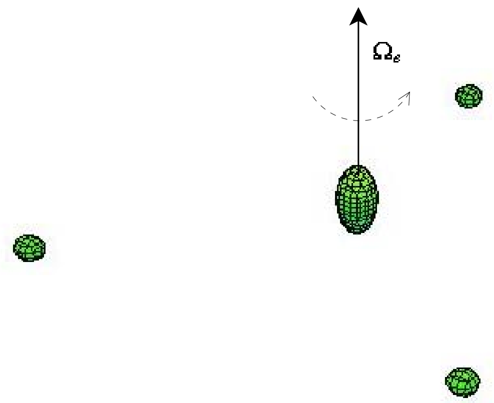

Figure 1: Relative Equilibrium $\mathbf{z}_e$

with $m = \sqrt{3}+3k-21\beta$, $n = -6\beta(13\beta+\sqrt{3}+3k)$ and the coefficients p, q, r, s expressed in Appendix A, $\widetilde{\omega}_e^2 = \frac{(3+k\sqrt{3})}{3} + \beta$, $\Phi = \frac{(A-C)\widetilde{\omega}_e+l}{A}$ and $k = \frac{m_0}{m}$.

If the roots of $(\lambda^4+m\lambda^2+n)(\lambda^8+p\lambda^6+q\lambda^4+r\lambda^2+s)$ are in the imaginary axis, then $\mathbf{z}_e$ is linearly stable since

$$Q(\lambda) = \lambda(\lambda^2+\Phi^2)(\lambda^2+\widetilde{\omega}_e^2)(\lambda^4+m\lambda^2+n)(\lambda^8+p\lambda^6+q\lambda^4+r\lambda^2+s)$$

is the minimum polynomial of the matrix $\mathfrak{U}(\mathbf{z}_e)$.

Denoting $h = \lambda^8+p\lambda^6+q\lambda^4+r\lambda^2+s$, if $m, n > 0$, $m^2-4n > 0$ and

$$r, s > 0, \;\; 3p^2-8q > 0, \;\; pr-16s > 0$$
$$p^2qr - 48sr - 9sp^3 + 32pqs - 4q^2r + 3pr^2 > 0$$
$$p^2q^2 - 3rp^3 - 6p^2s - 4q^3 + 14pqr + 16qs - 18r^2 > 0$$
$$discrim(h) > 0$$

with

$$discrim(h) = 18p^3rqs - 4p^3r^3 - 128q^2s^2 + 16q^4s$$
$$-4q^3r^2 - 27p^4s^2 - 80prq^2s + 256s^3 - 27r^4 - 6p^2r^2s$$
$$-192prs^2 + 18pr^3q + 144qp^2s^2 + q^2p^2r^2 - 4q^3p^2s + 144sr^2q$$

then $\mathbf{z}_e$ is linearly stable.

Let us consider Ω_i $(i = 1, 2, \ldots, 10)$ the regions of the parametric plane $k\beta$ associated to the previous inequalities, which are polynomial inequalities in the previous parametric plane.

With the help of the program Mathematica^{TM} we conclude $\cap\Omega_i \neq \varnothing$. Then $\mathbf{z}_e$ is linearly stable for all k, β in some certain region of the parametric plane $k\beta$.

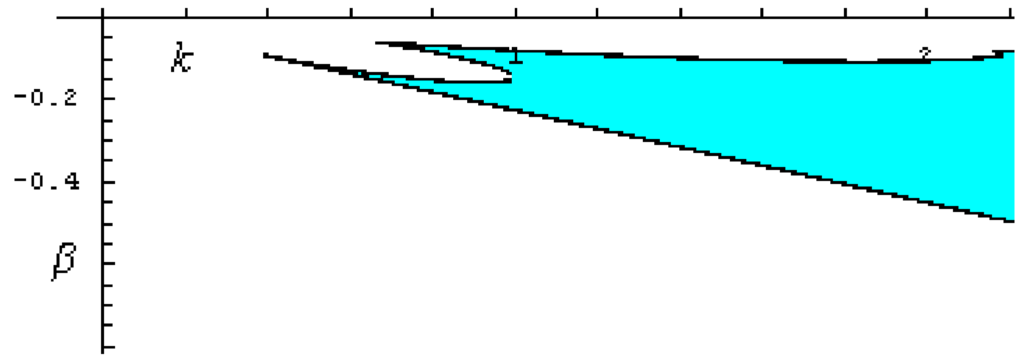

Figure 2: Region of Linear Stability

2.2.1 Stability of $\mathbf{z}_e$ in zero order

In zero order approximate dynamics ($\beta = 0$) the characteristic polynomial is

$$P(\lambda) = \lambda^5(\lambda^2 + \Phi^2)(\lambda^2 + \widetilde{\omega}_e^2)^3(\lambda^8 + p\lambda^6 + q\lambda^4 + r\lambda^2 + s)$$

with coefficients $p = (2 - (\sqrt{3} - 3)k)$, $q = (8k^2 + 2 - 17k)$ and

$$r = \frac{k(5\sqrt{3} - 18)(25k^2 - 219\sqrt{3} + 1494 + (235\sqrt{3} - 4383)k)}{2988},$$

$$s = \frac{(133 - 60\sqrt{3})((64\sqrt{3} + 81)k - 249)^2}{330\,672}$$

with $\Phi = \frac{(A-C)\widetilde{\omega}_e + l}{A}$ being $\widetilde{\omega}_e^2 = \frac{(3+k\sqrt{3})}{3}$.

The minimum polynomial of the matrix $\mathfrak{U}(\mathbf{z}_e)$ is

$$Q(\lambda) = \lambda^2(\lambda^2 + \Phi^2)(\lambda^2 + \widetilde{\omega}_e^2)(\lambda^8 + p\lambda^6 + q\lambda^4 + r\lambda^2 + s)$$

then $\mathbf{z}_e$ is linearly unstable.

Spectral stability is a necessary condition to obtain nonlinear Lyapunov stability of a equilibrium in Hamiltonian systems [7].

Each positive root of the polynomial $\sigma^4 - p\sigma^3 + q\sigma^2 - r\sigma + s$ corresponds to a couple of imaginary roots of the polynomial $\lambda^8 + p\lambda^6 + q\lambda^4 + r\lambda^2 + s$. Applying Descarte's rules of signs we conclude that the polynomial does not have all the roots in the positive real axis (see Appendix B). And by Sturm Theorem the polynomial $\sigma^4 - p\sigma^3 + q\sigma^2 - r\sigma + s$ has the maximum of three real roots in the positive real axis. Then $\mathbf{z}_e$ is unstable.

3 Conclusions

In this paper we used geometric-mechanics methods to study the relative equilibrium $\mathbf{z}_e$ obtained in previous works.

The relative equilibrium $\mathbf{z}_e$ in zero order approximate dynamics is unstable. As consequence of the previous result if S_0 is very close to a sphere that is, if $I_3 - I_1 \approx 0$, then for all k, $\mathbf{z}_e$ is unstable.

In order one approximate dynamics if $(k, \beta) \in \cap\Omega_i$ then $\mathbf{z}_e$ is linear stable. In a next work we will analyze the nonlinear stability of $\mathbf{z}_e$ be using the Energy-Casimir method [4,5].

A Coefficients of $h(\lambda)$ in approximate dynamics of order one.

The coefficients p, q, r, s comes given by the following formulas

$$p = \frac{(-2016k^2 - 3888k)\beta + 288k^3 - 432k^2 + 144\sqrt{3}k^2}{144k^2}$$

$$q = \frac{(-432k^2 + 26244)\beta^2 + (-12096k^2 - 3744k^3 + 5832k + 312\sqrt{3}k^2)\beta}{144k^2} + \frac{5043\sqrt{3}k^3 - 1728k^3 + 144k^4 + 576k^2}{144k^2}$$

$$r = \frac{(472392 + 52416k^2 + 291600k)\beta^3 + (-47628\sqrt{3}k + 85968k^2 + 320760k + 4608k^3}{144k^2} + \frac{-12984\sqrt{3}k^2 + 8748\sqrt{3})\beta^2 + (1944\sqrt{3}k + 59988k^2 - 2064\sqrt{3}k^3 - 1728k^4 + 432k^3}{144k^2} + \frac{-1620k - 14904\sqrt{3}k^2\beta) + 3972k^3 - 180k^2 - 1296k^4 + 196\sqrt{3}k^2 + 360\sqrt{3}k^4 - 1260\sqrt{3}k^3}{144k^2}$$

$$s = \frac{(97344k^2 + 1285956 + 707616k)\beta^4 + (44928k^3 + 1061424k - 34320\sqrt{3}k^2 + 455328k^2}{144k^2} + \frac{-183708\sqrt{3} - 175284\sqrt{3}k)\beta^3 + (-96228\sqrt{3}k + 19683 - 102780\sqrt{3}k^2 - 17280\sqrt{3}k^3 + 13122k}{144k^2} + \frac{346827k^2 + 5184k^4 + 101088k^3)\beta^2 + (-2160\sqrt{3}k^4 + 972\sqrt{3}k + 54630k^3 + 4644k^2 - 16512\sqrt{3}k^2}{144k^2} + \frac{7776k^4 + 4374k - 21276\sqrt{3}k^3)\beta + 279k^2 + 162k^3 - 1620\sqrt{3}k^4 + 3591k^4 - 792\sqrt{3}k^3 + 108\sqrt{3}k^2}{144k^2}$$

B Signs of coefficients of $h(\lambda)$ in approximate dynamics of order zero

Value of k	p	q	r	s	Value of k	p	q	r	s
$k < k_0$	$-$	$+$	$+$	$+$	$k_2 < k < k_3$	$-$	$+$	$-$	$+$
$k = k_0$	$-$	0	$+$	$+$	$k = k_3$	$-$	$+$	$-$	0
$k_0 < k < k_1$	$-$	$-$	$+$	$+$	$k_3 < k < k_4$	$-$	$+$	$-$	$+$
$k = k_1$	$-$	$-$	0	$+$	$k = k_4$	0	$+$	$-$	$+$
$k_1 < k < k_2$	$-$	$-$	$-$	$+$	$k > k_4$	$+$	$+$	$-$	$+$
$k = k_2$	$-$	0	$-$	$+$					

with

$$k_0=\frac{3}{2}-\frac{7\sqrt{3}}{16}-\frac{\sqrt{659-336-\sqrt{3}}}{16}, \qquad k_1=\frac{315\sqrt{3}-993+3\sqrt{130276-60654\sqrt{3}}}{2(49\sqrt{3}-45)}$$

$$k_2=\frac{3}{2}-\frac{7\sqrt{3}}{16}-\frac{\sqrt{659-336-\sqrt{3}}}{16}, \qquad k_3=\frac{88\sqrt{3}-18}{2(31+12\sqrt{3})}, \qquad k_4=\frac{3+\sqrt{3}}{3}$$

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