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Approximate Eigenvalues of Periodic Sturm-Liouville Problems Using Differential Quadrature Method

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Abstract

Computation of eigenvalues of regular Sturm-Liouville problems with periodic boundary conditions is considered. It is shown through numerical illustrations that both polynomial-based differential quadrature (PDQ) and Fourier expansion-based differential quadrature (FDQ) methods can be successfully used to accurately predict the first kth ($k = 1, 2, 3, \cdots$) eigenvalues of the problem using (at least) $2k$ mesh points in the computational domain. The errors in computed solutions are compared with other published results in the literature for the numerical illustrations considered in this work.

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1 Introduction

Problems in the fields of elasticity and vibration, including applications of the wave equations of modern physics, fall into a special class of boundary-value problems known as characteristic-value problems. Certain problems in statics also reduce to such problems.

In this work, we consider a special type of boundary-value problem called Sturm-Liouville problem. A general Sturm-Liouville problem consists of the linear homogeneous second-order differential equation of the form

$$\frac{d}{dx}\Big(p(x)\frac{dy}{dx}\Big) + \Big(r(x)\lambda - q_1(x)\Big)y = 0. \tag{1}$$

This problem can be easily reduced to the so-called Liouville normal form (or equivalently, the one-dimensional, time independent form of the all-pervading equation of physics, the Schrödinger equation)

$$y' + q(x)y = \lambda y. \tag{2}$$

Pryce [9] has provided an excellent review of the mathematical backround of Sturm-Liouville problems and their numerical solution, as well as a detailed discussion of applications, up to approximately 1993. He provides examples of Sturm-Liouville problems that have been considered by numerous authors.

Here, we consider the above differential equation with periodic boundary conditions of the type

$$y(0) = y(\pi)\ ,\ y'(0) = y'(\pi). \tag{3}$$

These types of boundary conditions arise in the study of planetary orbits and other periodic physical phenomena as well as when polar coordinates are used in partial differential equations solved by the method of seperation of variables. Problems with these boundary conditions in applications usually satisfy $q(0) = q(\pi)$ [7].

In general, it is not possible to obtain the eigenvalues of problem (2) analytically. However, there are various approximate methods as, for example, finite difference and finite elements methods. Paine et al. [8] considered computation of the eigenvalues of the regular Sturm-Liouville problem (2) with the boundary conditions

$$y(0) = y(\pi) = 0, \tag{4}$$

by the centered finite difference method with uniform mesh, and showed that when q is constant, the error in λ_k has the same asymptotic form as $k \to \infty$ as the error for general q. They also showed that the accuracy of the estimates obtained for high eigenvalues could be dramatically improved at negligible extra cost by using the known error for $q = 0$ to correct the estimates obtained for general q. Andersen and de Hoog [1] extended the analysis of [8] to problems with the general seperated boundary conditions. The correction technique was succesfully used by Andrew [2] for important non-seperated boundary conditions (periodic boundary conditions given by (3)). Vanden Berghe et al. [14] used a modified difference scheme for periodic and semi-periodic Sturm-Liouville problems and presented some numerical examples in which application of the same correction to the modified difference scheme gave more accurate results than [2]. Use of the correction technique in conjunction with the finite element method was studied in [3] for periodic and some other types of boundary conditions. Yücel [15] used differential quadrature (DQ) method to approximate the eigenvalues of problem (2) with boundary conditions (4) and obtained encouraging results when compared to other published results in the literature.

In this study, the DQ method is applied to obtain approximate eigenvalues of the Sturm-Liouville problem with periodic boundary conditions. In the DQ method, derivatives of a function with respect to a coordinate direction is expressed as linear weighted sums of all the functional values at all mesh (grid) points along that direction. The weighting coefficients in that weighting sum are determined using test functions. Among the many kinds of test functions, the Lagrange interpolation polynomial is widely employed since it has no limitation on the choice of the grid points. This leads to polynomial-based differential quadrature (PDQ) which is suitable in most engineering problems. For problems with periodic behaviours, polynomial approximation may not be the best choice for the true solution. In contrast, Fourier series expansion can be the best approximation giving the Fourier expansion-based differential quadrature (FDQ). The ease for computation of weighting coefficients in explicit formulations [13] for both cases is based on the analysis of function approximation and linear vector space.

Here is the outline of the paper. In Section 2 we summarize the DQ method. Its application to the periodic Sturm-Liouville problems is given in Section 3. Numerical results and conclusions are presented in Sections 4 and 5, respectively.

2 Differential Quadrature Method

Differential quadrature (DQ) method was presented for the first time by [4] for solving differential equations. The DQ method uses the basis of the quadrature method in driving the derivatives of a function. It follows that the partial derivative of a function with respect to a space variable can be approximated by a weighted linear combination of function values at some intermediate points in that variable.

In order to show the mathematical representation of DQ method, we consider a single variable function $f(x)$ on the domain $a \leq x \leq b$; then the nth order derivative of the function at an intermediate point (grid point) can be written as:

$$\left.\frac{d^n f}{dx^n}\right|_{x=x_i} = \sum_{j=1}^{N} r_j^n(x_i) f(x_j) \ , \ i = 1, 2, \cdots, N \ , \ n = 1, 2, \cdots, N-1, \quad (5)$$

where N is the number of grid points in the whole domain ($a = x_1, x_2, \cdots, x_N = b$) and $r_j^n(x_i)$ are the weighting coefficients of the nth derivative. As it can be seen from (5), two important factors control the quality of the approximation resulting from the application of DQ method. These are the values of weighting coefficients and the positions of the discrete variables. Once the weighting coefficients are determined, the bridge to link the derivatives in the governing differential equation and the functional values at the mesh points is established. In other words, with the weighting coefficients, one can easily use the functional values to compute the derivatives. Note that for multidimensional problems each derivative is approximated in the respective direction similarly.

In order to determine the weighting coefficients in (5), $f(x)$ must be approximated by some test functions.

2.1 Polynomial-based differential quadrature

When the function $f(x)$ is approximated by a higher order polinomial, Shu [13] and Shu and Richards [11] presented some explicit formulations to compute the weighting coefficients within the scope of a higher order polynomial approximation and a linear vector space. It is supposed that the solution of a one-dimensional differential equation is approximated by a $(N-1)th$ degree polynomial

$$f(x) = \sum_{k=0}^{N-1} c_k x^k. \tag{6}$$

If $r_k(x), k = 1, 2, , \cdots, N$, are the base polynomials in V_N (N-dimensional linear vector space), then $f(x)$ can be expressed by

$$f(x) = \sum_{k=1}^{N} d_k r_k(x). \tag{7}$$

Here, the base polynomials $r_k(x), k = 1, 2, \cdots, N$, are chosen as the Lagrange interpolated polynomials

$$r_k(x) = \frac{M(x)}{(x - x_k) M^{(1)}(x_k)}, \tag{8}$$

where

$$M(x) = \prod_{j=1}^{N} (x - x_j) \ , \ M^{(1)}(x_k) = \prod_{j=1, j\neq k}^{N} (x_k - x_j), \tag{9}$$

and $x_i, i = 1, 2, \cdots, N$, are the coordinates of grid points which may be chosen arbitrarily. Substituting (7) into (5) and using (8) result in the following weighting coefficients for the first- and second-order derivatives

$$r_j^{(1)}(x_i) = \frac{M^{(1)}(x_i)}{(x_i - x_j)M^{(1)}(x_j)}\ ,\ for j \neq i\ ,\ i,j = 1,\cdots,N$$
$$r_i^{(1)}(x_i) = -\sum_{j=1,i\neq j}^{N} r_j^{(1)}(x_i), \tag{10}$$

$$r_j^{(2)}(x_i) = 2r_j^{(1)}(x_i)\left(r_i^{(1)}(x_i) - \frac{1}{(x_i - x_j)}\right)\ ,\ for j \neq i\ ,\ i,j = 1,\cdots,N$$
$$r_i^{(2)}(x_i) = -\sum_{j=1,i\neq j}^{N} r_j^{(2)}(x_i). \tag{11}$$

It can be seen from the above equations that the weighting coefficients of the second-order derivative can be completely determined from those of the first-order derivative.

2.2 Fourier expansion-based differential quadrature

When the function $f(\mathrm{x})$ is approximated by a Fourier series expansion in the form

$$f(x) = c_0 + \sum_{k=1}^{N/2}\left(c_k \cos\frac{k\pi x}{L} + d_k \sin\frac{k\pi x}{L}\right), \tag{12}$$

and Lagrange interpolated trigonometric polynomials are taken as

$$r_k(x) = \frac{\sin\frac{x-x_0}{2L}\pi \ldots \sin\frac{x-x_{k-1}}{2L}\pi \sin\frac{x-x_{k+1}}{2L}\pi \ldots \sin\frac{x-x_N}{2L}\pi}{\sin\frac{x_k-x_0}{2L}\pi \ldots \sin\frac{x_k-x_{k-1}}{2L}\pi \sin\frac{x_k-x_{k+1}}{2L}\pi \ldots \sin\frac{x_k-x_N}{2L}\pi}, \tag{13}$$

the FDQ approach is obtained for $f(x)$ and its first- and second-order derivatives. So, the weighting coefficients $r_j^{(n)}(x_i)$, $(n = 1, 2)$ to be determined by the FDQ method are given by

$$r_j^{(1)}(x_i) = \frac{\pi}{2L}\frac{P(x_i)}{\sin\left((x_i - x_j)\frac{\pi}{2L}\right)P(x_j)}\ ,\ for j \neq i\ ,\ i,j = 1,\cdots,N$$
$$r_i^{(1)}(x_i) = -\sum_{j=1,i\neq j}^{N} r_j^{(1)}(x_i), \tag{14}$$

$$r_j^{(2)}(x_i) = r_j^{(1)}(x_i)\left(2r_i^{(1)}(x_i) - \frac{\pi}{L}\cot\left(\frac{x_i - x_j}{2L}\pi\right)\right), for j \neq i,\ i,j = 1,\cdots,N$$
$$r_i^{(2)}(x_i) = -\sum_{j=1,i\neq j}^{N} r_j^{(2)}(x_i), \tag{15}$$

where L is the length of the interval (physical domain) and

$$P(x_i) = \prod_{j=0, j\neq i}^{N} \sin \frac{x_i - x_j}{2L}\pi. \tag{16}$$

2.3 Choice of the grid point distributions

The selection of locations of the sampling points plays a significant role in the accuracy of the solution of the differential equations. Using equally spaced points (uniform grid) can be considered to be a convenient and easy selection method. For a domain specified by $a \leq x \leq b$ and discretized by N points, then the coordinate of any point i can be evaluated by

$$x_i = a + \frac{i-1}{N-1}(b-a). \tag{17}$$

Quite frequently, the DQ method delivers more accurate solutions with a set of unequally spaced points (non-uniform grid). The so-called Chebyshev-Gauss-Lobatto points, which are first used by [11] and whose advantage has been discussed by [5], are well accepted in the DQ method as follows:

$$x_i = a + \frac{1}{2}\left(1 - \cos\left(\frac{i-1}{N-1}\pi\right)\right)(b-a), \tag{18}$$

for a domain $a \leq x \leq b$ again.

2.4 Implementation of boundary conditions

Proper implementation of the boundary conditions is also very important for the accurate numerical solution of differential equations. Dirichlet, Neumann, mixed and periodic boundary conditions can be approximated by DQ method. Using the DQ method for solving differential equations, we actually satisfy the governing equations at each sampling point of the domain, so we have one equation for each point, for each unknown. To satisfy the boundary conditions, at the boundary points, the boundary condition equations are satisfied instead of the governing equations. In other words, in the resulting system of algebraic equations from DQ method, each boundary condition replaces the corresponding field equation. This procedure is straightforward when there is one boundary condition at each boundary and when we have distributed the sampling points so that there is one point at each boundary.

3 Applications to Sturm-Liouville problems

The DQ method has been applied to solve (2) with the periodic boundary conditions given by (3). We suppose that the number of grid points is N with coordinates of grid points given as $0 = x_1, x_2, \cdots, x_N = \pi$. By applying the PDQ or FDQ methods, equation (2) can be discretized as

$$-\sum_{j=1}^{N} r_j^{(2)}(x_i)y(x_j) + q(x_i)y(x_i) = \lambda y(x_i)\ ,\ i = 2, \cdots, N-1, \qquad (19)$$

where $r_j^{(2)}(x_i)$ the weighting coefficients of the second order derivative given by (11) for PDQ and (15) for FDQ. It is convenient to write (19) in the following form

$$-\sum_{j=1}^{N} r_{j,i}^{(2)} y_j + q_i y_i = \lambda y_i\ ,\ i = 2, \cdots, N-1. \qquad (20)$$

Equation (20) shows an eigenvalue equation system. It involves $N-2$ equations and N unknowns. Two unknows can be eliminated using the boundary conditions. The implementation of boundary conditions is shown as follows. From the first equation of (3), the numerical condition can be written as

$$y(x_1) = y_1 = y_N = y(x_N). \qquad (21)$$

Application of the DQ method to discritize the derivatives in the second equation of (3) gives

$$\sum_{j=1}^{N} r_{j,1}^{(1)} y_j = \sum_{j=1}^{N} r_{j,N}^{(1)} y_j\ , \qquad (22)$$

where $r_{j,1}^{(1)}$ and $r_{j,N}^{(1)}$ are the weighting coefficiens of the first order derivatives given by (10) for PDQ and (14) for FDQ. Equation (22) can be rewriten in the following form

$$r_{1,1}^{(1)} y_1 + r_{N,1}^{(1)} y_N + \sum_{j=2}^{N-1} r_{j,1}^{(1)} y_j = r_{1,N}^{(1)} y_1 + r_{N,N}^{(1)} y_N + \sum_{j=2}^{N-1} r_{j,N}^{(1)} y_j. \qquad (23)$$

After substituting (21) into (23), we can solve (23) for y_1 to obtain

$$y_1 = \alpha \sum_{j=2}^{N-1} \left(r_{j,N}^{(1)} - r_{j,1}^{(1)} \right) y_j, \qquad (24)$$

where

$$\alpha = \frac{1}{\left(r_{1,1}^{(1)} y_1 + r_{N,1}^{(1)} y_N \right) - \left(r_{1,N}^{(1)} y_1 + r_{N,N}^{(1)} y_N \right)}. \qquad (25)$$

Now, using (21) and (24) in (20) leads to the following $(N-2)\times(N-2)$ eigenvalue equation system

$$\mathbf{A}\mathbf{y} = \lambda\mathbf{y}. \tag{26}$$

Here, the elements of matrix $\mathbf{A}$ are completely determined from the weighting coefficients of first and second order derivatives given explicitly in the previous section.

From equation (26), the λ values can be obtained from the eigenvalues of matrix. This can be done by using various methods. Here, we use a FORTRAN IMSL Routine called DEVLRG. Routine DEVLRG computes the eigenvalues of a real matrix. The matrix is first balanced. Elementary or Gauss similarity transformations with partial pivoting are used to reduce this balanced matrix to a real upper Hessenberg matrix. A hybrid double-shifted LR-QR algorithm is used to compute the eigenvalues of the Hessenberg matrix. Computed eigenvalues is sorted out by routine DSVRGN. This routine sorts the elements of an array into ascending order by algebraic value.

4 Numerical illustrations

To demonstrate the efficiency and accuracy of the DQ method, the sample problem, i.e. equation (2) with $q(x) = 0$ and boundary conditions given by (3), which has exact eigenvalues, namely $\lambda_k = 4k^2$, $k = 1, 2, 3, \cdots$, is first choosen for study. Both the PDQ and FDQ methods with the grid points distribution given by (18) are applied. The performance of the DQ approach is measured by the relative error ε_k which is defined as

$$\varepsilon_k = \left| \frac{\lambda_k - \lambda_k^{DQ}}{\lambda_k} \right| ,\ k = 1, 2, 3, \cdots, \tag{27}$$

where λ_k^{DQ} indicates kth algebraic eigenvalues obtained by DQ method.

Table 1 lists the relative errors of the DQ results with different number of mesh points N. Note that even if we are only interested in non-negative eigenvalues, the routine that we use here gives all of the eigenvalues of the problem. In this sample problem, it can be seen that the algebraic multiplicity of the eigenvalues is 2. So, the errors are listed for $\lambda_1 = 4, \lambda_2 = 4, \lambda_3 = 16, \lambda_4 = 16, \cdots$ and so on. It can be observed from Table 1 that in order to have good approximations to the first kth eigenvalues, at least $2k$ grid points have to be used. It is also observed that the accuracy of the computed eigenvalues by the FDQ method is better than the PDQ approach. This is especially true for the higher eigenvalues. In both methods, the computed values for the lower eigenvalues have a better accuracy than those for the higher eigenvalues. As the number of grid points further increased to above $2k$, the accuracy of the DQ results especially for the higher eigenvalues can be further improved.

Table 1: Relative errors of the DQ results with $q(x) = 0$ for several values of the number of grid points N

k	ε_k $(N = 21)$		ε_k $(N = 31)$		ε_k $(N = 41)$	
	PDQ	FDQ	PDQ	FDQ	PDQ	FDQ
1	$.266E-14$	$.289E-14$	$.175E-13$	$.389E-14$	$.222E-14$	$.733E-14$
2	$.266E-14$	$.888E-15$	$.129E-13$	$.999E-14$	$.404E-13$	$.733E-14$
3	$.213E-10$	$.133E-14$	$.122E-14$	$.400E-14$	$.355E-14$	$.266E-14$
4	$.450E-09$	$.888E-15$	$.133E-14$	$.355E-14$	$.113E-13$	$.266E-14$
5	$.818E-06$	$.197E-15$	$.987E-15$	$.138E-14$	$.138E-14$	$.592E-15$
6	$.357E-07$	$.197E-15$	$.369E-13$	$.237E-14$	$.493E-14$	$.592E-15$
7	$.189E-05$	$.155E-14$	$.121E-09$	$.111E-14$	$.155E-14$	$.444E-15$
8	$.105E-03$	$.222E-15$	$.365E-11$	$.133E-14$	$.733E-14$	$.111E-14$
9	$.386E-02$	$.568E-15$	$.129E-08$	$.156E-14$	$.118E-13$	$.426E-15$
10	$.100E-02$	$.000E+00$	$.461E-07$	$.213E-14$	$.711E-15$	$.256E-14$
11	$.117E-01$	$.420E-02$	$.425E-05$	$.395E-15$	$.272E-12$	$.987E-15$
12	$.326E-01$	$.132E-01$	$.115E-06$	$.395E-15$	$.124E-10$	$.276E-14$
13	$.492E-01$	$.730E-02$	$.272E-05$	$.435E-15$	$.257E-08$	$.145E-15$
14	$.263E+00$	$.198E+00$	$.130E-03$	$.870E-15$	$.531E-10$	$.261E-14$
15	$.488E+00$	$.397E+00$	$.259E-02$	$.130E-04$	$.380E-08$	$.777E-15$
16	$.119E+01$	$.109E+01$	$.711E-03$	$.308E-04$	$.200E-06$	$.289E-14$
17	$.192E+01$	$.197E+01$	$.665E-02$	$.169E-02$	$.706E-05$	$.105E-14$
18	$.855E+01$	$.816E+01$	$.154E-01$	$.497E-02$	$.151E-06$	$.105E-14$
19	$.100E+01$	$.100E+01$	$.445E-01$	$.281E-02$	$.540E-05$	$.568E-15$
20	$.100E+01$	$.100E+01$	$.103E+00$	$.660E-01$	$.118E-03$	$.114E-14$

In order to facilitate comparision with the results of [2,6], we also choose the same functions $q(x)$ for our numerical examples, i.e. $q(x) = 10\cos(2x)$ and $q(x) = x^2(\pi - x)$.

In Table 2, we compare the errors in computed solutions of the present work for $q(x) = 10\cos(2x)$ with the corrected $\widetilde{\Lambda}_k^{(n)}$ non-negative eigenvalues of Andrew [2]. From left to right the columns of Table 2 show, for various k, the exact eigenvalue, the errors in the corrected estimates with $n = 40, 80$, the errors in the PDQ and FDQ calculations with $N = 41$, respectively. Note that equation (18) has been used for the grid point distributions in the DQ calculations. It can be observed from the table that the PDQ method used in this work to obtain the approximate non-negative eigenvalues of the periodic Sturm-Liouville problem produced very accurate results when compared to the corrected results of Andrew [2] for $n = 40$. This is true even for $n = 80$ case, except for the last couple of eigenvalues shown in the table. It can be also observed from the table that the FDQ method produce more accurate results than the PDQ method.

Table 2: Comparison of errors in computed solutions for $q(x) = 10\cos(2x)$

k	λ_k	$\lambda_k - \widetilde{\Lambda}_k^{(40)}$	$\lambda_k - \widetilde{\Lambda}_k^{(80)}$	$\lambda_k - \lambda_k^{PDQ(41)}$	$\lambda_k - \lambda_k^{FDQ(41)}$
2	2.09946	0.0175	0.0044	−0.0000004	−0.0000004
3	7.44911	0.0235	0.0058	0.0000003	0.0000003
4	16.64822	0.0169	0.0041	0.0000001	0.0000001
5	17.09658	0.0048	0.0011	−0.0000017	−0.0000017
6	36.35887	0.0134	0.0031	0.0000032	0.0000032
7	36.36090	0.0131	0.0031	0.0000000	0.0000000
8	64.19884	0.0145	0.0033	0.0000005	0.0000005
10	100.12637	0.0158	0.0034	0.0000008	0.0000008
12	144.08745	0.0175	0.0034	0.0000027	0.0000027
14	196.06412	0.0197	0.0036	0.0000209	0.0000039
16	256.04903	0.0225	0.0037	0.0000093	0.0000044
18	324.03870	0.0263	0.0038	0.0069764	0.0000016
20	400.03133	0.0313	0.0040	−0.0084761	0.0005062

In Table 3, we compare the errors in computed solutions of the present work for $q(x) = x^2(\pi - x)$ with the corrected eigenvalues of Andrew [2] and Condon [6]. From left to right the columns of Table 3 show, for various k, the exact eigenvalue, the errors in the corrected finite difference estimate of [2] with $n = 40$ and the errors in the corrected estimates of [6] with $n = 40$. Similarly, the next two colums show the errors in the corrected estimates of [2] and [6] with $n = 80$, respectively. The last column of Table 3 shows the errors in the FDQ calculations for $N = 40$. Here, we can also see that the FDQ method accurately predicts the eigenvalues of the periodic Sturm-Liouville problem. And, the errors in computed solutions for this approach is much smaller than the other errors listed in the table obtained using correction techniques in conjunction with finite differences [2,6].

5 Conclusions

The PDQ and FDQ methods have been applied to compute the eigenvalues of periodic Sturm-Liouville problems. Through test examples, it was found that in order to obtain accurate numerical results for the first kth $(k = 1, 2, \cdots)$

Table 3: Comparison of errors in computed solutions for $q(x) = x^2(\pi - x)$

k	λ_k	$\widetilde{\Lambda}_k^{(40)} - \lambda_k$	$\widetilde{\Gamma}_k^{(40)} - \lambda_k$	$\widetilde{\Lambda}_k^{(80)} - \lambda_k$	$\widetilde{\Gamma}_k^{(80)} - \lambda_k$	$\lambda_k^{FDQ(40)} - \lambda_k$
1	–	0.0040215	−0.0017898	0.0010033	−0.0004477	–
2	6.5005	0.0035214	−0.0015784	0.0008756	−0.0003975	−0.0000093
3	7.0151	0.0001520	0.0000792	0.0000389	0.0000206	−0.0000431
4	18.5848	0.0035783	−0.0023903	0.0008826	−0.0006079	−0.0000278
5	18.6655	0.0006379	0.0006105	0.0001615	0.0001547	−0.0000185
6	38.5816	0.0038340	−0.0023911	0.0009330	−0.0006215	0.0000279
7	38.6215	0.0006269	0.0006023	0.0001613	0.0001551	0.0000425
8	66.5821	0.0040092	−0.0023175	0.0009563	−0.0006233	−0.0000521
9	66.6054	0.0006047	0.0005879	0.0001593	0.0001550	−0.0000352
10	102.5825	0.0041890	−0.0021885	0.0009728	−0.0006192	0.0000262
11	102.5977	0.0005762	0.0005649	0.0001572	0.0001541	0.0000205
12	146.5829	0.0044024	−0.0020046	0.0009874	−0.0006116	−0.0000344
13	146.5935	0.0005407	0.0005333	0.0001557	0.0001528	0.0000230
14	198.5831	0.0046700	−0.0017563	0.0010023	−0.0006010	−0.0000017
15	198.5910	0.0004960	0.0004916	0.0001525	0.0001509	−0.0000240
16	258.5833	0.0050144	−0.0014259	0.0010185	−0.0005876	−0.0000393
17	258.5893	0.0004394	0.0004373	0.0001498	0.0001485	0.0000162
18	326.5834	0.0054656	−0.0009854	0.0010365	−0.0005716	−0.0000225
19	326.5882	0.0003670	0.0003665	0.0001467	0.0001456	−0.0000248
20	402.5835	0.0060690	−0.0003908	0.0010571	−0.0005525	−0.0000434

eigenvalues of the problem, the minimum number of grid points N used in the DQ calculations must be equal to $2k$. In addition, we found that Chebyshev-Gauss-Lobatto grid point distribution is suitable for both PDQ and FDQ approaches in the accurate computation of eigenvalues.

Comparison of errors in computed solutions showed that the DQ method has the capability of producing highly accurate results using considerably small number of grid points in the computational domain with minimal computational effort even if it produces dense, non-symmetric matrices. We can also conclude that the FDQ approach is very suitable for the periodic problems discussed here. The DQ method can be also applied for computing approximate eigenvalues of Sturm-Liouville problems with the Dirichlet, Neumann and mixed types of boundary conditions.

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