

Propagation of Torsional Surface Waves in Heterogeneous Half-Space with Irregular Free Surface

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Abstract

The effect of irregularity on the propagation of torsional surface waves in a heterogeneous, elastic half-space has been studied. The velocity equation has been derived. The velocities have been calculated numerically and are shown graphically. The study reveals that the surface irregularity has a notable effect on the propagation of torsional surface waves. It is also observed that the velocity of propagation of torsional surface waves depends on the heterogeneity present in the medium.

Keywords: Torsional surface waves, Irregularity, Heterogeneous medium

1. Introduction

Waves propagating along surfaces or interfaces are very important to seismologist, earthquake engineering in understanding of the causes of damage due to earthquakes. In fact, study of surface waves in homogenous, heterogeneous and layered media has been of central interest to theoretical seismologists until recently. The basic literature on the propagation of elastic waves is the monograph by Ewing et al.[6]. A large number of papers have been published in different journals after the publication of this book. Vrettos [17,18] gives much information on the effect of heterogeneity in the study of surface waves vibrations due to line-load. Although much information is available on the propagation of surface waves, such as, Rayleigh waves, Love waves,

and Stonely waves etc., the torsional wave has not drawn much attention and very little literature is available on propagation of this wave. Lord Rayleigh [10], in his remarkable paper, showed that the isotropic homogenous elastic half-space does not allow a torsional surface wave to propagate. Bhattacharya [1] has been investigated the torsional wave propagation in a two-layered circular cylinder with imperfect bond. The propagation of torsional wave in a finite piezoelectric cylindrical shell has been discussed by Paul and Sarma [14]. The propagation of torsional wave in an initially stressed cylinder has been discussed by Dey and Dutta [3]. As extension of this work Selim [16] has been investigated the torsional wave propagation in an initially stressed dissipative cylinder. The commendable works by Dey et al. [4] and Dey et al. [5] in the study of surface waves in a nonhomogenous medium may be cited. Studies of wave propagation in elastic media with irregular surfaces are very important, leading to better understanding of the behavior of surface wave propagating on the actual Earth surface. It is therefore interesting to study the torsional surface waves in medium with irregular surface. The problem of surface waves propagation in an irregular media using the perturbation technique indicated by Eringen and Suhubi [7] has been successfully employed by Mal [12], Kar et al. [9], Chattopadhyay et al. [2] and others.

It seems that no attempt has so far been made to find out the effect of irregular surface on the propagation of torsional surface waves in a heterogeneous medium. A similar problem for slightly curved elastic half-space has been analyzed by Handelman [8] and Pal et al. [13] for the case of Raleigh wave propagation. This paper has been framed out to show the effect of irregularity as well as heterogeneity on the propagation of torsional surface waves in a heterogeneous medium. The velocity equation has been derived and the results have been discussed. The study reveals that the surface irregularity as well as heterogeneity has a notable effect on the propagation of torsional surface waves.

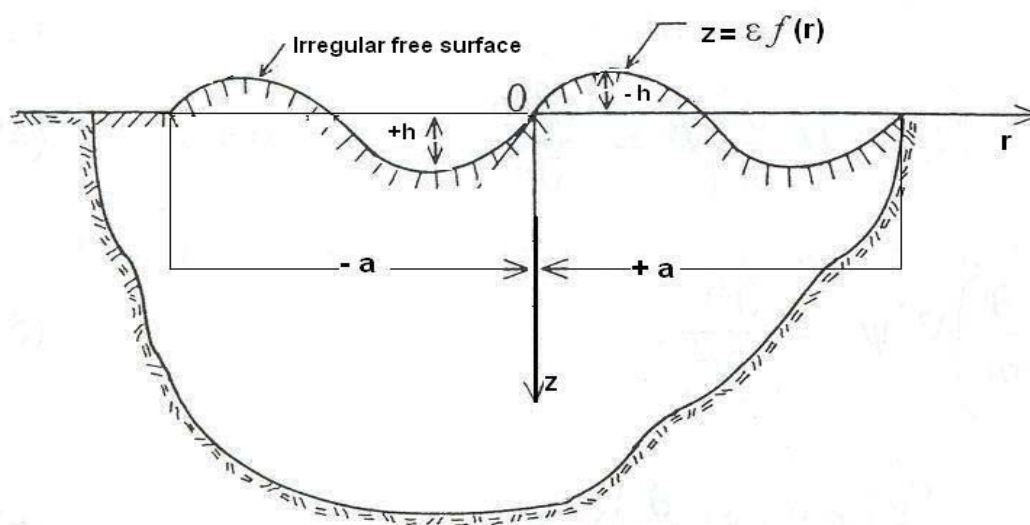


Fig. 1. Geometry of the problem.

2. Formulation of the problem and its solution

To study the torsional surface waves, a cylindrical coordinate with origin 0 on the surface and z directed normally to the interior of the half-space is considered. According to the problem, we assume the free surface may be taken as $z = \varepsilon f(r)$, where ε is small quantity whose second and higher powers may be neglected (Fig.1). The heterogeneity in the half-space is taken in the form

$$\begin{aligned} \mu &= \mu_0 e^{\delta z} && \text{for rigidity,} \\ \rho &= \rho_0 && \text{for density,} \end{aligned} \quad (1)$$

where δ is constant has dimension of inverse of length, μ_0 and ρ_0 are positive constants and the equation of irregularity has been taken as

$$\varepsilon f(r) = \begin{cases} \pm \chi r^2 & \text{for } |r| \leq a, \\ 0 & \text{for } |r| > a, \end{cases} \quad (2)$$

where $\chi = h/a$ (Fig. 1).

The dynamical equations of motion for the system are [11]

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} &= \rho_0 \frac{\partial^2 u_r}{\partial t^2}, \\ \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} &= \rho_0 \frac{\partial^2 u_\theta}{\partial t^2}, \\ \frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} &= \rho_0 \frac{\partial^2 u_z}{\partial t^2}, \end{aligned} \quad (3)$$

where ρ_0 is the density, $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{rz}, \sigma_{r\theta}$ and $\sigma_{\theta z}$ are the corresponding stress components in their conventional sense, u_r, u_θ, u_z are the displacement components in radial, circumferential and axial directions.

In cylindrical coordinates the tress-strain relations are taken as

$$\sigma_{ij} = \lambda \Omega \delta_{ij} + 2\mu e_{ij}, \quad (4)$$

where λ, μ are Lamé's coefficients and

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{and} \quad e_{ii} = \Omega, \quad i, j = 1, 2, 3. \quad (5)$$

Dynamical equation of motion for torsional wave may be written as:

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} = \rho_0 \frac{\partial^2 u_\theta}{\partial t^2}, \quad (6)$$

where $u_\theta(r, z, t)$ is the displacement along θ direction. For an elastic medium the stresses are related to the displacement component $u_\theta(r, z, t)$ by

$$\begin{aligned} \sigma_{r\theta} &= \mu(z) \left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right), \\ \sigma_{\theta z} &= \mu(z) \frac{\partial u_\theta}{\partial z} \end{aligned} \quad (7)$$

Inserting relations (7) in (6) one gets

$$\mu(z) \left(\frac{\partial}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) u_\theta + \frac{\partial}{\partial z} \left(\mu(z) \frac{\partial u_\theta}{\partial z} \right) = \rho_0 \frac{\partial^2 u_\theta}{\partial t^2} \quad (8)$$

Assuming harmonic wave solution $e^{i\omega t}$, the solution for circumferential displacement in the medium becomes

$$u_\theta(r, z, t) = U(z) V(r) e^{i\omega t}. \quad (9)$$

Using Eq.(9), Eq.(8) takes the form

$$r^2 \frac{d^2 V(r)}{dr^2} + r \frac{dV(r)}{dr} + (r^2 k^2 - 1) V(r) = 0, \quad (10)$$

Equation (10) called the parametric Bessel equation of first order and its general solution is

$$V(r) = AJ_1(kr) + BY_1(kr), \quad (11)$$

where $J_1(kr)$ and $Y_1(kr)$ denote Bessel's functions of the first order and of the first and second kind, respectively. The second part of Eq.(11) is left out from the complete solution for well-known physical reasons [15]. Then,

$$V(r) = AJ_1(kr) \quad (12)$$

where

$$k^2 = \frac{1}{U(z)} \left[\frac{d^2 U(z)}{dz^2} + \frac{\mu'(z)}{\mu(z)} \frac{dU(z)}{dz} + \frac{k^2 C^2}{\beta^2} U(z) \right], \quad (13)$$

and $\beta = \left(\frac{\mu(z)}{\rho_0} \right)^{\frac{1}{2}}$ and $C = \frac{\omega}{k}$ is torsional wave velocity in the medium.

From Eq.(12) in Eq.(9) the displacement component in the heterogeneous half-space is given by

$$u_{\theta}(r, z, t) = AU(z) J_1(kr) e^{i\omega t}, \tag{14}$$

where $U(z)$ is the solution to

$$\left[\frac{d^2U(z)}{dz^2} + \frac{\mu'(z)}{\mu(z)} \frac{dU(z)}{dz} - k^2 \left(1 - \frac{C^2}{\beta^2} \right) U(z) \right] \tag{15}$$

Using Eq. (1) and approximation $e^{\delta z} = (1 + \delta z)$, Eq.(15) take the form

$$\frac{d^2U(z)}{dz^2} + \delta \frac{dU(z)}{dz} - k^2 \left(1 - \frac{C^2}{C_1^2(1 + \delta z)} \right) U(z) = 0, \tag{16}$$

where $C_1 = \left(\frac{\mu_0}{\rho_0} \right)^{\frac{1}{2}}$.

Following the method of Dey et al., [4], we put $U(z) = \phi(z) e^{-\delta z/2}$ in Eq.(16) one gets

$$\frac{d^2\phi(z)}{dz^2} + k^2 \gamma^2 \left[\frac{C^2}{\gamma^2 C_1^2 (1 + \delta z)} - 1 \right] \phi(z) = 0. \tag{17}$$

where $\gamma^2 = (1 + \frac{\delta^2}{4k^2})$.

On putting $\alpha = \left(\frac{C^2 k}{2\gamma C_1^2} \right)$ and $\zeta = \frac{2k\gamma(1 + \delta z)}{\delta}$, in Eq.(17), we get

$$\frac{d^2\phi(\zeta)}{d\zeta^2} + \left[-\frac{1}{4} + \frac{\alpha}{\zeta} \right] \phi(\zeta) = 0, \tag{18}$$

Applying the surface wave condition, which is

$$U(z) \rightarrow 0 \text{ at } z \rightarrow \infty \quad \text{i.e.,} \quad \phi(\zeta) \rightarrow 0 \text{ at } \zeta \rightarrow \infty. \tag{19}$$

The solution of Eq.(18) may be taken as

$$\phi(\zeta) = A_1 W_{\alpha, 1/2}(\zeta), \tag{20}$$

where A_1 is constant and $W_{\alpha, 1/2}(\zeta)$ is Whittaker function. Hence,

$$U(z) = A_1 W_{\alpha, 1/2} \left(\frac{2\gamma k(1 + \delta z)}{\delta} \right) e^{-\delta z/2} \tag{21}$$

From Eq.(21) in Eq.(14), one gets

$$u_{\theta}(r, z, t) = BW_{\alpha, 1/2} \left(\frac{2\gamma k(1 + \delta z)}{\delta} \right) e^{-\delta z/2} J_1(kr) e^{i\omega t}, \quad (22)$$

where $B = AA_1$ is a constant.

3. Boundary Conditions

In the absence of any external body forces in the boundary, the surface of the medium is kept stress free at $z = \varepsilon f(r)$ and also we assume that the displacement at $r = 0$ is finite. The boundary conditions may be taken as

$$s_{z\theta} = \mu(z) \frac{du_{\theta}}{dz} \quad \text{at} \quad z = \varepsilon f(r) \quad (23)$$

Applying the boundary conditions (23), one gets

$$\frac{d[W_{\alpha, 1/2}(2\gamma k(1 + \delta z)/\delta)]}{dz} - \frac{\delta[W_{\alpha, 1/2}(2\gamma k(1 + \delta z)/\delta)]}{2} = 0, \quad z = \varepsilon f(r) \quad (24)$$

The boundary condition (23) takes the final form as

$$\left(\frac{4\gamma k}{\delta} \right) \left[\frac{d[W_{\alpha, 1/2}(2\gamma k(1 + \delta \varepsilon f(r))/\delta)]}{d\zeta} \right] - W_{\alpha, 1/2}(2\gamma k(1 + \delta \varepsilon f(r))/\delta) = 0 \quad (25)$$

Taking the asymptotic expansion of the Whittaker function and retaining up to linear term, Eq.(25) gives the velocity equation as

$$\frac{C}{C_1} = \left[\frac{4\delta}{k^2(3 + 2\delta\Delta)} - \frac{8\gamma^2(1 + \delta\Delta)}{\delta(3 - \delta\Delta)} - \frac{2\delta}{k^2(3 - \delta\Delta)} - \frac{4\gamma}{k(3 - \delta\Delta)} + \frac{2\gamma}{k} \right]^{\frac{1}{2}}, \quad (26)$$

where $\Delta = \pm \chi r^2$.

4. Numerical Results

In order to demonstrate the effect of irregularity on the propagation of torsional surface waves in the heterogeneous isotropic elastic half-space, numerical computations of Eq.(26) were performed. The effect of irregularity and heterogeneity on the phase velocity of torsional surface wave is shown in Figs 2, and 3. In these figures, the variation of C/C_1 against k/δ for different sizes of irregularity ($\chi = 0.0, \pm 0.2$ and ± 0.4) is displayed.

Figure 2 gives the variation of the phase velocity of torsional surface wave against k/δ for the positive values of $\chi = 0.2$ and 0.4 . Comparison with the case of $\chi = 0.0$, It is observed that the phase velocity of torsional surface waves decreases with an increase in irregularity parameter χ for the same value of k/δ . Figure 3 gives the same for the negative values of $\chi = -0.2$ and -0.4 .

The above numerical results for the propagation of torsional surface waves in the heterogeneous half-space for both positive and negative values of the irregularity

surface parameter χ reveal clearly that as k/δ increases the velocity of torsional surface decreases.

5. Conclusions

It is concluded that the surface irregularity has a notable effect on the propagation of torsional surface waves in heterogeneous medium with irregular free surface. Since the actual Earth has irregular surface space so it is more realistic to consider the irregularity effect in the study of the propagation of torsional surface waves in heterogeneous Earth medium. The study also shows that the heterogeneity present in the medium effect the propagation significantly.

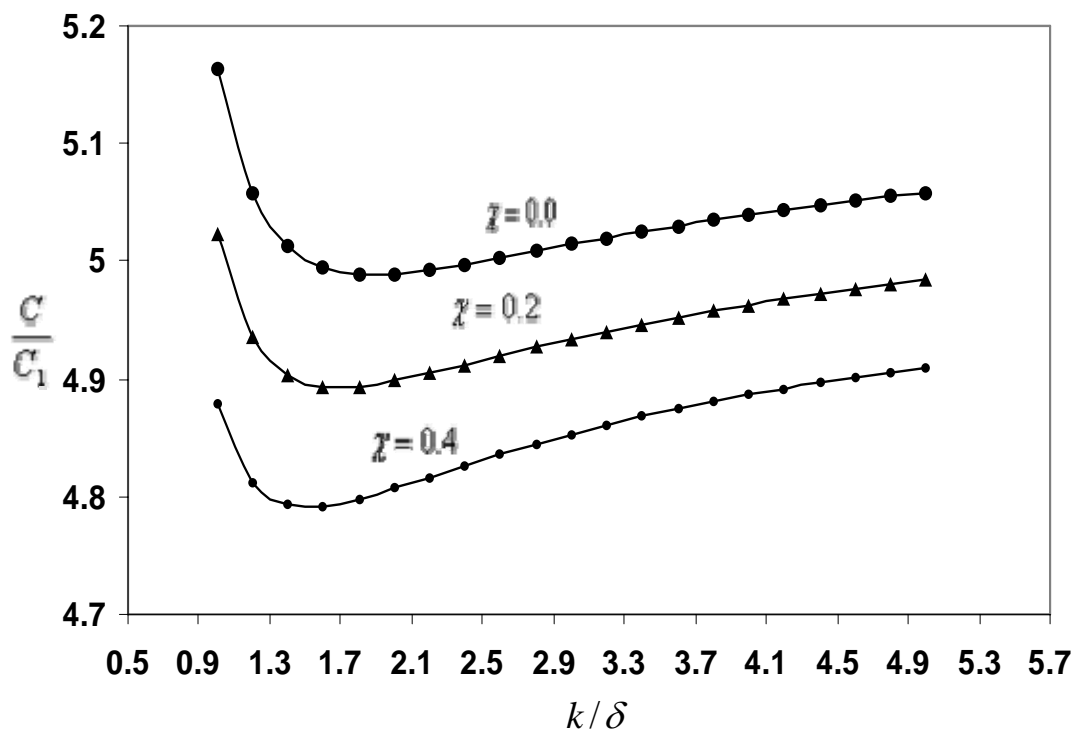


Fig. 2 Variation of C/C_1 versus k/δ at $\chi = 0.0, 0.2, 0.4$.

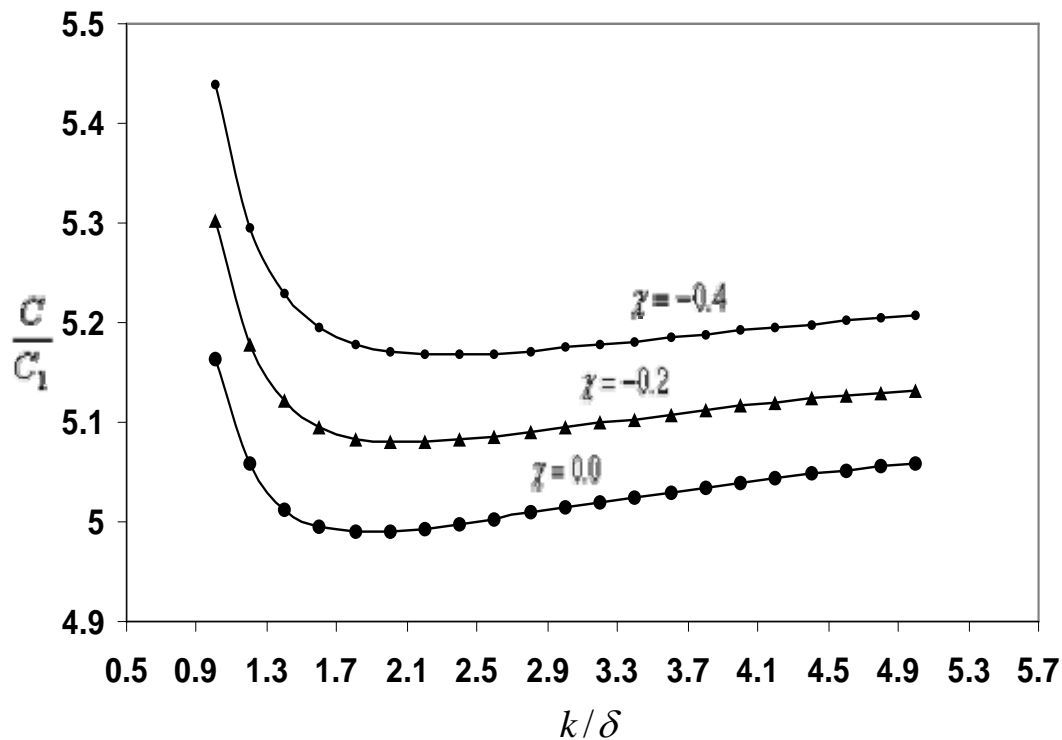


Fig. 3 Variation of C/C_1 versus k/δ at $\chi = 0.0, -0.2, -0.4$.

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