Applied Mathematical Sciences, Vol. 1, 2007, no. 24, 1181 - 1186

A Nonlinear Inverse Parabolic Problem

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Abstract

In this article we consider an inverse parabolic problem of linear heat equation with nonlinear boundary condition. We identify the temperature and the unknown radiation term from an overspecified condition on the boundary. We propose a numerical algorithm based on finite difference method and least square technique. Results show that an excellent estimation can be obtained for unknown functions.

Mathematics Subject Classification: 35R30

Keywords: Inverse heat conduction problem, Radiation term, Finite difference method, Least-squares method

1 Introduction

Mathematically, the inverse problems belong to the class of ill-posed or ill-conditioned problems; that is, their solutions do not satisfy the general requirements of existence, uniqueness, and stability under small change to the input data. Recently much attention has been given in the literature to the development, analysis and implementation of accurate methods for the numerical solution of parabolic inverse problems. So, the problem of determining unknown parameters in parabolic differential equations has bean treated by many authors [1-8]. Usually these type problems involve the determination of a single unknown parameter from overspecified boundary data. In some applications, however, it is desirable to be able to determine more than one parameter from the given boundary data [2,6]. It is well known that the radiative heat is a function of temperature. In certain radiative heat transfer it is of interest to devise methods for evaluating radiation function by using only measurements taken outside the medium.

2 Mathematical formulation

In this paper, we consider the problem of determining an unknown function $P(u)$ which is defined on [0,1], and a function $u(x,t)$ satisfying

$$\begin{aligned} u_t &= u_{xx}, \qquad 0 < x < 1, \quad 0 < t < T, &(1)\\ u(x,0) &= f(x), \qquad 0 \leq x \leq 1, &(2)\\ u(0,t) &= g(t), \qquad 0 < t < T, &(3)\\ u_x(1,t) - P(u(1,t)) &= \chi(t), \qquad 0 < t < T, &(4) \end{aligned}$$

and the overspecified condition

$$u(1,t) = \phi(t), \qquad 0 < t < T, \tag{5}$$

where T is a given positive constant and f(x), g(t), $\chi(t)$ and $\phi(t)$ are piecewise-continuous functions on their domains. The equation (1) may be used to describe the flow of heat in a rod. Hence, we might think of this problem as the problem of determining the unknown radiation term in a rod.

If the function $P(u)$ is given, then there may be no solution for problem (1)-(5). For an unknown $P(u)$, we must therefore provide additional information namely (5) to provide a unique solution $(u, P(u))$ to the inverse problem (1)-(5) [2]. The nonlinear inverse problem (1)-(5) have been previously treated by many authors [1-6].

2.1 Theorem

For any piecewise-continuous functions f, g, χ, and ϕ there is a unique solution pair (u, P), for the inverse problem (1)-(5), [8].

3 Numerical procedure

The application of the present numerical method to find the solution of problem (1)-(5) can be described as follows.

First, for linearized nonlinear term in equation (4) we used Taylorś series expansion. Therefore the function $P(u)$ in equation (4) can be linearized by Taylorś series expansion, as follows

$$P(u) = P(\bar{u}) + (\frac{\partial}{\partial u} P(u))_{u=\bar{u}}(u - \bar{u}), \tag{6}$$

where $\bar{u} = \begin{pmatrix} \bar{u}_0, & \bar{u}_1, & . & . & ., & \bar{u}_N \end{pmatrix}$ denotes the previously iterated solution.

The discretized forms of problem obtained by using the finite difference approximation and then the matrix form is given as

$$\Lambda U = \Theta, \tag{7}$$

where

$$\Lambda = \begin{pmatrix} 1+2r & -r & 0 & 0 \\ -r & 1+2r & -r & 0 \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 0 & -r & 1+2r & -r \\ 0 & 0 & -2r & 1+2r-2rh\frac{\partial P}{\partial u}(\bar{u}_N) \end{pmatrix},$$

and

$$U^t = \begin{pmatrix} u_{1,\nu+1} & u_{2,\nu+1} & . & . & . & u_{N-1,\nu+1} & u_{N,\nu+1} \end{pmatrix},$$

$$\Theta^t = \begin{pmatrix} u_{1,\nu} + u_{0,\nu+1} & u_{2,\nu} & . & . & . & u_{N-1,\nu} & u_{N,\nu} + \Omega(\nu) \end{pmatrix},$$

where

$$\Omega(\nu) = 2rhP(\bar{u}_N) - 2rh\bar{u}_N\frac{\partial P}{\partial u}(\bar{u}_N) + 2rh\chi(\nu k + k).$$

Note that equation (7) is a linear equation.

The LU-Decomposition algorithm is used to solve

$$U^t = \begin{pmatrix} u_{1,\nu+1} & u_{2,\nu+1} & . & . & . & u_{N,\nu+1} \end{pmatrix}.$$

These updated values of U are used to calculate Λ and Θ for iteration. This computational procedure is performed repeatedly until desired convergence is achieved. In this work the polynomial form proposed for the unknown $P(u)$ before performing the inverse calculation. Therefore $P(u)$ approximated as

$$P(u) = a_0 + a_1 u + a_2 u^2 + ... + a_q u^q, \tag{8}$$

where $\{a_0, a_1, ..., a_q\}$ are constants which remain to be determined simultaneously.

To minimize the sum of the squares of the deviations between $u_{N,\nu+1}$ (calculated) and $\phi(\nu(k+1))$, we use least-squares method. The error in the estimate

$$E(a_0, a_1, ..., a_q) = \sum_{\nu=0}^{N}(u_{N,\nu+1} - \phi((\nu+1)k))^2, \tag{9}$$

which remain to be minimized. The estimated values of a_i are determined until the value of $E(a_0, a_1, ..., a_q)$ is minimized.

4 Numerical results

In this section, by giving an example we are going to demonstrate some results for unknown radiation term in the inverse problem (1)-(5), numerically. All the computations are performed on the PC.

Example. In this example let us consider the following inverse problem

$$u_t = u_{xx}, \qquad 0 < x < 1, \quad t > 0, \tag{10}$$

$$u(x,0) = \cos(x), \qquad 0 < x < 1, \tag{11}$$

$$u(0,t) = \exp(-t) \qquad 0 < t < T, \tag{12}$$

$$u_x(1,t) - P(u(1,t)) = -1 - (\cos 1 + \sin 1)\exp(-t), \quad 0 < t < T, \tag{13}$$

with the overspecified condition

$$u(1,t) = \cos(1)\exp(-t), \qquad 0 < t < T. \tag{14}$$

The exact solution of this problem is

$$u(x,t) = \cos(x)\exp(-t),$$

and

$$P(u) = 1 + u.$$

To solve the problem (25)-(29), the unknown function $P(u)$ defined as the following form

$$P(u) = a_0 + a_1 u.$$

	Numerical	*Exact*	*Numerical*	*Exact*	*Numerical*	*Exact*
μ	$u_{\mu,1}$	$u_{\mu,1}$	$u_{\mu,2}$	$u_{\mu,2}$	$u_{\mu,3}$	$u_{\mu,3}$
1	0.893831	0.892299	0.809370	0.807368	0.732600	0.730553
2	0.857181	0.855032	0.776548	0.773665	0.703057	0.700041
3	0.796150	0.794070	0.721328	0.718504	0.653101	0.650129
4	0.712498	0.711100	0.645324	0.643430	0.584192	0.582200
5	0.608461	0.608424	0.550660	0.550525	0.498301	0.498135
6	0.491868	0.488886	0.449049	0.442362	0.410603	0.450266

Table 1.

	Numerical	*Exact*	*Numerical*	*Exact*
μ	$u_{\mu,4}$	$u_{\mu,4}$	$u_{\mu,5}$	$u_{\mu,5}$
1	0.662979	0.661032	0.599892	0.598126
2	0.636300	0.633424	0.575729	0.573145
3	0.591074	0.588261	0.534741	0.532281
4	0.528620	0.526796	0.478105	0.476665
5	0.450730	0.450731	0.407436	0.407839
6	0.375915	0.362175	0.344561	0.327710

Table 2.

The estimated values of a_0, a_1 are $a_0 = 0.988109$ and $a_1 = 1$. Tables 1 and 2, respectively, shown the values of U in $x = \mu h$ and $t = \nu k$ when $k = \frac{1}{10}$, $h = \frac{1}{6}$.

5 Conclusion

A numerical method to estimate unknown radiation term is proposed for an inverse problem of linear heat equation with nonlinear boundary condition and from the illustrated example it can be seen that the proposed numerical method is efficient and accurate to estimate the unknown radiation term.

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Received: October 7, 2007