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On Some Properties of a High Order Fractional Differential Operator which is not in General Selfadjoint

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Abstract. A fractional differential operator of order $\alpha \in \mathbb{R}_+$ is introduced and some of its properties are studied. This operator is a generalization of the operators of Riesz-Feller, of Riemann-Liouville, of the fractional power of the Laplacian and of a class of the Jacob pseudodifferential operators.

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1. Introduction

Recently, fractional calculus is used more and more in modern technology. Phenomena in several domains are characterized to be anomalous. In the probabilistic point of view, these phenomena can not be covered by the Gaussian approach. For example, diffusions in disordered, fractal and chaotic mediums, kinematics in viscoelastic medium, relaxation processes in complex systems, propagation of seismic waves, pollution and transport of data across the internet (see e.g. [1] [10] [12] [19] [20] [23] [24] [29] [31] [32] [37] [39] [40] [41] [42] and the part II of [36] and the references therein). The fractional calculus allows the modeling of these anomalies and describes precisely material properties. In particular, fractional differential operators are introduced in the modeling. It is known that different definitions of fractional differential operators, not necessary equivalent, have been given. The Riemann-Liouville operator, the fractional power of the Laplacian and the operators given by Riesz-Bessel potential are largely used. The fractional heat equation of order less than 2 is connected with stable Lévy processes (see [4] [5] [7] [11][18]). The stable laws associated to the Riemann-Liouville, or Nishimoto fractional differential operators are totally skewed, and the stable laws associated to the fractional power of the Laplacian are symmetric. Feller idea was to introduce an operator for which all stable laws will be covered. In [7], he gave the fractional derivative operator as the infinitesimal generator of the stable semigroup. A representation of this operator via Fourier transform is given by Gorenflo and Mainardi [11]. In [18], Komatsu, dealing with other problems, gave an explicit form to the infinitesimal generator of the stable semigroup in the multidimensional case. In [15], [16], Jacob introduced a class of pseudodifferential operators which are connected with Lévy processes. In spite of the importance of these results, the order of these operators is less than two. Furthermore, in practice partial differential equations of high order are used (see the references above and [9]).

We note that the question of establishing a link between high order partial differential equations, even of integer order, and stochastic processes is not obvious. Only few equations of high order have been treated in a probabilistic context (see e.g. [2], [3]). Krylov has introduced a new approach, called quasiprobabilistic [18]. This approach allowed the generalization of the stochastic calculus of Brownian motion to pseudoprocesses (see e.g. [8] [13] [14] [17] [18] [25] [26] [27] [28] and the references therein).

The aim of this work is to extend the definitions of the fractional differential operators connected with stable laws to higher order and to study its functional properties. The operator introduced in this paper is given via Fourier transform on $L^2(\mathbb{R}^d)$ and the second characteristic function of the stable law. The formula of this function still makes sense when $\alpha > 2$, but the function is not negative definite anymore [15]. In particular, we prove that this operator is the infinitesimal generator of an analytic semigroup. A finite positive measure on the unit sphere in $\mathbb{R}^d$ is used in the definition as an auxiliary parameter. For special values of it, the defined operator coincides with the fractional Laplacian given in [22] and [38]. Further, when the operator is of order less than 2, it coincides with the operators given in [7] and [11] and with the pseudodifferential operator, without Gaussian part, [15] [16]. We prove also that the totally skewed case of this operator coincides with Liouville fractional differential operator and it keeps most of the properties known for the differentiation. Furthermore, it will allow in its turn to extend the stochastic calculus of stable Lévy processes to pseudoprocesses. Only few works are done in this direction. In [3], the authors are interested in the heat type equation of order 4. They remarked that the subordination method remains valid when using stable pseudoprocesses with "stability index" less than 4, but they did not give any indication on the associated differential equations. In [5], the authors represented the solutions of high order fractional heat type equations driven by

Liouville or Nishimoto operators via Lévy motion. In [4], a probabilistic and a quasiprobabilistic approach of high order fractional Fokker-Planck equations are studied. The fractional differential operators used are the Liouville and the Nishimoto operator and the onedimensional operator defined in this paper. For some cases the solutions, are represented, in the probabilistic context, as functionals of subordinators.

The paper is organized as follows. In the next section we introduce the fractional differential operator on $L^2(\mathbb{R}^d)$, $\quad d \geq 1$ and study its properties. Some of them can not be applied when $d = 1$. Therefore, we devote section 3 to the study of the onedimensional case using a relevant representation for the definition. Further, good illustrations of multidimensional results can be seen through the onedimensional operator. At the end of the section, we discuss the partial fractional differentiation. On one hand, it generalizes the entire partial differentiation and on the other hand, it connects the multidimensional and the onedimensional operator. We note that we can define in a similar way fractional differential operators on $L^2(D)$ for bounded $D \subset \mathbb{R}^d$.

2. The multidimensional Fractional Differential Operator

Let $\alpha \in \mathbb{R}_+ \backslash \mathbb{N}$ and let Γ be a finite positive measure on the unit sphere in $\mathbb{R}^d$, $d \geq 1$; $S^{d-1} = \{s \in \mathbb{R}^d \setminus |s|_d = 1\}$. We define the function ${}_\Gamma\psi_\alpha : \mathbb{R}^d \to \mathbb{C}$ by

$$ {}_\Gamma\psi_\alpha(\lambda) = -\int_{S^{d-1}} |\lambda.s|^\alpha \Big(1 - isgn(\lambda.s)\tan\frac{\alpha\pi}{2}\Big)\Gamma(ds), \tag{1} $$

where $\lambda.s$ denotes the scalar product in $\mathbb{R}^d$.

It is known that when $0 < \alpha < 2$ and $\alpha \neq 1$, ${}_\Gamma\psi_\alpha(.)$ is the second characteristic function of an $\alpha-$stable random vector. We recall that it is proved that the characteristic function of a stable law is strictly positive, so its logarithm exists, it is called the second characteristic function, see e.g. [21], [34], [35] and [36]. First, we give some properties of this function for $\alpha \in \mathbb{R}_+ \backslash \mathbb{N}$. It is easy to prove the following Lemma using the homogeneity property of the function ${}_\Gamma\psi_\alpha$ and the fact that the measure Γ is bounded on the sphere.

Lemma 1. *The function ${}_\Gamma\psi_\alpha$ is continuous on $\mathbb{R}^d$ and satisfies the inequality*

$$ \big|{}_\Gamma\psi_\alpha(\lambda)\big| \leq K_{\alpha,\Gamma}|\lambda|_d^\alpha, \tag{2} $$

for all $\lambda \in \mathbb{R}^d$, where $K_{\alpha,\Gamma}$ is a constant depending on α and Γ.

Lemma 2. *If the measure Γ satisfies the assumption*

(3) *There exists $0 \neq m(.) \in C^d(S^{d-1})$ such that $\Gamma(ds) = m(s)\sigma(ds)$,*

then ${}_\Gamma\psi_\alpha(.) \in C^{d+1}(S^{d-1})$.

Proof. Similar to the proof of Lemma 1.1 in [18]. □

Corollary 1. *The function ${}_\Gamma\psi_\alpha(.)$ is $d+1$- differentiable on $\mathbb{R}^d \backslash \{0\}$.*

Lemma 3. *If the measure* Γ *satisfies the assumption*

(4)
$$\text{There exists a constant } c_{\alpha,\Gamma} > 0 \text{ such that } \int_{S^{d-1}} |\xi.s|^{\alpha}\Gamma(ds) \geq c_{\alpha,\Gamma}, \forall \xi \in S^{d-1},$$

then ${}_{\Gamma}h_{\alpha}(t,.) : \mathbb{R}^d \to \mathbb{C}$ *defined by* ${}_{\Gamma}h_{\alpha}(t,\lambda) = \exp[t{}_{\Gamma}\psi_{\alpha}(\lambda)]$ *belongs to* $L^p(\mathbb{R}^d), \forall p \in (0,\infty]$ *and* $\forall t > 0$.

Let us give the Fourier transform and its inverse on $L^2(\mathbb{R}^d)$ by

$$\begin{array}{rl} \mathcal{F}\{\varphi(x);\lambda\} = \hat{\varphi}(\lambda) = \int_{\mathbb{R}^d} \exp(ix.\lambda)\varphi(x)dx, \\ \mathcal{F}^{-1}\{\varphi(\lambda);x\} = \check{\varphi}(x) = (2\pi)^{-d}\int_{\mathbb{R}^d} \exp(-ix.\lambda)\varphi(\lambda)d\lambda, \end{array} \qquad (5)$$

where $\varphi \in L^2(\mathbb{R}^d)$. It is known that the Fourier transform is an isometric on $L^2(\mathbb{R}^d)$.

Lemma 4. *If the measure* Γ *satisfies (4), then the function* ${}_{\Gamma}p_{\alpha}(t,x) = \mathcal{F}^{-1}\{{}_{\Gamma}h_{\alpha}(t,\lambda);x\}$ *satisfies the following properties for all* $\alpha \in \mathbb{R}_+\backslash\mathbb{N}$ *and for all* $t > 0$

(i) $\int_{\mathbb{R}^d} {}_{\Gamma}p_{\alpha}(t,x)dx = 1$,

(ii) ${}_{\Gamma}p_{\alpha}(t,.)$ *is real and symmetric relatively to* x, *when* Γ *is symmetric, i.e. when* $\Gamma(B) = \Gamma(-B)$, *for all Borel subsets of* S^{d-1},

(iii) ${}_{\Gamma}p_{\alpha}(t,x) = t^{-\frac{d}{\alpha}}{}_{\Gamma}p_{\alpha}(1, t^{-\frac{1}{\alpha}}x)$ *(Scaling Property)*,

(iv) ${}_{\Gamma}p_{\alpha}(t,.) \in \{f \in C^{\infty}(\mathbb{R}^d)$ *and* $D^{\beta}f$ *is bounded and tends to zero when* $|x|_d$ *tends to* ∞ *,* $\forall$ *multi-index* β $\}$,

(v) If Γ *satisfies (3), then for* $\alpha > 1$, ${}_{\Gamma}p_{\alpha}(t,.) \in L^p(\mathbb{R}^d), \forall\, p \in [1,\infty]$,

(vi) If Γ *satisfies (3), then* ${}_{\Gamma}p_{\alpha}(t,.)$ *satisfies the semi-group property, or the Chapman Kolmogorov Equation i.e. for* $s,t \geq 0$

$${}_{\Gamma}p_{\alpha}(t+s,x) = \int_{\mathbb{R}^d} {}_{\Gamma}p_{\alpha}(t,\xi){}_{\Gamma}p_{\alpha}(s,x-\xi)d\xi.$$

Proof

It is easy to see $(i) - (iv)$.

(v) Let $\nu = (\nu_1,\nu_2,...,\nu_d) \in \mathbb{N}^d$ be a multi-index with $|\nu| \leq d+1$. From Lemma 2 and using the fact that ${}_{\Gamma}\psi_{\alpha}(.)$ is homogeneous, we obtain (see e.g.[18])

$$|\partial_{\lambda}^{\nu} e^{{}_{\Gamma}\psi_{\alpha}(\lambda)}| \leq Ce^{-c_{\alpha,\Gamma}|\lambda|_d^{\alpha}}(|\lambda|_d^{\alpha-d} + |\lambda|_d^{(d+1)\alpha-d}),$$

where $\partial_{\lambda}^{\nu} = \partial_{\lambda_1}^{\nu_1}\partial_{\lambda_2}^{\nu_2}...\partial_{\lambda_d}^{\nu_d}$. Consequently, $\partial_{\lambda}^{\nu}e^{{}_{\Gamma}\psi_{\alpha}(\lambda)} \in L^1(\mathbb{R}^d)$ for $\alpha > 1$. But $|x^{\nu}{}_{\Gamma}p_{\alpha}(1,x)| = |\mathcal{F}^{-1}\{\partial_{\lambda}^{\nu}e^{{}_{\Gamma}\psi_{\alpha}(\lambda)};x\}|$, hence $|x^{\nu}{}_{\Gamma}p_{\alpha}(1,x)| \leq C$. In particular, taking $\nu = \nu_k = (0,0,..,d+1,...,0)$, we obtain $|{}_{\Gamma}p_{\alpha}(1,x)| \leq C|x_k|^{-d-1}$, so $|{}_{\Gamma}p_{\alpha}(1,x)| \leq C\min_{1\leq k\leq d}|x_k|^{-d-1}$. Thanks to the equivalence of norms in $\mathbb{R}^d$, we conclude that ${}_{\Gamma}p_{\alpha}(1,x) = O(|x|_d^{-d-1})$, as $|x|_d \to \infty$. By (iv), we get the result.

(vi) From (v) and (i), we have ${}_{\Gamma}p_{\alpha}(t,.)$, ${}_{\Gamma}\hat{p}_{\alpha}(t,.) \in L^1(\mathbb{R}^d)$, so ${}_{\Gamma}p_{\alpha}(t,.) * {}_{\Gamma}p_{\alpha}(s,.) \in L^1(\mathbb{R}^d)$. Further, $\mathcal{F}({}_{\Gamma}p_{\alpha}(t,.) * {}_{\Gamma}p_{\alpha}(s,.)) = {}_{\Gamma}\hat{p}_{\alpha}(t,.){}_{\Gamma}\hat{p}_{\alpha}(s,.) = {}_{\Gamma}h_{\alpha}(t+s,.)$. By applying Fourier inverse, we obtain the result see [15] p 89. $\square$

In the sequel of this section, we suppose that, Γ satisfies assumptions (3) and (4). We denote the norm respectively the scalar product in $L^2(\mathbb{R}^d)$ by $||.||_2$ respectively $\langle f, g\rangle_{L^2}$. Using the properties of the function ${}_\Gamma p_\alpha(t,.)$, we get

Proposition 1. *The family of operators $\{{}_\Gamma T_\alpha(t), t \geq 0\}$ defined on $L^2(\mathbb{R}^d)$ by*

$$ {}_\Gamma T_\alpha(t)f(x) = \int_{\mathbb{R}^d} {}_\Gamma p_\alpha(t, x-y)f(y)dy, $$

is a uniformly strongly continuous semigroup.

Let us now define the fractional differential operator.

Definition 1. *The fractional differential operator D^α_Γ, $\alpha \in \mathbb{R}_+\backslash\mathbb{N}$, is defined on $L^2(\mathbb{R}^d)$ by*

$$ D^\alpha_\Gamma f(x) = \mathcal{F}^{-1}\{{}_\Gamma\psi_\alpha(\lambda)\mathcal{F}\{f(x);\lambda\};x\}, \tag{6} $$

where ${}_\Gamma\psi_\alpha(\lambda)$ is given by (1) and $\mathcal{F}$ (respectively $\mathcal{F}^{-1}$) is the Fourier (respectively Fourier inverse) transform defined on $L^2(\mathbb{R}^d)$.

The domain of definition of D^α_Γ is given by

$$ D(D^\alpha_\Gamma) = \{f \in L^2(\mathbb{R}^d)/{}_\Gamma\psi_\alpha(.)\hat{f} \in L^2(\mathbb{R}^d)\}. \tag{7} $$

It is easy to see that $\mathcal{S}^\infty \subset D(D^\alpha_\Gamma)$ and is invariant by D^α_Γ, where $\mathcal{S}^\infty$ is the set of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$, hence $D(D^\alpha_\delta)$ is dense in $L^2(\mathbb{R}^d)$. Further $\{f \in L^2(\mathbb{R}^d)/|\lambda|^\alpha\hat{f} \in L^2(\mathbb{R}^d)\} \subset D(D^\alpha_\Gamma)$. For some measures Γ, the imaginary part of ${}_\Gamma\psi_\alpha(.)$ can vanish. This allows to the operator D^α_Γ to be symmetric. However, we can define selfadjoint fractional differential operators without any supplementary condition on the measure Γ.

Definition 2. *The fractional differential operator ${}_0D^\alpha_\Gamma$, $\alpha \in \mathbb{R}_+$, is defined on $L^2(\mathbb{R}^d)$ by*

$$ {}_0D^\alpha_\Gamma f(x) = \mathcal{F}^{-1}\{{}_\Gamma\psi^0_\alpha(\lambda)\mathcal{F}\{f(x);\lambda\};x\}, \tag{8} $$

where

$$ {}_\Gamma\psi^0_\alpha(\lambda) = -\int_{S^{d-1}} |\lambda.s|^\alpha\Gamma(ds). $$

The domain of definition of ${}_0D^\alpha_\Gamma$ is given by (7) replacing ${}_\Gamma\psi_\alpha(\lambda)$ by ${}_\Gamma\psi^0_\alpha(\lambda)$.

Remark 1.

- *It is clear that Definition 1 is also applicable when $\alpha = 2n$. Then it generalizes the notion of the classical differential operator. In this work, we deal only with non entire orders.*
- *If Γ satisfies, under normalization, the identity: $\int_{S^{d-1}} |\xi.s|^\alpha\Gamma(ds) = 1, \forall\xi \in S^{d-1}$, then ${}_0D^\alpha_\Gamma$ coincides with the fractional power of the Laplacian* [22].
- *We note that the study of the operator D^α_Γ is also applicable for ${}_0D^\alpha_\Gamma$ with the expectation of the properties connected to the selfadjointness.*

Theorem 1. *The operator D^α_Γ with $\alpha > \frac{1}{2}$ is the infinitesimal generator of the semigroup of convolution $\{{}_\Gamma T_\alpha(t), t \geq 0\}$.*

Proof

Let A and $(\mathfrak{R}_a, a > 0)$ be the infinitesimal generator respectively the resolvent associated to the semigroup ${}_\Gamma T_\alpha(t), t \geq 0$, i.e. A is defined on $D(A) = \{f \in L^2(\mathbb{R}^d) / \lim_{t\downarrow 0} \frac{1}{t}({}_\Gamma T_\alpha(t)f - f) \text{ exists}\}$, by the formula $Af = \lim_{t\downarrow 0} \frac{1}{t}({}_\Gamma T_\alpha(t)f - f)$ and $\mathfrak{R}_a$ is the bounded operator $(a - A)^{-1} : L^2(\mathbb{R}^d) \to D(A)$. The resolvent determines uniquely the associated operator and is represented via the semigroup by the formula

$$\mathfrak{R}_a = \int_0^{+\infty} e^{-as}{}_\Gamma T_\alpha(s) ds \tag{9}$$

see e.g. [22], p.10, [38], p.240, [30], p8. Hence to prove that D^α_Γ is the infinitesimal generator of the semi group $\{{}_\Gamma T_\alpha(t)\}_{t\geq 0}$, it is sufficient to prove that

$$\mathfrak{R}_a(a - D^\alpha_\Gamma) = Id_{D(D^\alpha_\Gamma)} \qquad \text{and} \qquad (a - D^\alpha_\Gamma)\mathfrak{R}_a = Id_{L^2}, \tag{10}$$

where Id_X is the identity on X. Let $f \in D(D^\alpha_\Gamma)$,

$$\mathfrak{R}_a((a - D^\alpha_\Gamma)f) = a\mathfrak{R}_a(f) - \mathfrak{R}_a(D^\alpha_\Gamma f). \tag{11}$$

Using the representation of the resolvent, the definition of D^α_Γ, Fubini's theorem and the integration by parts to calculate the second term, we get

$$\begin{aligned}
\mathfrak{R}_a(D^\alpha_\Gamma f)(x) &= \int_0^{+\infty} e^{-as}{}_\Gamma T_\alpha(s)(D^\alpha_\Gamma f)(x)ds \\
&= \int_0^{+\infty} e^{-as}\mathcal{F}^{-1}(\exp[{}_\Gamma\psi_\alpha(.)s]{}_\Gamma\psi_\alpha(.)\hat{f})(x)ds \\
&= (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-i\lambda.x}\Big(\int_0^{+\infty} e^{-as}\exp[{}_\Gamma\psi_\alpha(\lambda)s]{}_\Gamma\psi_\alpha(\lambda)ds\Big)\hat{f}(\lambda)d\lambda \\
&= -\mathcal{F}^{-1}(\hat{f})(x) + a\int_0^{+\infty} e^{-as}\mathcal{F}^{-1}\Big(\exp[{}_\Gamma\psi_\alpha(\lambda)s]\hat{f}(\lambda)\Big)(x)ds \\
&= -f(x) + a\int_0^{+\infty} e^{-as}({}_\Gamma p_\alpha(s,.) * f)(x)ds \\
&= -f(x) + a(\mathfrak{R}_a f)(x)
\end{aligned}$$

Inserting this in (11), we obtain the first identity in (10).

Now let $f \in L^2(\mathbb{R}^d)$. First, we prove that $\mathfrak{R}_a f \in D(D^\alpha_\Gamma)$, then we calculate the term $D^\alpha_\Gamma(\mathfrak{R}_a f)$ using the same tools as above.

$$\begin{aligned}
{}_\Gamma\psi_\alpha(\lambda)\mathcal{F}\Big(\mathfrak{R}_a f\Big)(\lambda) &= {}_\Gamma\psi_\alpha(\lambda)\mathcal{F}\Big(\int_0^{+\infty} e^{-as}{}_\Gamma T_\alpha(s)f(x)ds\Big)(\lambda) \\
&= \Big(\int_0^{+\infty} {}_\Gamma\psi_\alpha(\lambda)e^{-as}\exp[{}_\Gamma\psi_\alpha(\lambda)s]ds\Big)(\hat{f})(\lambda) \\
&= \Big(-1 + a\int_0^{+\infty} e^{-as}\exp[{}_\Gamma\psi_\alpha(\lambda)s]ds\Big)(\hat{f})(\lambda) \\
&= -\hat{f}(\lambda) + a\int_0^{+\infty} e^{-as}\Big(\exp[{}_\Gamma\psi_\alpha(\lambda)s]\hat{f}(\lambda)\Big)ds \\
&= -\hat{f}(\lambda) + a\mathcal{F}\Big(\int_0^{+\infty} e^{-as}{}_\Gamma T_\alpha(s)f ds\Big)(\lambda) \in L^2(\mathbb{R}^d)
\end{aligned}$$

On the other hand using the result in the above equality, we get

$$\begin{aligned} D^\alpha_\Gamma(\Re_a f)(x) &= \mathcal{F}^{-1}\Big({}_\Gamma\psi_\alpha(\lambda)\mathcal{F}\big(\Re_a f\big)(\lambda)\Big)(x) \\ &= \mathcal{F}^{-1}\Big(-\hat{f}(\lambda)+a\mathcal{F}\Big(\int_0^{+\infty} e^{-as}{}_\Gamma T_\alpha(s) f ds\Big)(\lambda)\Big)(x) \\ &= -f(x)+a\int_0^{+\infty} e^{-as}{}_\Gamma T_\alpha(s) f(x) ds, \end{aligned}$$

hence the result. We note that the Fubini's Theorem is applicable thanks to the inequalities

$$\begin{aligned} &\int_0^{+\infty} e^{-as}\Big(\int_{\mathbb{R}^d} |\exp[{}_\Gamma\psi_\alpha(\lambda)s]||{}_\Gamma\psi_\alpha(\lambda)||\hat{f}(\lambda)|d\lambda\Big)ds \\ &\quad\le K\int_0^{+\infty} e^{-as}\Big(\int_{\mathbb{R}^d}\exp[-c_{\alpha,\Gamma}|\lambda|^\alpha s)]|{}_\Gamma\psi_\alpha(\lambda)||\hat{f}(\lambda)|d\lambda\Big)ds \\ &\quad\le K\Big(\int_{\mathbb{R}^d}|{}_\Gamma\psi_\alpha(\lambda)|^{2\alpha}|\hat{f}(\lambda)|^2 d\lambda\Big)^{\frac{1}{2}}\int_0^{+\infty} e^{-as}\Big(\int_{\mathbb{R}^d}\exp[-2c_{\alpha,\Gamma}|\lambda|^\alpha s)]d\lambda\Big)^{\frac{1}{2}}ds \\ &\quad\le K\Big(\int_{\mathbb{R}^d}\exp[-2c_{\alpha,\Gamma}|\lambda|^\alpha)]d\lambda\Big)^{\frac{1}{2}}\Big(\int_0^{+\infty} e^{-as}s^{\frac{-1}{2\alpha}}ds\Big), \end{aligned}$$

$\int_{\mathbb{R}^d}|{}_\Gamma\psi_\alpha(\lambda)|^{2\alpha}|\hat{f}(\lambda)|^2 d\lambda<\infty$, because $f\in D(D^\alpha_\Gamma)$ and

$$\begin{aligned} &\int_0^{+\infty} e^{-as}\Big(\int_{\mathbb{R}^d} |\exp[{}_\Gamma\psi_\alpha(\lambda)s]||{}_\Gamma\psi_\alpha(\lambda)||\hat{f}(\lambda)|d\lambda\Big)ds \\ &\quad\le K\int_0^{+\infty} e^{-as}\Big(\int_{\mathbb{R}^d}\exp[-c_{\alpha,\Gamma}|\lambda|^\alpha s)]|\hat{f}(\lambda)|d\lambda\Big)ds \\ &\quad\le K||f||_2\Big(\int_{\mathbb{R}^d}\exp[-2c_{\alpha,\Gamma}|\lambda|^\alpha)]d\lambda\Big)^{\frac{1}{2}}\Big(\int_0^{+\infty} e^{-as}s^{\frac{-1}{2\alpha}}ds\Big). \end{aligned}$$

The final bounds in these chains of inequalities are finite since $\alpha>\frac{1}{2}$. □

Remark 2. *We note that for $\alpha<\frac{1}{2}$, we can prove, using other methods, that the infinitesimal generator of the semigroup ${}_\Gamma T_\alpha$ is D^α_Γ, see e.g.* [7], [11] *and* [18]. *In particular, in the last paper, an integral representation is also given to this generator for the case $0<\alpha<2$, see also* [4] *and* [6].

Corollary 2. *The operator D^α_δ is closed with dense domain.*

Proposition 2. *Let D^α_Γ and ${}_0D^\alpha_{\Gamma'}$ be the fractional differential operators defined above where $\alpha\in\mathbb{R}_+\backslash\mathbb{N}$ and Γ,Γ' satisfying assumptions (3) and (4). Then*

- *$D^{\alpha*}_\Gamma=D^\alpha_{\Gamma_-}$, where Γ_- is given by $\Gamma_-(B)=\Gamma(-B)$, for all Borel subsets of S^{d-1}, and ${}_0D^\alpha_{\Gamma'}$ is selfadjoint.*
- *$\Re(D^\alpha_\Gamma):=\frac{1}{2}(D^\alpha_\Gamma+D^{\alpha*}_\Gamma)={}_0D^\alpha_\Gamma$. $\Re(D^\alpha_\Gamma)$ is called the real part of the operator D^α_Γ.*
- *$\langle -D^\alpha_\Gamma f,f\rangle_{L^2}\ge 0$, $\forall f\in D(D^\alpha_\Gamma)\bigcap L^2(\mathbb{R}^d;\mathbb{R})$, where $L^2(\mathbb{R}^d;\mathbb{R})$ is the subspace of square integrable real functions.*
- *Let $f,g\in D(D^\alpha_\Gamma)\bigcap L^2(\mathbb{R}^d;\mathbb{R})$, then the integration by parts formula holds;*

$$\int_{\mathbb{R}^d} D^\alpha_\Gamma f(x)g(x)dx=\int_{\mathbb{R}^d} f(x)D^\alpha_{\Gamma_-}g(x)dx,$$

$$\int_{\mathbb{R}^d} {}_0D^\alpha_{\Gamma'}f(x)g(x)dx=\int_{\mathbb{R}^d} f(x){}_0D^\alpha_{\Gamma'}g(x)dx,$$

- *The associated sesquilinear form,*

$$\mathcal{E} : D(D^{\alpha}_{\Gamma}) \times D(D^{\alpha}_{\Gamma}) \to \mathbb{C}$$

is given for all $f, g \in D(D^{\alpha}_{\Gamma})$ *by*

$$\mathcal{E}(f,g) := \langle D^{\alpha}_{\Gamma} f, g\rangle_{L^2} = \int_{\mathbb{R}^d} {}_{\Gamma}\psi_{\alpha}(\lambda)\hat{f}(\lambda)\overline{\hat{g}(\lambda)}d\lambda.$$

- *The operator* D^{α}_{0} *is self adjoint.*

Proof

The proof follows from the equalities: $\overline{{}_{\Gamma}\psi_{\alpha}(\lambda)} = {}_{\Gamma-}\psi_{\alpha}(\lambda)$ and

$$\begin{aligned}\Im\langle -D^{\alpha}_{\Gamma} f, f\rangle = \tan\frac{\alpha\pi}{2}\Big[&\int_{\mathbb{R}^d}\int_{\{s\in S^{d-1}\backslash \lambda.s<0\}} |\lambda.s|^{\alpha}\Gamma(ds)|\hat{f}(\lambda)|^2 d\lambda \\ &- \int_{\mathbb{R}^d}\int_{\{s\in S^{d-1}\backslash \lambda.s>0\}} |\lambda.s|^{\alpha}\Gamma(ds)|\hat{f}(\lambda)|^2 d\lambda\Big] \\ = \tan\frac{\alpha\pi}{2}&\int_{\mathbb{R}^d}\int_{\{s\in S^{d-1}\backslash \lambda.s>0\}} |\lambda.s|^{\alpha}\Gamma(ds)\Big(|\hat{f}(-\lambda)|^2 - |\hat{f}(\lambda)|^2\Big)d\lambda = 0,\end{aligned}$$

where $\Im$ means the imaginary part. □

Remark 3. *It is obvious that the sesquilinear* $\mathcal{E}$ *is not Markovian* [22].

The operator D^{α}_{Γ} is not, in general, selfadjoint, so complex values in its spectrum $\sigma(D^{\alpha}_{\Gamma})$ are expected. In the theorem below, we prove that the spectrum of D^{α}_{Γ} is given by the values of the function ${}_{\Gamma}\psi_{\alpha}$.

Theorem 2. $\sigma(D^{\alpha}_{\Gamma}) = \{{}_{\Gamma}\psi_{\alpha}(\lambda), \lambda \in \mathbb{R}^d\}$.

Proof

Let $\xi \in \mathbb{C}$, we solve the equation $(\xi - D^{\alpha}_{\Gamma})f = g$, in $D(D^{\alpha}_{\Gamma})$ for all $g \in L^2(\mathbb{R}^d)$ when ${}_{\Gamma}\psi_{\alpha}(\lambda) \neq \xi, \forall\lambda \in \mathbb{R}^d$. Using Fourier transform, we get $\hat{f}(\lambda) = (\xi - {}_{\Gamma}\psi_{\alpha}(\lambda))^{-1}\hat{g}(\lambda)$ and by the fact that $|\xi - {}_{\Gamma}\psi_{\alpha}(\lambda)| \geq \min_{\lambda\in\mathbb{R}^d}|\xi - {}_{\Gamma}\psi_{\alpha}(\lambda)| = c(\xi) > 0$, we conclude that $(\xi - {}_{\Gamma}\psi_{\alpha}(\lambda))^{-1}\hat{g} \in L^2(\mathbb{R}^d)$ and $||(\xi - {}_{\Gamma}\psi_{\alpha}(\lambda))^{-1}\hat{g}||_2 \leq c^{-1}(\xi)||\hat{g}||_2$. Further, $\frac{{}_{\Gamma}\psi_{\alpha}(\lambda)}{\xi - {}_{\Gamma}\psi_{\alpha}(\lambda)}$ is continuous and bounded on $\mathbb{R}^d$, so $(\xi - {}_{\Gamma}\psi_{\alpha}(\lambda))^{-1}\hat{g} \in D(D^{\alpha}_{\Gamma})$. □

Corollary 3. *The spectra of the operator* D^{α}_{Γ} *is situated inside the closed cone* $C_{\delta'} := \{z \in \mathbb{C} : |\arg z| \geq \pi - \frac{\delta'\pi}{2}\}$, *where* $\delta' = \min\{2 - \alpha + [\alpha]_2, \alpha - [\alpha]_2\}$ *and* $[\alpha]_2$ *is the largest integer less than* α *(even part).*

Theorem 3. *The semigroup* ${}_{\Gamma}T_{\alpha}(t)$ *associated to the operator* D^{α}_{Γ} *can be extended to an analytic semigroup on the sector* $\Delta_{\delta'} = \{z \in \mathbb{C} : |\arg z| < \frac{\pi}{2}(1-\delta')\}$ *and* $||{}_{\Gamma}T_{\alpha}(z)||$ *is uniformly bounded in every closed subsector* Δ_{θ} *of* $\Delta_{\delta'}$ *(*δ' *is given in Corollary 3) .*

Proof. It is sufficient to prove that for all $f \in L^2(\mathbb{R}^d)$, $t \mapsto {}_\Gamma T_\alpha(t)f$ is differentiable for all $t > 0$ and there exists a constant C such that $||D^\alpha_\Gamma {}_\Gamma T_\alpha(t)|| \le ct^{-1}$(the bounded operator norm)(see [30]). In fact, ${}_\Gamma T_\alpha(t)f$ is differentiable iff ${}_\Gamma T_\alpha(t)f \in D(D^\alpha_\Gamma)$. But ${}_\Gamma\psi_\alpha(\lambda)(\widehat{{}_\Gamma T_\alpha(t)f})(\lambda) = {}_\Gamma\psi_\alpha(\lambda)e^{{}_\Gamma\psi_\alpha(\lambda)t}\hat{f}(\lambda) \in L^2(\mathbb{R}^d)$, thanks to the fact that $f \in L^2(\mathbb{R}^d)$ and $|{}_\Gamma\psi_\alpha(\lambda)e^{{}_\Gamma\psi_\alpha(\lambda)t}|$ is bounded for all $t > 0$. Further

$$||D^\alpha_\Gamma {}_\Gamma T_\alpha(t)f||_2^2 = ||{}_\Gamma\psi_\alpha(.)e^{{}_\Gamma\psi_\alpha(\lambda)t}\hat{f}(.)||_2^2 = \int_{\mathbb{R}^d} |{}_\Gamma\psi_\alpha(\lambda)e^{{}_\Gamma\psi_\alpha(\lambda)t}|^2|\hat{f}(\lambda)|^2 d\lambda.$$

By scaling Property and change of variable, we get

$$\begin{aligned} ||D^\alpha_\Gamma {}_\Gamma T_\alpha(t)f||_2^2 &= t^{-2-\frac{1}{\alpha}} \int_{\mathbb{R}^d} |{}_\Gamma\psi_\alpha(\lambda)e^{{}_\Gamma\psi_\alpha(\lambda)}|^2|\hat{f}(t^{-\frac{1}{\alpha}}\lambda)|^2 d\lambda \\ &\le Ct^{-2}\int_{\mathbb{R}^d} |\hat{f}(\lambda)|^2 d\lambda. \quad \square \end{aligned}$$

3. The Onedimensional Fractional Operator

It is clear that when d=1, the measure Γ used in section 2 is Dirac measure concentrated on the points $+1, -1$. Consequently, it does not satisfy assumption (3). In this section, we define the fractional differential operator via an equivalent representation which is more relevant. Let f be a function in $L^2(\mathbb{R})$,

Definition 3. *Let $\alpha \in \mathbb{R}_+$. The $\alpha-$fractional derivative of the function f in the point $x \in \mathbb{R}$, when it exists, is given by*

$$D^\alpha_\delta f(x) = \mathcal{F}^{-1}\{{}_\delta\psi_\alpha(\lambda)\mathcal{F}\{f(x);\lambda\};x\}, \tag{12}$$

where

$${}_\delta\psi_\alpha(\lambda) = -\,|\lambda|^\alpha\, e^{-i\delta\frac{\pi}{2}sgn\lambda}, \tag{13}$$

$|\delta| \le \min\{\alpha - [\alpha]_2, 2 + [\alpha]_2 - \alpha\}$, $[\alpha]_2$ *is the even part of* α, *and* $\delta = 0$ *when* $\alpha \in 2\mathbb{N}+1$ *and* $\mathcal{F}$ *is the Fourier transform in* $L^2(\mathbb{R})$.

The operator D^α_δ given by (12) and (13) is defined on

$$D(D^\alpha_\delta) = \{f \in L^2 / |\lambda|^\alpha \hat{f}(\lambda) \in L^2\}.$$

It is easy to see that $\mathcal{S}^\infty \subset D(D^\alpha_\delta)$.

Let us introduce as in section 2, the function ${}_\delta h_\alpha(t,\lambda) = e^{-t|\lambda|^\alpha e^{-i\frac{\delta\pi}{2}sgn\lambda}}$. It is clear that ${}_\delta h_\alpha(t,.) \in L^p(\mathbb{R}), \forall 1 \le p \le \infty$. Let ${}_\delta p_\alpha(t,x)$ its Fourier inverse. In the same way as in Lemma 4, we can prove properties $(i), (iii), (iv)$. Further, we have

Lemma 5. $\forall\alpha \in \mathbb{R}_+\backslash\mathbb{N}$, $\forall t > 0$,

(i) ${}_\delta p_\alpha(t,x)$ *is real and is not symmetric relatively to* x *when* $\delta \neq 0$,

(ii) ${}_{-\delta} p_\alpha(t,x) = {}_\delta p_\alpha(t,-x)$,

(iii) ${}_{\delta}p_{\alpha}^{(l)}(1,x) =$
$\frac{1}{\pi}\sum_{j=1}^{n}|x|^{-\alpha j-(l+1)}\frac{(-1)^{j+l}}{j!}\Gamma(\alpha j+l+1)\sin j\frac{(\alpha+\delta)}{2}\pi + O(|x|^{-\alpha(n+1)-(l+1)})$, *when* $|x|$ *is large, where* $p_{\alpha}^{(l)}(1,.)$ *is the derivative of order* l *of* $p_{\alpha}(1,.)$,

(iv) $\lim_{t\to 0}{}_{\delta}p_{\alpha}(t,x) = \delta_0(x)$.

Proof

It is easy to see (i) and (ii).

(iii) It is sufficient to prove this property for the function ${}_{\delta}p_{\alpha}(1,x)$ when $x>0$. In fact, using property (ii), we get (iii) for $x<0$. Moreover, using the representation

$$ {}_{\delta}p_{\alpha}^{(l)}(1,x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}(-i\lambda)^{l}\exp\left[-i\lambda x - |\lambda|^{\alpha}e^{-i\frac{\delta\pi}{2}sgn\lambda}\right]d\lambda, $$

we can see that the proof for $l>0$ is similar to the proof for the case $l=0$.

The function ${}_{\delta}p_{\alpha}(1,x)$ can be written as

$$ {}_{\delta}p_{\alpha}(1,x) = \frac{1}{\pi}\Re\{\int_{0}^{+\infty}\exp\left[-i\lambda x - \lambda^{\alpha}e^{-i\frac{\delta\pi}{2}}\right]d\lambda\}. $$

Let $0<r,R<\infty$ and let the curve C_{δ}: $[r,R]\vee\{Re^{i\delta\theta}, 0\le\theta\le\frac{\pi}{2\alpha}\}\vee\{\lambda e^{i\frac{\delta\pi}{2\alpha}}, r\le\lambda\le R\}^{*}\vee\{re^{i\delta\theta}, 0\le\theta\le\frac{\pi}{2\alpha}\}^{*}$, where $[r,R]$ designs the segment in the real axis between r and R, the symbol $\vee$ means followed by and $*$ means that the curve is taken in the opposite direction. By the Cauchy Theorem the integral of the function $\exp\left[-izx - z^{\alpha}e^{-i\frac{\delta\pi}{2}}\right]$ over C_{δ} vanishes, further the integrals over the two arcs tend to zero when R tends to infinity and r tends to zero, so

$$ \int_{0}^{+\infty}\exp\left[-i\lambda x - \lambda^{\alpha}e^{-i\frac{\delta\pi}{2}}\right]d\lambda = e^{i\frac{\pi\delta}{2\alpha}}\int_{0}^{+\infty}\exp\left[-i\lambda x e^{i\frac{\pi\delta}{2\alpha}} - \lambda^{\alpha}\right]d\lambda. $$

By integrating the function $e^{i\frac{\pi\delta}{2\alpha}}\exp\left[-izxe^{i\frac{\pi\delta}{2\alpha}} - z^{\alpha}\right]$ over the curve C_{-1} when δ is positive and over C_{1} when δ is negative, we get

$$ {}_{\delta}p_{\alpha}(1,x) = \frac{1}{\pi}\Re\{\int_{0}^{+\infty}e^{i\frac{\pi(\delta-1)}{2\alpha}}\exp\left[-\lambda x e^{i\frac{\pi(\alpha+\delta-1)}{2\alpha}} - \lambda^{\alpha}e^{-i\frac{\pi}{2}}\right]d\lambda\}. $$

Making the change of variable $\xi = x\lambda$, and then expanding the exponential containing x in Taylor series, we find

$$ \begin{aligned} {}_{\delta}p_{\alpha}(1,x) &= \frac{1}{\pi x}\Re\{e^{i\frac{\pi(\delta-1)}{2\alpha}}\int_{0}^{+\infty}\exp\left[-\xi e^{i\frac{\pi(\alpha+\delta-1)}{2\alpha}} - x^{-\alpha}\xi^{\alpha}e^{-i\frac{\pi}{2}}\right]d\xi\} \\ &= \frac{1}{\pi x}\Re\{e^{i\frac{\pi(\delta-1)}{2\alpha}}\sum_{j=0}^{n}\frac{(-1)^{j}}{j!}x^{-\alpha j}e^{i\frac{j\pi}{2}}E_{\alpha}(j)\} \\ &\quad + \frac{1}{\pi x}\Re\{e^{i\frac{\pi(\delta-1)}{2\alpha}}\theta\frac{(-1)^{n+1}}{(n+1)!}x^{-\alpha(n+1)}e^{i\frac{(n+1)\pi}{2}}E_{\alpha}(n+1)\}, \end{aligned} $$

where $E_\alpha(j) = \int_0^{+\infty} \exp\left[-\xi e^{i\frac{\pi(\alpha+\delta-1)}{2\alpha}}\right] \xi^{\alpha j} d\xi$ and $|\theta| < 1$. By the same technique we find $E_\alpha(j) = \exp\left[-i\frac{\pi(\alpha+\delta-1)j}{2} - i\frac{\pi(\alpha+\delta-1)}{2\alpha}\right] \Gamma(\alpha j+1), j \in \overline{1(n+1)}$. Inserting in the formula above, we find the series in (iii) for $l = 0$.

(iv) We use the scaling property and property (iii), to prove that ${}_\delta p_\alpha(t,x)$ tends to zero, when $x \neq 0$, and to infinity when $x = 0$. □

Remark 4. *The results in Lemma 5 generalize the properties of the density of stable laws where $0 < \alpha \leq 2$ (see [21] and [36]).*

Corollary 4.

- ${}_\delta p_\alpha(t,.) \in L^p(\mathbb{R}), \forall 1 \leq p \leq \infty$ *and* $\forall t > 0$.
- ${}_\delta p_\alpha(.,.)$ *satisfies the semigroup property.*

As a result of these properties we can use the technique of section 2 to prove

Theorem 4. *The operator D_δ^α is the infinitesimal generator of the analytic semigroup $\{{}_\delta T_\alpha(t), t \geq 0\}$ defined on $L^2(\mathbb{R})$ by*

$$ {}_\delta T_\alpha(t) f(x) = \int_{\mathbb{R}} {}_\delta p_\alpha(t, x-y) f(y) dy. $$

Further, we get

Proposition 3. *Let $\alpha, \beta \in \mathbb{R}_+ \backslash \mathbb{N}$, $\delta, \delta' \in \mathbb{R}$ such that $|\delta| \leq \min\{\alpha - [\alpha]_2, 2 + [\alpha]_2 - \alpha\}$ and $|\delta'| \leq \min\{\beta - [\beta]_2, 2 + [\beta]_2 - \beta\}$. Then*

- $D_{\delta'}^\beta D_\delta^\alpha = D_\delta^\alpha D_{\delta'}^\beta = D_{\delta+\delta'}^{\alpha+\beta}$,
- *Let $\alpha \geq \beta$ and $f \in D(D_\delta^\alpha)$. Then $f \in D(D_{\delta'}^\beta)$, further $D_\delta^\alpha f = D_{\delta-\delta'}^{\alpha-\beta} D_{\delta'}^\beta f$ under the condition that $|\delta - \delta'| = \min\{(\alpha-\beta) - [\alpha-\beta]_2, 2 + [\alpha-\beta]_2 - (\alpha-\beta)\}$. In particular, the condition is satisfied for the interesting case $\beta = \frac{\alpha}{2}$ and $\delta' = \frac{\delta}{2}$,*
- *$D(D_\delta^{\alpha *}) = D(D_\delta^\alpha)$ and $D_\delta^{\alpha *} = D_{-\delta}^\alpha$, ($D_\delta^\alpha$ is not a selfadjoint operator),*
- $D_\delta^\alpha D_\delta^{\alpha *} = D_\delta^{\alpha *} D_\delta^\alpha = D_0^{2\alpha}$,
- $\Re(D_\delta^\alpha) = \cos(\frac{\delta\pi}{2}) D_0^\alpha$,
  $\Im(D_\delta^\alpha)(\varphi) := \frac{1}{2i}(D_\delta^\alpha - D_\delta^{\alpha *})(\varphi) = \sin(\frac{\delta\pi}{2}) \mathcal{F}^{-1}(-sgn\lambda|\lambda|^\alpha \hat{\varphi}(\lambda))$,
- $\langle -D_\delta^\alpha f, f\rangle_{L^2} \geq 0$, $\forall f \in D(D_\delta^\alpha) \bigcap L^2(\mathbb{R};\mathbb{R})$,
- *Let $f, g \in D(D_\delta^\alpha) \bigcap L^2(\mathbb{R};\mathbb{R})$, then*

$$ \int_{-\infty}^{+\infty} D_\delta^\alpha f(x) g(x) dx = \int_{-\infty}^{+\infty} f(x) D_{-\delta}^\alpha g(x) dx \quad \text{(Integration by parts).} $$

- *the associated sesquilinear form $\mathcal{E} : D(D_\delta^\alpha) \times D(D_\delta^\alpha) \to \mathbb{C}$ is given by*

$$ \mathcal{E}(f,g) = \langle D_{\frac{\delta}{2}}^{\frac{\alpha}{2}} f, D_{-\frac{\delta}{2}}^{\frac{\alpha}{2}} g\rangle_{L^2}. $$

In particular, when $f \in D(D_\delta^\alpha) \bigcap L^2(\mathbb{R};\mathbb{R})$,

$$ \mathcal{E}(f,f) = -2\cos(\frac{\delta\pi}{2}) \int_0^{+\infty} |\lambda|^\alpha |\hat{f}(\lambda)|^2 d\lambda. $$

- $\sigma(D_\delta^\alpha) = \{z \in \mathbb{C} : \arg z = \pi(1 - \frac{|\delta|}{2})\} \cup \{z \in \mathbb{C} : \arg z = \pi(1 + \frac{|\delta|}{2})\}$.

Let us now compare the operator D_δ^α with the Liouville fractional differential operator D^α, defined by:

$$(D^\alpha f)(x) = \frac{1}{\Gamma([\alpha] + 1 - \alpha)} \frac{d^{[\alpha]+1}}{dx^{[\alpha]+1}} \int_{-\infty}^{x} (x - t)^{[\alpha]-\alpha} f(t) dt,$$

where Γ is the *Gamma*$-$function and $[\alpha]$ is the integer part of α. It is known that Liouville fractional differential operator is applied on smooth functions. In the following theorem, we prove the equality between D^α and D_δ^α on a class of smooth functions, which is a dense subset in $L^2(\mathbb{R})$.

Theorem 5. *Let the function $f \in C_0^{[\alpha]+2}$, where $C_0^{[\alpha]+2}$ is the set of $[\alpha] +$ 2-continuously derivable functions on $\mathbb{R}$ with compact support. Then $f \in D(D^\alpha) \cap D(D_\delta^\alpha)$, for all δ given in Definition 3. Further, $D^\alpha f = D_{\delta'}^\alpha f$, where $\delta' = \alpha - [\alpha]_2$ when $\frac{[\alpha]_2}{2}$ is even and $\delta' = \alpha - [\alpha]_2 - 2$ when $\frac{[\alpha]_2}{2}$ is odd.*

Proof
It is easy to see that $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and

$$I^{[\alpha]-\alpha+1} f(x) = \frac{1}{([\alpha] - \alpha + 1)\Gamma([\alpha] - \alpha + 1)} \int_0^{+\infty} \tau^{[\alpha]-\alpha+1} f'(x - \tau) d\tau,$$

where f' is the first derivative of f. Further $I^{[\alpha]-\alpha+1} f(x) \in C^{[\alpha]+1}$, hence $f \in D(D^\alpha)$ and $\mathcal{F}\{D^\alpha f; \lambda\} = (-i\lambda)^\alpha \mathcal{F}\{f; \lambda\}$ [33](Theorem 7.1, p137–139). On the other hand since $f \in C_0^{[\alpha]+2}$ then $|\lambda|^\alpha \hat{f}(\lambda) \in L^2$ and consequently, $f \in D(D^\alpha) \cap D(D_\delta^\alpha), \forall \delta$. Let us suppose, $\frac{[\alpha]_2}{2}$ even. Then

$$\begin{aligned} \mathcal{F}\{D^\alpha f; \lambda\} &= |\lambda|^\alpha e^{\frac{-i\alpha\pi}{2} sgn\lambda} \hat{f}(\lambda) \\ &= |\lambda|^\alpha e^{\frac{-i\pi}{2}([\alpha]_2 + (\alpha - [\alpha]_2)) sgn\lambda} \hat{f}(\lambda) \\ &= |\lambda|^\alpha e^{\frac{-i}{2}(\alpha - [\alpha]_2)\pi sgn\lambda} \hat{f}(\lambda). \end{aligned}$$

If $\frac{[\alpha]_2}{2}$ is odd, we prove in the same way that

$$\mathcal{F}\{D^\alpha f; \lambda\} = |\lambda|^\alpha e^{\frac{-i}{2}(\alpha - [\alpha]_2 - 2)\pi sgn\lambda} \hat{f}(\lambda).$$

Further we have seen that $|\lambda|^\alpha e^{\frac{-i}{2}(\alpha - [\alpha]_2 - 2)\pi sgn\lambda} \hat{f}(\lambda) \in L^2$, hence

$$(D^\alpha f)(x) = \mathcal{F}^{-1}\{|\lambda|^\alpha e^{\frac{-i}{2}\delta' \pi sgn\lambda} \hat{f}; x\} = (D_{\delta'}^\alpha f)(x). \quad \square$$

Remark 5.
This calculus can also be applied to the other representations of the Riemann-Liouville fractional derivative operators.

Let us finally comment on mixed fractional derivatives.

Definition 4. *Let f be a function on $\mathbb{R}^d$, such that $f_{x_1,..x_{j-1},x_{j+1}...,x_n}(.) \in L^2(\mathbb{R})$ and let $\alpha_j \in \mathbb{R}_+$ and δ_j such that $|\delta_j| \leq \min\{\alpha_j - [\alpha_j]_2, 2 + [\alpha_j]_2 - \alpha_j\}$ and equal to 0, when $\alpha_j \in 2\mathbb{N}+1$. The α_j-fractional derivative of f with respect to x_j with parameter δ_j is defined by*

$$_{x_j}\partial^{\alpha_j}_{\delta_j} f(x_1,..,x_j...,x_n) = \mathcal{F}^{-1}\{_{\delta_j}\psi_{\alpha_j}(\lambda_j)\mathcal{F}\{f_{x_1,..x_{j-1},x_{j+1}...,x_n}(x_j);\lambda_j\};x_j\},$$

where $_{\delta_j}\psi_{\alpha_j}(\lambda_j)$ is given by (13).

This Definition coincides with Definition 1 when the support of the measure Γ is concentrated on the intersection points of S^{d-1} and the $j-$ axis. The following Proposition gives sufficient condition for joint fractional partial derivatives.

Proposition 4. *Let $\alpha_k, \alpha_j \in \mathbb{R}_+ \backslash \mathbb{N}$, $1 \leq k, j \leq d$, and let δ_k, δ_j such that $|\delta_k| \leq \min\{\alpha_k - [\alpha_k]_2, 2+[\alpha_k]_2 - \alpha_k\}$ and $|\delta_j| \leq \min\{\alpha_j - [\alpha_j]_2, 2+[\alpha_j]_2 - \alpha_j\}$, and let the function f defined on $\mathbb{R}^d$ such that its restriction on the variables x_k, x_j $(1 \leq k, j \leq d)$ is such that*

- $|\lambda_k|^{\alpha_k}|\lambda_j|^{\alpha_j}\hat{f}(\lambda_k,\lambda_j)$ *are square integrable on* $\mathbb{R}^2$,
- $\int_{\mathbb{R}^2}|\lambda_k|^{\alpha_k}|\hat{f}(\lambda_k,x_j)|d\lambda_k dx_j < \infty$,
- $\int_{\mathbb{R}^2}|\lambda_j|^{\alpha_j}|\hat{f}(x_k,\lambda_j)|d\lambda_j dx_k < \infty$,

where $\hat{f}(\lambda_k,x_j)$ and $\hat{f}(x_k,\lambda_j)$ are respectively the Fourier transform of $f(.,x_j)$ in λ_k and the Fourier transform of $f(x_k,.)$ in λ_j. Then f is α_k-differentiable with respect to x_k and its α_k-derivative is α_j-differentiable with respect to x_j, and we have

$$_{\delta_j}\partial^{\alpha_j}_{x_j}\,{}_{\delta_k}\partial^{\alpha_k}_{x_k} f(x_1,x_2,..,x_d) = {}_{\delta_k}\partial^{\alpha_k}_{x_k}\,{}_{\delta_j}\partial^{\alpha_j}_{x_j} f(x_1,x_2,..,x_d).$$

Proof. We can obtain the result using Definition 4 and Fubini's Theorem.

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