

Characterization of α -convex Functions via Weakly α -monotone Bifunctions

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Abstract

In this paper, we give some properties of α -convex functions and we characterize this class of functions via some weakly α -monotone bifunctions.

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1 Introduction

It is very natural in nonsmooth analysis to characterize generalized convex functions in terms of generalized directional derivatives. Recently, several contributions related to this question have been made. Let us just mention Komlosi in [6], Sach and Penot in [5].

The main purpose of this note is to give an analogous characterization involving α -convex functions.

The paper is organized as follows: we recall first the definition of this class of functions (see Avriel in [1]) and we give some properties of regularity of such functions. Finally, we characterize α -convex functions via a new family of monotone bifunctions called class of weakly α -monotone bifunctions.

2 α -convex functions

Let X be a convex set of R^n and f be a function acting from X into R .

- f is said to be convex if for all $u, v \in X$ and $\lambda \in [0, 1]$ one has:

$$f(\lambda u + (1 - \lambda)v) \leq \lambda f(u) + (1 - \lambda)f(v).$$

- f is said to be concave if $-f$ is convex.
- f is said to be quasiconvex if for all $u, v \in X$ and $\lambda \in [0, 1]$ one has:

$$f(\lambda u + (1 - \lambda)v) \leq \max\{f(u), f(v)\}.$$

We define α -convex functions as follows:

Definition 2.1 *let f be a function acting from X into R and α be a real number.*

If $\alpha = 0$, then f is said to be α -convex if f is convex.

If $\alpha \neq 0$, then f is said to be α -convex if $\alpha e^{-\alpha f}$ is concave.

In order to give a link between the class of α -convex functions and those of quasiconvex functions, we need to use a powerful tool like the convexity index of a function, which is defined in [3] by the following way:

Definition 2.2 *Given a nonempty convex subset X of R^n and a real valued function f on X . Set $r_{\lambda, f}(x) = e^{-\lambda f(x)}$ for $x \in X$ and $\lambda \in R$. Then the convexity index $c(f)$ of f is defined as follows: if there exists $\mu < 0$ such that $r_{\mu, f}$ is not convex, then*

$$c(f) = \sup\{\lambda : \lambda < 0, r_{\lambda, f} \text{ is convex}\}.$$

Otherwise,

$$c(f) = \sup\{\lambda : \lambda \geq 0, r_{\lambda, f} \text{ is concave}\}.$$

We can easily see that f is α -convex if and only if $c(f) - \alpha$ is non negative. Assume now that X is an open convex subset of R^n . Consider the bifunction f_α , with $\alpha \in R^*$, defined on $X \times]0, +\infty[$ by:

$$f_\alpha(x, u) = f(x) + \alpha^{-1} \ln(u).$$

Then we have the following result:

Proposition 2.3 *Let f be a non constant function acting from X into R and $\alpha \in R^*$. Then the following assertions are equivalent:*

- i) f is α -convex.
- ii) f_α is quasiconvex.

Proof. By ([3], theorem 5), f_α is quasi convex iff $c(f) - \alpha$ is non negative. On the other hand, $c(f) - \alpha$ is non negative is equivalent to say that f is α -convex. Therefore, (i) is equivalent to (ii). Thus, we achieve the proof.

Proposition 2.4 *Let f be a function acting from X into R and α be a real number. Assume that f is a α -convex function. Then, f is locally Lipschitzian.*

Proof. By the definition of $c(f)$, we can easily see that there exists $\beta > 0$ such that the function $h \equiv e^{\beta f}$ is convex. Therefore, h is locally lipschitzian. Consequently, f is continuous. Let $\bar{x} \in X$. Then, there are $k > 0$ and $\lambda > 0$ such that for all $x, y \in \bar{B}(\bar{x}, \lambda)$

$$|h(x) - h(y)| \leq k \|x - y\|.$$

By the continuity of f , we deduce that f is bounded. This implies that there is $a > 0$ such that $a \leq e^{\beta f(x)}$ for all x close to \bar{x} . Hence,

$$|f(x) - f(y)| \leq \frac{k}{a} \beta^{-1} \|x - y\|.$$

Thus, we achieve the proof.

Let us recall that the directional derivative of every function f is defined, when it exists for $x \in X$ and $v \in R^n$ by:

$$f'(x, v) = \lim_{t \rightarrow 0^+} \frac{f(x + tv) - f(x)}{t}.$$

While the Clarke directional derivative is defined for $x \in X$ and $v \in R^n$ by:

$$f^0(x, v) = \limsup_{t \rightarrow 0^+, y \rightarrow x} \frac{f(y + tv) - f(y)}{t}.$$

Next, we prove that every α -convex function is regular in the following sense:

Definition 2.5 A function f is said to be regular at x if :

- i) $f'(x, v)$ exists for all $v \in R^n$.
- ii) $f'(x, v) = f^0(x, v)$ for all $v \in R^n$.

Recall that f is regular on X if it is regular at every x in X .

Proposition 2.6 Let f be a function acting from X into R . Assume that f is α -convex. Then f is regular.

Proof. By the definition of $c(f)$, there is β such that the function $h \equiv e^{\beta f}$ is convex.

Let $\bar{x} \in X$, $v \in R^n$ and $\lambda > 0$. We have

$$\frac{f(\bar{x} + \lambda v) - f(\bar{x})}{\lambda} = \beta^{-1} \frac{\ln(h(\bar{x} + \lambda v)) - \ln(h(\bar{x}))}{\lambda}.$$

Therefore,

$$\frac{f(\bar{x} + \lambda v) - f(\bar{x})}{\lambda} = (\lambda\beta)^{-1} (\ln(1 + u(\lambda)))$$

where,

$$u(\lambda) = \frac{h(\bar{x} + \lambda v) - h(\bar{x})}{h(\bar{x})}.$$

By the continuity of h , we deduce

$$\lim_{\lambda \rightarrow 0^+} \frac{f(\bar{x} + \lambda v) - f(\bar{x})}{\lambda} = \frac{h'(\bar{x}, v)}{\beta h(\bar{x})}.$$

Thus,

$$f'(\bar{x}, v) = \frac{h'(\bar{x}, v)}{\beta h(\bar{x})}. \quad (1.1)$$

Let us now show that $f'(\bar{x}, v) = f^0(\bar{x}, v)$.

Since h is convex, then h is regular at \bar{x} . By the definition of Clarke directional derivative, there are sequences $t_n \downarrow 0^+$ and $y_n \rightarrow \bar{x}$ such that

$$\limsup_{t \rightarrow 0^+, y \rightarrow \bar{x}} \frac{f(y + tv) - f(y)}{t} = \lim_n \frac{f(y_n + t_n v) - f(y_n)}{t_n}.$$

On the other hand,

$$f^0(\bar{x}, v) = \beta^{-1} \limsup_{t \rightarrow 0^+, y \rightarrow \bar{x}} t^{-1} \ln(1 + u(t, y)),$$

where $u(t, y) = \frac{h(y+tv)-h(y)}{h(y)}$. Therefore, since $u(t_n, y_n) \rightarrow 0$ and $\ln(1 + s) \leq s$ for s small enough, then $f^0(\bar{x}, v) \leq \beta^{-1} \frac{h^0(\bar{x}, v)}{h(\bar{x})} = \beta^{-1} \frac{h'(\bar{x}, v)}{h(\bar{x})} = f'(\bar{x}, v)$. Hence, $f'(x, v) = f^0(x, v)$. Thus, we achieve the proof.

Next, we show that the class of α -convex functions is stable under positive scalar multiplication and summation under certain conditions.

Proposition 2.7 *Let f be a function acting from X into R , $\lambda \in R_+^*$, $k \in R$, $\alpha \in R$. Assume that f is α -convex. Then the function $h \equiv \lambda f + k$ is $(\alpha\lambda^{-1})$ -convex.*

Proof. Without loss of generality, we can assume that $\alpha \neq 0$. Since f is α -convex, then the function $g \equiv \alpha e^{-\alpha f}$ is concave. Therefore, $g_\lambda \equiv (\lambda^{-1}\alpha)e^{-\alpha\lambda^{-1}(\lambda f)}$ is also concave. Consequently, h is $(\alpha\lambda^{-1})$ -convex.

Definition 2.8 *Assume that X and Y are non-empty open convex subsets of R^n and R^p respectively, f and g are real-valued functions on X and Y respectively. we define the direct sum $f \oplus g$ to be the function s acting from $X \times Y$ into R and defined by $s(x, y) = f(x) + g(y)$.*

Theorem 2.9 *Let X and Y be non-empty open convex subsets of R^n and R^p respectively, f and g be non constant real-valued functions on X and Y respectively. Let $\alpha_1, \alpha_2 > 0$. Assume that f and g are respectively α_1 and α_2 -convex. Then $(f \oplus g)$ is r -convex with $r = (\alpha_1^{-1} + \alpha_2^{-1})^{-1}$.*

Proof. Since $c(f) + c(g) \geq 0$ and f, g are non constant, then by ([3], theorem 5), $f \oplus g$ is quasiconvex. Consequently, using ([3], theorem 9), we deduce $\frac{1}{c(f \oplus g)} = \frac{1}{c(f)} + \frac{1}{c(g)}$. On the other hand, f is α_1 -convex and g is α_2 -convex. Hence, by ([3], theorem 8), it follows that $(f \oplus g)_r$ is quasiconvex. Therefore, $f \oplus g$ is r -convex. Thus, we achieve the proof.

3 α -monotone and weakly α -monotone bifunctions

Let X be a subset of R^n and let F be a bifunction acting from $X \times R^n$ into R . F is said to be monotone if for any $x, y \in X$ one has

$$F(x, y - x) - F(y, y - x) \leq 0.$$

F is said to be quasimonotone if for any $x, y \in X$ one has

$$F(x, y - x) > 0 \Rightarrow F(y, y - x) \geq 0.$$

F is said to be pseudomonotone if for any $x, y \in X$ one has

$$F(x, y - x) > 0 \implies F(y, y - x) > 0.$$

Recall that $M \implies PM \implies QM$.

In this section, we introduce a new class of monotone bifunctions called class

of weakly α -monotone bifunctions and we characterize α -convex functions via this new kind of monotone bifunctions.

Let F be a bifunction acting from $X \times R^n$ into R , α be a real number such that $\alpha \neq 0$. By definition, F_α is the function acting from $X \times]0, +\infty[\times R^n \times R$ into R and defined by: $F^\alpha(x, t, v, u) = F(x, v) + \frac{u}{\alpha t}$.

Definition 3.1 Let F be a bifunction acting from $X \times R^n$ into R . F is said to be anti-quasimonotone if the following implication holds:
 $F(x, x - y) \geq 0 \Rightarrow F(y, x - y) \geq 0$. That is $-F$ is quasimonotone.

Definition 3.2 Let F be a bifunction acting from $X \times R^n$ into R . F is said to be α -monotone if $\forall x, y \in X$ $F(x, y - x) - F(y, y - x) \geq \alpha$.

Definition 3.3 Let F be a bifunction acting from $X \times R^n$ into R . F is said to be weakly α -monotone if F^α is anti-quasimonotone as a function acting from $(X \times R^n) \times R^{n+1}$ into R .

We have the following results which are immediate consequences of definitions 11 and 12.

Proposition 3.4 If F is s -monotone and $s \geq r$, then F is r -monotone.

Proposition 3.5 Let F be a bifunction acting from $X \times R^n$ into R . Assume that F is α -monotone. Then F is weakly α -monotone.

4 Characterization of α -convex functions via weakly α -monotone bifunctions

Let X be an open convex subset of R^n and $\alpha \in R^*$, f be a function acting from X into R . Recall that the upper and lower Dini directional derivatives are respectively defined by :

$$f^{D+}(x, v) = \limsup_{t \rightarrow 0_+} \frac{f(x + tv) - f(x)}{t},$$

$$f^{D-}(x, v) = \liminf_{t \rightarrow 0_+} \frac{f(x + tv) - f(x)}{t}.$$

Analogously, the upper and lower Hadamard directional derivatives are respectively defined by :

$$f^{H+}(x, v) = \limsup_{t \rightarrow 0_+, u \rightarrow v} \frac{f(x + tu) - f(x)}{t}.$$

$$f^{H^-}(x, v) = \liminf_{t \rightarrow 0^+, u \rightarrow v} \frac{f(x + tu) - f(x)}{t}.$$

The class of α -convex functions are characterized via weakly α -monotone bi-functions as follows:

Theorem 4.1 *Let f be a continuous function acting from X into R , $\alpha \in R^*$. Then we have the following implications:*

- a) f is α -convex $\Rightarrow f^{D^-}$ is weakly α -monotone.
- b) f^{H^-} is weakly α -monotone $\Rightarrow f$ is α -convex.

Proof. a) Suppose that f is α -convex. First let us remark that $(f_\alpha)^{D^-} = (f^{D^-})^\alpha$. Since f is α -convex, then by proposition 3, f_α is quasiconvex. Consequently, applying theorem 2.1 of [5], we deduce that $(f_\alpha)^{D^+}$ is quasimonotone. Therefore, $(f_\alpha)^{D^-}$ is anti-quasimonotone. On the other hand, $(f_\alpha)^{D^-} = (f^{D^-})^\alpha$. Hence, f^{D^-} is weakly α -monotone.

b) Assume now that f^{H^-} is weakly α -monotone. Then, $(f^{H^-})^\alpha$ is anti-quasimonotone. On the other hand, $(f^{H^-})^\alpha = (f_\alpha)^{H^-}$. Consequently, $(f_\alpha)^{H^+}$ is quasimonotone. Therefore, by ([5], theorem 2.1), we deduce that f_α is quasiconvex. Hence, by proposition 3, f is α -convex. Thus, we achieve the proof.

Corollary 4.2 *Let f be a continuous function acting from X into R , $\alpha \in R^*$. Assume that f^{H^-} is α -monotone. That is $\forall x, y \in X \quad f^{H^-}(x, y - x) - f^{H^-}(y, y - x) \geq \alpha$. Then f is α -convex.*

Proof. Using proposition 14 and (theorem 15, b)), we deduce the result.

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