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Some Remarks on Minimal Spline Interpolation

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Abstract

Interpolation of $2n+k$ data by spline spaces of order k with n knots is studied. Some conditions on existence of interpolating splines are given. Numerical methods for constructing a solution are suggested.

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0. Introduction

Grandine [1] has given a characterization of spline spaces of order k with a set of n simple knots, i.e., linear spaces of dimension $n+k$, from which Lagrange interpolation at $2n+k$ data might be possible. Recently, we have extended his results to Hermite interpolation by splines with multiple knots and have obtained some results on uniqueness of interpolating splines [2].

In this paper we give some conditions on existence of interpolating splines. In the case of linear splines we establish a simple characterization of the spline solution. Finally, we make some comments on determining the solutions numerically applying the Newton method for vector-valued functions.

1. Characterization of Minimal Spline Interpolation

The considered Hermite problem of interpolating $2n+k$ data by a spline of order k ($k \geq 2$) with n knots ($n \geq 1$) counting multiplicities is as follows: Let be given real data points

$$x_0 < \ldots < x_l,$$

each of them of multiplicity $m_i + 1$ ($m_i \geq 0$), $i = 0, \ldots, l$, such that

$$\rho = \max_{0 \leq i \leq l} m_i \leq k-2, \quad \sum_{i=0}^{l} (m_i + 1) = 2n + k,$$

and a data vector $\{y_{ji}\}_{j=0}^{m_i}{}_{i=0}^{l}$. We arrange the data points according to their multiplicities by

$$\{z_0, \ldots, z_{2n+k-1}\} = \{\underbrace{x_0, \ldots, x_0}_{m_0+1}, \ldots, \underbrace{x_l, \ldots, x_l}_{m_l+1}\}, \tag{1.1}$$

and set, for $h \in C^\rho[x_0, x_l]$, the jth divided difference of h with respect to $\{z_i, \ldots, z_{i+j}\}$ by

$$\mu_i^j h = [z_i, \ldots, z_{i+j}]h, \ i, j \geq 0$$

where

$$[z_p, \ldots, z_{p+q}]h = \frac{1}{q!} h^{(q)}(z_p)$$

if $z_p = \ldots = z_{p+q}$. Replacing h by a function $y \in C^\rho[x_0, x_l]$ such that

$$y^{(j)}(x_i) = y_{ji}, \ j = 0, \ldots, m_i, \ i = 0, \ldots, l,$$

we analogously define the jth divided difference of the given data vector $\{y_{ji}\}_{j=0}^{m_i}{}_{i=0}^{l}$ with respect to $\{z_i, \ldots, z_{i+j}\}$ by

$$\mu_i^j y = [z_i, \ldots, z_{i+j}]y, \ i, j \geq 0.$$

The spline space $S_k(T)$ of order k of interest is defined by a knot vector

$$T : t_0 = x_0 < t_1 < \ldots < t_r < x_l = t_{r+1} \tag{1.2}$$

where each t_i has multiplicity $1 \leq k_i \leq k - 1 - \rho$, $i = 1, \ldots, r$, such that

$$\sum_{i=1}^{r} k_i = n,$$

i.e.,

$$\begin{aligned} S_k(T) \ = \ & \{s : [t_0, t_{r+1}] \to \mathbb{R} : s_-^{(j)}(t_i) = s_+^{(j)}(t_i), \ j = 0, \ldots, k - k_i - 1, \\ & i = 1, \ldots, r, \ s|_{[t_i, t_{i+1}]} \in \Pi_{k-1}, \ i = 0, \ldots, r\} \end{aligned}$$

(Π_{k-1} denotes the linear space of polynomials of degree at most $k-1$). It is well-known that $\dim S_k(T) = n + k$.

We are interested in solving the minimal Hermite interpolation problem (H): Under what conditions on the data does a knot vector T of the above type exist such that

$$s^{(j)}(x_i) = y_{ji}, \ j = 0, \ldots, m_i, \ i = 0, \ldots, l, \tag{H}$$

for some $s \in S_k(T)$?

An answer to this question is closely related to certain properties of the class $\{B_i\}_{i=0}^{2n-1}$ of B-splines of order k to the knots $\{z_j\}_{j=0}^{2n+k-1}$. These functions satisfy the well-known representation

$$B_i(t) = (-1)^k k[z_i, \ldots, z_{i+k}](t - \cdot)_+^{k-1},\ i = 0, \ldots, 2n-1$$

where the kth divided difference is evaluated with respect to the function $x \to (t-x)_+^{k-1}$ (for details see e. g. [2]).

The main result in [2] can now be stated as follows.

Theorem 1. *Consider any set of $2n+k$ ordered data points as in (1.1), any data vector $\{y_{ji}\}_{j=0}^{m_i}{}_{i=0}^{l}$ and any knot vector T as in (1.2). Then the following statements are equivalent:*

(1) There exists a solution $s \in S_k(T)$ of problem (H).

(2) The vector of the kth divided differences of the data, i.e.,

$$\widetilde{\mathbf{y}} = (\mu_0^k y, \mu_1^k y, \ldots, \mu_{2n-1}^k y)^t$$

can be written as a linear combination of the columns of

$$B(T) = \begin{pmatrix} B_0(t_1) & \cdots & B_0^{(k_1-1)}(t_1) & \cdots & B_0(t_r) & \cdots & B_0^{(k_r-1)}(t_r) \\ B_1(t_1) & \cdots & B_1^{(k_1-1)}(t_1) & \cdots & B_1(t_r) & \cdots & B_1^{(k_r-1)}(t_r) \\ \vdots & & \vdots & & \vdots & & \vdots \\ B_{2n-1}(t_1) & \cdots & B_{2n-1}^{(k_1-1)}(t_1) & \cdots & B_{2n-1}(t_r) & \cdots & B_{2n-1}^{(k_r-1)}(t_r) \end{pmatrix}.$$

Remarks. (1) Theorem 1 is an extension of the main result in [1] where a characterization is given for the case when $m_i = 0$ for all i in (1.1) and $k_i = 1$ for all i in (1.2), i.e., Lagrange interpolation by splines with simple knots. In addition, it is supposed in [1] that the solution knot vector T satisfies an interlacing property.
(2) As we have shown in [2], problem (H) is not uniquely solvable, in general. But assuming an interlacing property

$$z_{2i-1} < \hat{t}_i < z_{2i+k-2},\ i = 1, \ldots, n \tag{IP}$$

for every solution knot vector $T = \{t_i\}_{i=0}^{r+1}$ where

$$\hat{t}_0 = t_0,\ \hat{t}_1 = \ldots = \hat{t}_{k_1} = t_1,\ \ldots\ , \hat{t}_{n-k_r+1} = \ldots = \hat{t}_n = t_r, \hat{t}_{n+1} = t_{r+1},$$

in [2] we have shown the uniqueness of the interpolating spline $s \in S_k(T)$.

An important role for the following arguments plays a result on the sign behavior of $\widetilde{\mathbf{y}}$ given in [2].

Theorem 2. *If statement (2) of Theorem 1 is true, it is necessary that the vector* $\widetilde{\mathbf{y}}$ *has at most* $n-1$ *sign changes, i.e., there do not exist indices* $0 \le i_0 < \ldots < i_n \le 2n-1$ *such that*

$$\mu^k_{i_j} y \cdot \mu^k_{i_{j+1}} y < 0,\ j = 0, \ldots, n-1.$$

2. Necessary Conditions for Spline Solutions

Using the arguments in the proof of Theorem 2 we obtain additional information on the sign behavior of the vector of the kth divided differences.

Let us assume that problem (H) has a solution $s \in S_k(T)$. This implies that

$$\widetilde{\mathbf{y}} = (\mu^k_0 y, \mu^k_1 y, \ldots, \mu^k_{2n-1} y)^t = (\mu^k_0 s, \mu^k_1 s, \ldots, \mu^k_{2n-1} s)^t.$$

To simplify notations we omit the upper index k of the components of $\widetilde{\mathbf{y}}$ in the following statements.

Theorem 3. *Suppose that for some indices* $0 \le \rho_0 < \ldots < \rho_j \le 2n-1$ $(j \ge 1)$ *the following is true:*

- $I = \bigcup_{p=0}^{j} I_{\rho_p}$ *is an open interval where* $I_{\rho_p} = (z_{\rho_p}, z_{\rho_p+k})$, $p = 0, \ldots, j$;
- $I \cap T = \{\widetilde{t}_0, \ldots, \widetilde{t}_{j-1}\} = \{\underbrace{t_m, \ldots, t_m}_{k_m}, \underbrace{t_{m+1}, \ldots, t_{m+1}}_{k_{m+1}}, \ldots, \underbrace{t_q, \ldots, t_q}_{k_q}\}$, *for some* $1 \le m \le q \le r$;
- $\widetilde{t}_p \in I_{\lambda_p}$ *(resp.* $\widetilde{t}_p \in I_{\lambda_p} \cap I_{\lambda_{p+1}}$*),* $p = 0, \ldots, j-1$, *for some indices* $\rho_0 \le \lambda_0 < \lambda_1 < \ldots < \lambda_{j-1} \le \rho_j$.

Then the vector

$$(\mu_{\rho_0} s, \mu_{\rho_1} s, \ldots, \mu_{\rho_j} s)$$

has at most $j-1$ *sign changes (resp. at most* $j-1$ *weak sign changes or each component is zero).*

Proof. First note that

$$j = \sum_{i=m}^{q} k_i \le n$$

by (1.2). We follow the lines in the proof of proposition 3 in [2]. In view of our hypotheses, property (3.13) in that proof is satisfied (with $\sigma = 0$). Let B_{ρ_p} be the B-spline of order k with the knots $\{z_i\}_{i=\rho_p}^{\rho_p+k}$, i.e.,

$$\operatorname{supp} B_{\rho_p} = [z_{\rho_p}, z_{\rho_p+k}],\ p = 0, \ldots, j.$$

Suppose first that

$$\widetilde{t}_p \in I_{\lambda_p} = (z_{\lambda_p}, z_{\lambda_p+k}) = \operatorname{int} \operatorname{supp} B_{\lambda_p},\ p = 0, \ldots, j-1.$$

Now using properties (2.12) and (3.14) in [2] we obtain

- $\det M(\rho_0, \ldots, \rho_{p-1}, \rho_{p+1}, \ldots, \rho_j) \geq 0,\ p = 0, \ldots, j,$
- $\det M(\lambda_0, \ldots, \lambda_{j-1}) > 0,$
- $0 = \sum_{p=0}^{j} (-1)^{j+p}\, \mu_{\rho_p} s \, \det M(\rho_0, \ldots, \rho_{p-1}, \rho_{p+1}, \ldots, \rho_j).$ (2.1)

$M(\nu_1, \ldots, \nu_j)$ denotes a $(j \times j)$-submatrix of $B(T)$ where $\rho_0 \leq \nu_1 < \nu_2 < \ldots < \nu_j \leq \rho_j$ defined by

$$M(\nu_1, \ldots, \nu_j) =$$

$$\begin{pmatrix} B_{\nu_1}(t_m) & \cdots & B_{\nu_1}^{(k_m-1)}(t_m) & \cdots & B_{\nu_1}(t_q) & \cdots & B_{\nu_1}^{(k_q-1)}(t_q) \\ \vdots & & & & & & \\ B_{\nu_j}(t_m) & \cdots & B_{\nu_j}^{(k_m-1)}(t_m) & \cdots & B_{\nu_j}(t_q) & \cdots & B_{\nu_j}^{(k_q-1)}(t_q) \end{pmatrix}.$$

Assume now that $(\mu_{\rho_0} s, \ldots, \mu_{\rho_j} s)$ has j sign changes, i.e.,

$$\mu_{\rho_{p-1}} s \cdot \mu_{\rho_p} s < 0,\ p = 1, \ldots, j. \tag{2.2}$$

This, in particular, implies that $\mu_{\rho_p} s \neq 0$ for all p.
Hence each summand in (2.1) has the same weak sign (nonpositive or nonnegative). Since the resulting sum is zero, each term in (2.1) must vanish which in view of (2.2) implies that

$$\det M(\rho_0, \ldots, \rho_{p-1}, \rho_{p+1}, \ldots, \rho_j) = 0,\ p = 0, \ldots, j.$$

This, however, contradicts the fact that

$$\det M(\lambda_0, \ldots, \lambda_{j-1}) > 0$$

for some indices $\rho_0 \leq \lambda_0 < \ldots < \lambda_{j-1} \leq \rho_j$.
Thus we have shown that $(\mu_{\rho_0} s, \ldots, \mu_{\rho_j} s)$ has at most $j-1$ sign changes.
To prove the second statement assume that the vector

$$(\mu_{\rho_0} s, \ldots, \mu_{\rho_j} s)$$

has j weak sign changes, i.e.,

$$\mu_{\rho_{p-1}} s \cdot \mu_{\rho_p}\, s \leq 0,\ p = 1, \ldots, j,$$

and

$$\mu_{\rho_l} s \neq 0$$

for some $l \in \{0, \ldots, j\}$. Since by assumption

$$\widetilde{t}_p \in I_{\lambda_p} \cap I_{\lambda_{p+1}} = \text{int supp}\, B_{\lambda_p} \cap \text{int supp}\, B_{\lambda_{p+1}},$$

it follows from (2.12) in [2] that

$$\det M(\rho_0, \ldots, \rho_{p-1}, \rho_{p+1}, \ldots, \rho_j) > 0,\ p = 0, \ldots, j.$$

Now analogously arguing as in the first case we can verify that each summand in (2.1) must vanish. But in view of the above properties, we have got

$$\mu_{\rho_l} s\ \det M(\rho_0, \ldots, \rho_{l-1}, \rho_{l+1}, \ldots, \rho_j) \neq 0,$$

a contradiction.

Thus we have shown that the vector $(\mu_{\rho_0} s, \ldots, \mu_{\rho_j} s)$ is either zero or has at most $j-1$ weak sign changes. □

Remarks. (1) While the vector $(\mu_0 y, \ldots, \mu_{2n-1} y)$ has at most $n-1$ sign changes by Theorem 2 (if a solution exists), the number of weak sign changes may not be bounded by $n-1$. To show it let $n = k = 2$, $x_i = i$, $i = 0, \ldots, 5$ and

$$(\mu_0^2 y, \ldots, \mu_3^2 y) = (0, 0, 2, 1).$$

It is easily seen that

$$\begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} B_0(t_1) \\ \vdots \\ B_3(t_1) \end{pmatrix} + c_2 \begin{pmatrix} B_0(t_2) \\ \vdots \\ B_3(t_2) \end{pmatrix} = c_1 \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 0 \\ 2/3 \\ 1/3 \end{pmatrix}$$

where $c_1 \in \mathbb{R}$ arbitrarily, $t_1 = 3/2$ and $t_2 = 10/3$. Hence by Theorem 1, $T = \{0, 3/2, 10/3, 5\}$ is a solution knot vector. Moreover, the vector $(0, 0, 2, 1)$ has n $(= 2)$ weak sign changes. This is no contradiction to Theorem 3, because neither $(0, 0, 2)$ nor $(0, 0, 1)$ satisfy all the hypotheses of the theorem. In the first case we obtain $I = (x_0, x_4) = (0, 4)$, $I \cap T = \{t_1, t_2\}$, however $t_2 \notin I_1 \cap I_2 = (1, 3) \cap (2, 4) = (2, 3)$. A similar argument holds in the second case.

Note that T satisfies property (IP).

(2) The vector of the kth divided differences can even have the maximal number of $2n-1$ weak sign changes: As in (1) let $n = k = 2$, $x_i = i$, $i = 0, \ldots, 5$ and

$$(\mu_0^2 y, \ldots, \mu_3^2 y) = (1, 0, 0, -1).$$

Again using Theorem 1 it is easily verified that $T = \{0, 1, 4, 5\}$ is a solution knot vector. Nevertheless, it follows that the vector $(1, 0, 0, -1)$ has the maximal number of $2n-1$ $(= 3)$ weak sign changes.

In the case when $k = 3$ we obtain some more information on the number of sign changes of the vector of divided differences. Moreover, if $k = 2$, we are even able to give a simple construction of a solution knot vector (if it exists).

Corollary 4. *Let $k = 3$ and assume that $m_i = 0$, i.e., $z_i = x_i$, $i = 0, \ldots, 2n+2$ (simple knots). Moreover, assume that problem (H) has a solution $s \in S_3(T)$, where T satisfies property (IP). Then the following statements hold:*

(i) *The vectors* $(\mu_0 s, \mu_1 s, \mu_2 s)$ *and* $(\mu_{2n-3} s, \mu_{2n-2} s, \mu_{2n-1} s)$ *have at most one sign change;*

(ii) *if* $t_2 \in (x_3, x_4)$ *resp.* $t_{n-1} \in (x_{2n-2}, x_{2n-1})$, *then* $(\mu_0 s, \mu_1 s, \mu_2 s)$ *resp.* $(\mu_{2n-3} s, \mu_{2n-2} s, \mu_{2n-1} s)$ *has at most one weak sign change or each component is zero;*

(iii) *the vector* $(\mu_{2i-1} s, \mu_{2i} s, \mu_{2i+1} s, \mu_{2i+2} s)$ *has at most two sign changes,* $i = 1, \ldots, n-2$*;*

(iv) *if* $t_i \in (x_{2i}, x_{2i+1})$ *and* $t_{i+2} \in (x_{2i+3}, x_{2i+4})$, *then* $(\mu_{2i-1} s, \ldots, \mu_{2i+2} s)$ *has at most two weak sign changes or each component is zero,* $i = 1, \ldots, n-2$.

Proof. To simplify the proof we may assume that the solution vector T consists of simple knots only. Hence, $T = \{x_0, t_1, \ldots, t_n, x_{2n+2}\}$, $t_i < t_j$, if $i < j$. Since T satisfies property (IP), it follows that

$$x_{2i-1} < t_i < x_{2i+1}, \; i = 1, \ldots, n. \tag{2.3}$$

To prove statement (i) we apply Theorem 3: Set $\rho_p = p$, $p = 0, 1, 2$. Then $I_p = (x_p, x_{p+3})$, $p = 0, 1, 2$, and $I = \bigcup_{p=0}^{2} I_p = (x_0, x_5)$. Moreover, it follows from (2.3) that

$$T \cap (x_0, x_5) = \{t_1, t_2\}$$

and

$$t_1 \in (x_1, x_3) \subset I_0 \cap I_1, \; t_2 \in (x_3, x_5) \subset I_2.$$

Hence by Theorem 3, $(\mu_0 s, \mu_1 s, \mu_2 s)$ has at most one sign change. The same property follows analogously for $(\mu_{2n-3} s, \mu_{2n-2} s, \mu_{2n-1} s)$.
This proves (i).
To prove statement (ii) assume that $t_2 \in (x_3, x_4)$. This implies that

$$t_1 \in I_0 \cap I_1, \; t_2 \in I_1 \cap I_2,$$

and by Theorem 3, $(\mu_0 s, \mu_1 s, \mu_2 s)$ has at most one weak sign change or each component is zero.
The statement for $(\mu_{2n-3} s, \mu_{2n-2} s, \mu_{2n-1} s)$ follows analogously.
This proves (ii).
To prove (iii) resp. (iv) we again apply Theorem 3: Set $\rho_p = 2i - 1 + p$, $p = 0, \ldots, 3$. Then $I_p = (x_{2i-1+p}, x_{2i+2+p})$, $p = 0, \ldots, 3$, and $I = \bigcup_{p=0}^{3} I_p = (x_{2i-1}, x_{2i+5})$. From (2.3) it follows that

$$T \cap I = \{t_i, t_{i+1}, t_{i+2}\}$$

and

$$t_i \in (x_{2i-1}, x_{2i+1}) \subset I_0,\ t_{i+1} \in (x_{2i+1}, x_{2i+3}) \subset I_1 \cap I_2,$$
$$t_{i+2} \in (x_{2i+3}, x_{2i+5}) \subset I_3.$$

Hence statement (iii) follows from Theorem 3. Statement (iv) follows analogously, because then

$$t_i \in (x_{2i}, x_{2i+1}) \subset I_0 \cap I_1,\ t_{i+2} \in (x_{2i+3}, x_{2i+4}) \subset I_2 \cap I_3.$$

This completes the proof of Corollary 4. □

In practise one would be interested in an algorithm to construct a solution vector T of the considered interpolation problem. In the case when $n = 1$ (one interior knot) Grandine [1] has given a simple condition for an optimal knot $x_1 < t_1 < x_k$. If it exists, it is a solution of

$$f(x) = \det \begin{pmatrix} B_0(x) & \mu_0^k y \\ B_1(x) & \mu_1^k y \end{pmatrix} = 0.$$

In the following we establish a method to determine optimal knots in the case when $k = 2$: Let $2n+2$ ordered data points $\{(x_i, y_i)\}_{i=0}^{2n+1}$ be given. We are interested in a solution of

$$s(x_i) = y_i,\ i = 0, \dots, 2n+1 \tag{2.4}$$

by a spline $s \in S_2(T)$ where

$$T : t_0 = x_0 < t_1 < \dots < t_n < x_{2n+1} = t_{n+1}.$$

To obtain a characterization we suppose a weak interlacing property for T:

$$x_{2i-1} \le t_i \le x_{2i},\ i = 1, \dots, n. \tag{2.5}$$

It is easily verified (see the definition in Section 1) that the linear B-splines $\{B_i\}_{i=0}^{2n-1}$ to the knots $\{x_i\}_{i=0}^{2n+1}$ are given by

$$B_i(x) = \beta_i \cdot \begin{cases} \frac{x - x_i}{x_{i+1} - x_i}, & \text{if } x_i \le x \le x_{i+1}, \\ \frac{x_{i+2} - x}{x_{i+2} - x_{i+1}}, & \text{if } x_{i+1} < x \le x_{i+2}, \\ 0, & \text{elsewhere,} \end{cases}$$

where $\beta_i = \frac{2}{x_{i+2} - x_i}$, $i = 0, \dots, 2n-1$. Then the matrix $B(T)$ in Theorem 1 has the form

$$B(T) = \begin{pmatrix} B_0(t_1) & \cdots & B_0(t_n) \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \vdots & & \vdots \\ B_{2n-1}(t_1) & \cdots & B_{2n-1}(t_n) \end{pmatrix} = \begin{pmatrix} \beta_0 \frac{x_2 - t_1}{x_2 - x_1} & 0 & \cdots & \cdots \\ \beta_1 \frac{t_1 - x_1}{x_2 - x_1} & 0 & \cdots & \cdots \\ 0 & \beta_2 \frac{x_4 - t_2}{x_4 - x_3} & 0 & \cdots \\ \vdots & \beta_3 \frac{t_2 - x_3}{x_4 - x_3} & 0 & \cdots \\ \vdots & 0 & \ddots & \cdots \end{pmatrix}.$$

Assume now that

$$\widetilde{\mathbf{y}} = (\mu_0 y, \dots, \mu_{2n-1} y)^t,$$

the vector of the second divided differences of the data, and suppose that for a vector $\mathbf{c} = (c_1, \dots, c_n)^t$

$$\widetilde{\mathbf{y}} = B(T)\mathbf{c}$$

or, equivalently,

$$c_i \beta_{2i-2} \frac{x_{2i} - t_i}{x_{2i} - x_{2i-1}} = \mu_{2i-2} y, \quad c_i \beta_{2i-1} \frac{t_i - x_{2i-1}}{x_{2i} - x_{2i-1}} = \mu_{2i-1} y, \quad i = 1, \dots, n.$$

Thus for each i we have a system of two equations for c_i and t_i which are independent of the corresponding system with $j \neq i$. To obtain solutions we distinguish several cases.

(i) Assume that $\mu_{2i-2} y = \mu_{2i-1} y = 0$. We set $c_i = 0$, and take any $t_i \in [x_{2i-1}, x_{2i}]$.

(ii) Assume that $\mu_{2i-2} y = 0$ and $\mu_{2i-1} y \neq 0$. We set $c_i = \frac{\mu_{2i-1} y}{\beta_{2i-1}}$ and $t_i = x_{2i}$.

(iii) Assume that $\mu_{2i-2} y \neq 0$ and $\mu_{2i-1} y = 0$. We set $c_i = \frac{\mu_{2i-2} y}{\beta_{2i-2}}$ and $t_i = x_{2i-1}$.

(iv) Assume that $\mu_{2i-2} y \cdot \mu_{2i-1} y \neq 0$. Then, since $\beta_i > 0$ for each i, and by (2.5),

$$\beta_{2i-2} \frac{x_{2i} - t_i}{x_{2i} - x_{2i-1}} > 0, \quad \beta_{2i-1} \frac{t_i - x_{2i-1}}{x_{2i} - x_{2i-1}} > 0$$

which implies that a solution (c_i, t_i) exists if and only if $\mu_{2i-2} y \cdot \mu_{2i-1} y > 0$. In this case we obtain a unique solution

$$t_i = \frac{1}{\beta_{2i-1} \mu_{2i-2} y + \beta_{2i-2} \mu_{2i-1} y} [x_{2i-1} \beta_{2i-1} \mu_{2i-2} y + x_{2i} \beta_{2i-2} \mu_{2i-1} y].$$

(This formula remains true if only one of the terms $\{\mu_{2i-2} y, \mu_{2i-1} y\}$ is nonzero. The resulting knot is the same one as in the corresponding cases (ii) or (iii).)

Hence we have shown the following statement.

Theorem 5. *Under the above hypotheses there exists a solution $s \in S_2(T)$ of (2.4) if and only if*

$$\mu_{2i-2} y \cdot \mu_{2i-1} y \geq 0, \ i = 1, \dots, n.$$

In this case we obtain a solution knot vector T as follows:

If $\mu_{2i-2} y = \mu_{2i-1} y = 0$, then any $t_i \in [x_{2i-1}, x_{2i}]$, otherwise,

$$t_i = \frac{1}{\beta_{2i-1} \mu_{2i-2} y + \beta_{2i-2} \mu_{2i-1} y} [x_{2i-1} \beta_{2i-1} \mu_{2i-2} y + x_{2i} \beta_{2i-2} \mu_{2i-1} y].$$

Remark. If the vector $\widetilde{\mathbf{y}}$ has a strong sign change from $\mu_{2i-2}y$ to $\mu_{2i-1}y$ for some i, then there does not exist any solution $s \in S_2(T)$ of (2.4). On the other hand, $\widetilde{\mathbf{y}}$ can have $n-1$ strong sign changes, but each of them from $\mu_{2i-1}y$ to $\mu_{2i}y$, and a solution would exist by Theorem 5.

3. Some Numerical Tests

Grandine [1] has presented some numerical examples on optimal spline interpolation having computed the knots as solutions of a nonlinear least-squares problem. In the following we also study the problem of determining a solution $s \in S_k(T)$ of (H) numerically (at least for the case of Lagrange interpolation). We apply the Newton method for vector-valued functions. To develop it let us assume that for a given set $\{(x_i, y_i)\}_{i=0}^{2n+k-1}$ of ordered data points there exists a knot vector

$$T : t_0 = x_0 < t_1 < \ldots < t_n < t_{n+1} = x_{2n+k-1}$$

and an $s \in S_k(T)$ such that

$$s(x_i) = y_i,\ i = 0, \ldots, 2n+k-1.$$

Moreover, assume that T satisfies property (IP), i.e.,

$$x_{2i-1} < t_i < x_{2i+k-2},\ i = 1, \ldots, n.$$

Then, in view of Theorem 4.3 in [3], the solution s is unique under this assumption, and the vector $\widetilde{\mathbf{y}} = (\mu_0^k y, \ldots, \mu_{2n-1}^k y)^t = (\mu_0^k s, \ldots, \mu_{2n-1}^k s)^t$ satisfies

$$\underbrace{\begin{pmatrix} B_0(t_1) & \cdots & B_0(t_n) \\ \vdots & & \vdots \\ B_{2n-1}(t_1) & \cdots & B_{2n-1}(t_n) \end{pmatrix}}_{B(T)} \underbrace{\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}}_{\mathbf{c}} = \begin{pmatrix} \mu_0^k y \\ \vdots \\ \mu_{2n-1}^k y \end{pmatrix}$$

resp.

$$B(T)\,\mathbf{c} - \widetilde{\mathbf{y}} = \mathbf{0} \in \mathbb{R}^{2n},$$

B_i the B-spline of order k to the knots $\{x_j\}_{j=i}^{i+k}$, $i = 0, \ldots, 2n-1$.
To apply the Newton method we set

$$\begin{aligned} F(\mathbf{c}, \mathbf{t}) &= F(c_1, \ldots, c_n, t_1, \ldots, t_n) := B(T)\,\mathbf{c} - \widetilde{\mathbf{y}} \\ &= \begin{pmatrix} \sum\limits_{j=1}^{n} c_j B_0(t_j) - \mu_0^k y \\ \vdots \\ \sum\limits_{j=1}^{n} c_j B_{2n-1}(t_j) - \mu_{2n-1}^k y \end{pmatrix} = \begin{pmatrix} F_0(\mathbf{c}, \mathbf{t}) \\ \vdots \\ \vdots \\ F_{2n-1}(\mathbf{c}, \mathbf{t}) \end{pmatrix}. \end{aligned}$$

This defines a function $F : D \to \mathbb{R}^{2n}$ with some $D \subset \mathbb{R}^{2n}$. We still need the matrix of the partial derivatives,

$$\begin{aligned} F'(\mathbf{c},\mathbf{t}) &= \begin{pmatrix} \frac{\partial F_0}{\partial c_1} & \cdots & \frac{\partial F_0}{\partial c_n} & \frac{\partial F_0}{\partial t_1} & \cdots & \frac{\partial F_0}{\partial t_n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial F_{2n-1}}{\partial c_1} & \cdots & \frac{\partial F_{2n-1}}{\partial c_n} & \frac{\partial F_{2n-1}}{\partial t_1} & \cdots & \frac{\partial F_{2n-1}}{\partial t_n} \end{pmatrix} \\ &= \begin{pmatrix} B_0(t_1) & \cdots & B_0(t_n) & c_1 B_0'(t_1) & \cdots & c_n B_0'(t_n) \\ \vdots & & \vdots & \vdots & & \vdots \\ B_{2n-1}(t_1) & \cdots & B_{2n-1}(t_n) & c_1 B_{2n-1}'(t_1) & \cdots & c_n B_{2n-1}'(t_n) \end{pmatrix}. \end{aligned}$$

Now starting with a vector $(\mathbf{c}^{(0)},\mathbf{t}^{(0)})$, in the m-th step we determine $(\mathbf{c}^{(m+1)},\mathbf{t}^{(m+1)})$ iteratively by

$$F(\mathbf{c}^{(m)},\mathbf{t}^{(m)}) + F'(\mathbf{c}^{(m)},\mathbf{t}^{(m)})[(\mathbf{c}^{(m+1)},\mathbf{t}^{(m+1)}) - (\mathbf{c}^{(m)},\mathbf{t}^{(m)})] = \mathbf{0}, \qquad \text{(N)}$$

$m = 0, 1, \ldots$. In view of (IP), it seems to be reasonable setting

$$t_i^{(0)} = \begin{cases} x_{2i+(k-3)/2}, & \text{if } k \text{ is odd,} \\ \frac{1}{2}(x_{2i-2+k/2} + x_{2i-1+k/2}), & \text{if } k \text{ is even.} \end{cases} \qquad (3.1)$$

We have tested the algorithm with Maple 9.5 for the cases $k = 3$ and $k = 4$, and have chosen the vector $\mathbf{c}^{(0)} = (c_1^{(0)}, \ldots, c_n^{(0)})^t$ as follows.

The case $k = 3$: Since $t_i^{(0)} = x_{2i}$ for each i by (3.1), it is easily seen that

$$B_{2j-2}(t_i^{(0)}) = B_{2j-1}(t_i^{(0)}) = 0, \ j \neq i, \ i,j = 1, \ldots, n.$$

This implies that

$$F(\mathbf{c}^{(0)},\mathbf{t}^{(0)}) = \begin{pmatrix} c_1^{(0)} B_0(t_1^{(0)}) - \mu_0^3 y \\ c_1^{(0)} B_1(t_1^{(0)}) - \mu_1^3 y \\ \vdots \\ c_n^{(0)} B_{2n-2}(t_n^{(0)}) - \mu_{2n-2}^3 y \\ c_n^{(0)} B_{2n-1}(t_n^{(0)}) - \mu_{2n-1}^3 y \end{pmatrix}.$$

Hence we suggest setting

$$c_i^{(0)} = \frac{\mu_{2i-2}^3 y}{B_{2i-2}(t_i^{(0)})} \quad \text{or} \quad c_i^{(0)} = \frac{\mu_{2i-1}^3 y}{B_{2i-1}(t_i^{(0)})}, \ i = 1, \ldots, n.$$

The case $k = 4$: Since $t_i^{(0)} = \frac{1}{2}(x_{2i} + x_{2i+1})$ for each i by (3.1), we obviously obtain

$$F(\mathbf{c}^{(0)},\mathbf{t}^{(0)}) = \begin{pmatrix} c_1^{(0)} B_0(t_1^{(0)}) - \mu_0^4 y \\ c_1^{(0)} B_1(t_1^{(0)}) + c_2^{(0)} B_1(t_2^{(0)}) - \mu_1^4 y \\ \vdots \\ c_{n-1}^{(0)} B_{2n-2}(t_{n-1}^{(0)}) + c_n^{(0)} B_{2n-2}(t_n^{(0)}) - \mu_{2n-2}^4 y \\ c_n^{(0)} B_{2n-1}(t_n^{(0)}) - \mu_{2n-1}^4 y \end{pmatrix}.$$

We suggest setting

$$c_i^{(0)} = \frac{\mu_{2i-2}^4 y}{B_{2i-2}(t_i^{(0)})},\ i=1,\dots,n-1,\ c_n^{(0)} = \frac{\mu_{2n-1}^4 y}{B_{2n-1}(t_n^{(0)})},$$

or

$$c_1^{(0)} = \frac{\mu_0^4 y}{B_0(t_1^{(0)})},\ c_i^{(0)} = \frac{\mu_{2i-1}^4 y}{B_{2i-1}(t_i^{(0)})},\ i=2,\dots,n.$$

Our numerical experiences with the algorithm are different. In the case when $k=4,\ n=5,\ x_i=i,\ i=0,\dots,13$ we have taken the data $y_i=s(x_i)$, $i=0,\dots,13$ from the test splines

$$s(x) = x^3 - (x-2)_+^3 + 2(x-4)_+^3 + 1/2(x-11/2)_+^3 - (x-8)_+^3 - 2(x-11)_+^3$$

resp.

$$\widetilde{s}(x) = x^3 - (x-2)_+^3 + 2(x-4)_+^3 - 3(x-7)_+^3 + (x-9)_+^3 - 2(x-11)_+^3,$$

and have started the Newton method (N) with the above defined initial vectors. In the first case the solution knot vector $(2,4,11/2,8,11)$ has been computed by 8 iterations within an accuracy of 10^{-7} while in the second case the computation of the knot vector $(2,4,7,9,11)$ has needed 12 iterations.

In the case when $k=3,\ n=5,\ x_i=i,\ i=0,\dots,12$, we have tested our algorithm (N) with several vectors $\widetilde{\mathbf{y}}$ of third divided differences satisfying the sign properties given in Corollary 4. In most cases the algorithm works satisfactory and determines a solution knot vector within few iterations.

References

[1] T. A. Grandine, Minimal function spaces for spline interpolation, Constr. Approx. 17 (2001), 157 - 168.

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[3] M. Sommer, Interpolation by uni- and multivariate spline functions, in: "Program Systems: Theory and Applications" (Proceedings of an International Conference in Pereslavl-Zalessky), Institute for Program Systems, Russian Academy of Sciences, 2006, 337 - 352.

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