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The Second Gradient Operator and Integral Theorems for Tensor Fields on Curved Surfaces

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Abstract. On the basis of the second gradient operator defined on curved surfaces, the second category of integral theorems for tensor fields, including the second divergence theorems, the second gradient theorems, the second curl theorems and the second circulation theorems, are systematically demonstrated. Simple conservation laws about the mean curvature and Gauss curvature are deduced from the integral theorems.

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1. Introduction

Recently, progress is made in the study of biomembranes. Either for closed biomembranes or for open ones with free edges, the equilibrium differential equations in the normal directions of membranes have the unified form (Y.J. Yin et al., 2005, Y.J. Yin, J. Yin and D. Ni, 2005)

$$\nabla^2\varphi+\bar{\bar{\nabla}}^2\psi+f=0 \quad \text{or} \quad \nabla\cdot\left(\nabla\varphi+\bar{\bar{\nabla}}\psi\right)+f=0 \tag{1}$$

In Eq.(1), $\varphi=\varphi(H,K)$, $\psi=\psi(H,K)$ and $f=f(H,K)$ are three scalar functions derived from the free energy density of the biomembrane. H and K are respectively the mean curvature and Gauss curvature. $\nabla^2=\nabla\cdot\nabla$ is the conventional Laplace-Beltrami operator and $\bar{\bar{\nabla}}^2=\nabla\cdot\bar{\bar{\nabla}}$ is another scalar differential operator on curved surfaces. It is found that two differential operators control the equilibrium configurations and topological structures of biomembranes: One is the classical 2D gradient operator ∇ (K.Z. Huang, M.D. Xue and M.W. Lu, 2003), and another is a new 2D gradient operator $\bar{\bar{\nabla}}$ (Y.J Yin et al., 2005, Y.J. Yin, J. Yin and D. Ni, 2005). In previous researches (Y.J. Yin, 2005), ∇ and $\bar{\bar{\nabla}}$ are termed "the first and the second gradient operators" respectively, because they are dominated respectively by the first and second fundamental tensors.

The mathematical characteristics of the second gradient operator $\bar{\bar{\nabla}}$ are worthy to be explored systematically. The reasons are as follows:

First, gradient is an important concept in science and technology. A gradient is physically a “force” that drives various dynamics in macro or micro scales. Without pressure gradient, deformation gradient, temperature gradient and electromagnetic gradient, there would be no fluid dynamics, solid mechanics, thermal dynamics and electromagnetism. Now that there are two gradients, they may be of equal importance. As a conventional gradient operator, ∇ and its mathematical characteristics are well known to scientists and engineers. Nevertheless as a new gradient operator, $\bar{\nabla}$ and its mathematical characteristics are still very unfamiliar to researchers.

Second, similar to ∇, $\bar{\nabla}$ may also be a fundamental and universal differential invariant. $\bar{\nabla}$ has “appeared” in various soft matters with curved surfaces. It plays dominant roles in biomembranes as well as in liquid crystals. For example, when the smectic-*A* phase liquid crystal grows from the isotropic one (H. Naito, O. Okuda and Z.C. Ou-Yang, 1993), its equilibrium differential equation (H. Naito, O. Okuda and Z.C. Ou-Yang, 1995) may be transformed into be same form as Eq.(1). Hence $\bar{\nabla}$ may be universally applicable to various soft matters with curved structures. Besides, the applicable scope of $\bar{\nabla}$ is far beyond soft matters. For some condensed matters with curved surfaces such as shells in mechanics, $\bar{\nabla}$ may also be very useful.

In short, $\bar{\nabla}$ may be widespread used in various matters or structures with curved surfaces. It’s necessary to reveal the operator’s general mathematical characteristics.

This paper will concentrate on the operator's integral characteristics. In the past, various classical integral theorems called "the first category of integral theorems" can be derived from ∇. Similarly, various new integral theorems named "the second category of integral theorems" may also be obtained from $\overline{\nabla}$ (Y.J Yin, 2005). Nevertheless, these integral theorems are just applicable to vector or scalar fields on curved surfaces. In physics and mechanics, tensor fields are very popular. For example, once the displacement vector $\boldsymbol{v}$ in a deformed shell or cell membrane is given, many important physical quantities will be definable through the displacement gradient $\nabla\boldsymbol{v}$ or $\overline{\nabla}\boldsymbol{v}$. Therefore, the integral theorems for tensor fields will be focused in this paper.

2. The Gradient Operators

The first and second gradient operators are defined respectively by (Y.J. Yin et al., 2005, Y.J. Yin, J. Yin and D. Ni, 2005)

$$\nabla = g^{ij}\boldsymbol{g}_i \frac{\partial}{\partial u^j} \qquad (i, j = 1, 2) \tag{2}$$

$$\overline{\nabla} = \hat{L}^{ij}\boldsymbol{g}_i \frac{\partial}{\partial u^j} \qquad (i, j = 1, 2) \tag{3}$$

In Eq.(2) and Eq.(3), u^i is the Gauss parameter coordinate. $\boldsymbol{g}_i$ is the covariant base vector. g^{ij} is the contravariant component of the first fundamental tensor $\boldsymbol{G}$. $\hat{L}^{ij}$ is the contravariant component of the tensor $\hat{\boldsymbol{L}} = K\boldsymbol{L}^{-1}$ with $\boldsymbol{L}$ the second fundamental tensor.

$\overline{\overline{\nabla}}$ may be of special importance in small-scale curved structures. Because $\overline{\overline{\nabla}}$ is constructed from the second fundamental tensor, it may be influenced intensively by the curvatures of curved surfaces. For example, on a spherical surface with radius R, there is the relation $\overline{\overline{\nabla}} = -\frac{1}{R}\nabla$. Thus the smaller is the radius, the larger is the effect of $\overline{\overline{\nabla}}$. This viewpoint should draw the attentions of researchers. It's well known that the physics on nano-scale curved surfaces are very different from those on macro-scale ones. To understand better the physics on small-scale surfaces, the roles played by $\overline{\overline{\nabla}}$ should not be neglected.

3. The Second Category of Integral Theorems for Tensor Fields

Various theorems connecting line integrals round a closed curve drawn on the surface, with surface integrals taken over the enclosed region, will be proven. Let $\boldsymbol{n}$ be the outward unit normal of the surface and C be a smooth and closed curve drawn on the surface. At any point of this curve, let $\boldsymbol{m}$ be the unit vector tangential to the surface and normal to the curve, drawn outward from the region enclosed by C. Let $\boldsymbol{t}$ be the unit tangent to the curve, in that sense for which $\boldsymbol{m}$, $\boldsymbol{t}$, $\boldsymbol{n}$ form a right-handed system of unit vectors (Fig.1), so that $\boldsymbol{m} = \boldsymbol{t} \times \boldsymbol{n}$,

$\boldsymbol{t} = \boldsymbol{n} \times \boldsymbol{m}$ and $\boldsymbol{n} = \boldsymbol{m} \times \boldsymbol{t}$. The sense of $\boldsymbol{t}$ is the positive sense for a description of the curve. With the aid of the unit vectors, three element vectors may be formulated as $d\boldsymbol{s} = \boldsymbol{m}ds$, $d\boldsymbol{r} = \boldsymbol{t}ds$ and $d\boldsymbol{A} = \boldsymbol{n}dA$. Here ds is the length of an element of the curve. $d\boldsymbol{r}$ is the displacement along the curve in the positive sense. A is the area enclosed by C.

3.1 The second divergence theorem

Without losing universality, any tensor $\boldsymbol{T}$ with rank k on a curved surface can be expressed as

$$\boldsymbol{T} = \boldsymbol{g}^i \boldsymbol{R}_i + \boldsymbol{n}\boldsymbol{S} \qquad (i = 1, 2) \tag{4}$$

where $\boldsymbol{R}_i = \boldsymbol{g}_i \cdot \boldsymbol{T}$ and $\boldsymbol{S} = \boldsymbol{n} \cdot \boldsymbol{T}$ are tensors with rank $k-1$. The second divergence of the tensor $\boldsymbol{T}$ can be proven to be:

$$\bar{\nabla} \cdot \boldsymbol{T} = \frac{1}{\sqrt{g}} \frac{\partial\left(\sqrt{g}\,\hat{L}^{ij} \boldsymbol{R}_i\right)}{\partial u^j} - 2K\boldsymbol{S} \tag{5}$$

Here $g = \left| g_{ij} \right|$ is the determinant of the first fundamental tensor. The surface integral is taken over the region enclosed by C

$$\iint_A \bar{\nabla}\cdot\boldsymbol{T}dA = \iint_A \frac{\partial\left(\sqrt{g}\,\hat{L}^{ij}\boldsymbol{R}_i\right)}{\partial u^j}du^1du^2 - \iint_A 2K\boldsymbol{S}dA \tag{6}$$

The right-hand side may be further written as

$$\iint_A \frac{\partial\left(\sqrt{g}\,\hat{L}^{ij}\boldsymbol{R}_i\right)}{\partial u^j}du^1du^2 = \oint_C d\boldsymbol{s}\cdot\hat{\boldsymbol{L}}\cdot\boldsymbol{T}\ ,\quad \iint_A 2K\boldsymbol{S}dA = \iint_A 2Kd\boldsymbol{A}\cdot\boldsymbol{T} \tag{7}$$

Then Eq.(6) becomes

$$\iint_A \bar{\nabla}\cdot\boldsymbol{T}dA = \oint_C d\boldsymbol{s}\cdot\hat{\boldsymbol{L}}\cdot\boldsymbol{T} - \iint_A 2Kd\boldsymbol{A}\cdot\boldsymbol{T} \tag{8}$$

Eq.(8) is the second divergence theorem for a tensor field. This theorem reveals the conservation between a tensor field and its second divergence on curved surfaces.

3.2 The second gradient theorem

Consider tensor $\boldsymbol{c}_0\boldsymbol{T}$. Here $\boldsymbol{c}_0$ is a constant vector and $\boldsymbol{c}_0\boldsymbol{T}$ is a tensor with rank $k+1$. By substituting $\boldsymbol{T}$ for $\boldsymbol{c}_0\boldsymbol{T}$ and making use of the relation $\bar{\nabla}\cdot\left(\boldsymbol{c}_0\boldsymbol{T}\right) = \boldsymbol{c}_0\cdot\bar{\nabla}\boldsymbol{T}$, one may change Eq.(8) into

$$\iint_A \boldsymbol{c}_0 \cdot \bar{\bar{\nabla}} \boldsymbol{T} dA = \oint_C d\boldsymbol{s} \cdot \hat{\boldsymbol{L}} \cdot \boldsymbol{c}_0 \boldsymbol{T} - \iint_A 2K d\boldsymbol{A} \cdot \boldsymbol{c}_0 \boldsymbol{T} \tag{9}$$

Note $d\boldsymbol{s} \cdot \hat{\boldsymbol{L}} \cdot \boldsymbol{c}_0 = \boldsymbol{c}_0 \cdot \left(d\boldsymbol{s} \cdot \hat{\boldsymbol{L}} \right)$. Thus Eq.(9) may be rewritten as

$$\boldsymbol{c}_0 \cdot \left(\iint_A \bar{\bar{\nabla}} \boldsymbol{T} dA - \oint_C d\boldsymbol{s} \cdot \hat{\boldsymbol{L}} \boldsymbol{T} + \iint_A 2K d\boldsymbol{A} \boldsymbol{T} \right) = \boldsymbol{0} \tag{10}$$

Let $\boldsymbol{c}_m$ $(m = 1, 2, \ldots k)$ be a series of constant vectors. One may define a multiple dot product $\left\{ \left[\left(\boldsymbol{T} \cdot \boldsymbol{c}_1 \right) \cdot \boldsymbol{c}_2 \right] \ldots \right\} \cdot \boldsymbol{c}_k = \boldsymbol{T} \cdot \ldots \cdot \boldsymbol{c}$. Here $\boldsymbol{c} = \boldsymbol{c}_1 \boldsymbol{c}_2 \ldots \boldsymbol{c}_k$ is a constant tensor with rank k and $\boldsymbol{T} \cdot \ldots \cdot \boldsymbol{c}$ is a scalar. If $\boldsymbol{T}$ in Eq.(10) is replaced by $\boldsymbol{T} \cdot \ldots \cdot \boldsymbol{c}$, then one has $\bar{\bar{\nabla}} \left(\boldsymbol{T} \cdot \ldots \cdot \boldsymbol{c} \right) = \left(\bar{\bar{\nabla}} \boldsymbol{T} \right) \cdot \ldots \cdot \boldsymbol{c}$. At last Eq.(10) becomes

$$\boldsymbol{c}_0 \cdot \left(\iint_A \bar{\bar{\nabla}} \boldsymbol{T} dA - \oint_C d\boldsymbol{s} \cdot \hat{\boldsymbol{L}} \boldsymbol{T} + \iint_A 2K d\boldsymbol{A} \boldsymbol{T} \right) \cdot \ldots \cdot \boldsymbol{c} = 0 \tag{11}$$

Since this is true for all constant $\boldsymbol{c}_0$ and $\boldsymbol{c}$, it follows

$$\iint_A \bar{\bar{\nabla}} \boldsymbol{T} dA = \oint_C d\boldsymbol{s} \cdot \hat{\boldsymbol{L}} \boldsymbol{T} - \iint_A 2K d\boldsymbol{A} \boldsymbol{T} \tag{12}$$

Here $\bar{\bar{\nabla}} \boldsymbol{T}$ is the second gradient of tensor $\boldsymbol{T}$. Eq.(12) is the second gradient theorem for a tensor field. This theorem displays the conservation between a tensor field and its second gradient on curved surfaces.

3.3 The second curl theorem

Apply Eq.(8) to tensor $\boldsymbol{c}_0 \times \boldsymbol{T}$. For constant $\boldsymbol{c}_0$ one has $\bar{\nabla} \cdot (\boldsymbol{c}_0 \times \boldsymbol{T}) = -\boldsymbol{c}_0 \cdot (\bar{\nabla} \times \boldsymbol{T})$. Thus Eq.(8) may be reconstructed as

$$-\iint_A \boldsymbol{c}_0 \cdot (\bar{\nabla} \times \boldsymbol{T})\, dA = \oint_C ds\, \hat{\boldsymbol{L}} \cdot (\boldsymbol{c}_0 \times \boldsymbol{T}) - \iint_A 2K d\boldsymbol{A} \cdot (\boldsymbol{c}_0 \times \boldsymbol{T}) \tag{13}$$

Note $ds\, \hat{\boldsymbol{L}} \cdot (\boldsymbol{c}_0 \times \boldsymbol{T}) = -\boldsymbol{c}_0 \cdot (ds\, \hat{\boldsymbol{L}} \times \boldsymbol{T})$ and $d\boldsymbol{A} \cdot (\boldsymbol{c}_0 \times \boldsymbol{T}) = -\boldsymbol{c}_0 \cdot (d\boldsymbol{A} \times \boldsymbol{T})$. Eq.(13) may be rewritten as

$$\boldsymbol{c}_0 \cdot \left(\iint_A \bar{\nabla} \times \boldsymbol{T} dA - \oint_C ds\, \hat{\boldsymbol{L}} \times \boldsymbol{T} + \iint_A 2K d\boldsymbol{A} \times \boldsymbol{T} \right) = \boldsymbol{0} \tag{14}$$

For a series of constant vectors $\boldsymbol{c}_m$ $(m = 1, 2, ..., k-1)$, a multiple dot product $\{[(\boldsymbol{T} \cdot \boldsymbol{c}_1) \cdot \boldsymbol{c}_2]...\} \cdot \boldsymbol{c}_{k-1} = \boldsymbol{T} \cdot .. \cdot \boldsymbol{c}$ may be defined. Here $\boldsymbol{c} = \boldsymbol{c}_1 \boldsymbol{c}_2 ... \boldsymbol{c}_{k-1}$ is a constant tensor with rank $k-1$ and $\boldsymbol{T} \cdot .. \cdot \boldsymbol{c}$ is vector. If $\boldsymbol{T}$ in Eq.(14) is replaced by $\boldsymbol{T} \cdot .. \cdot \boldsymbol{c}$, then one has $\bar{\nabla} \times (\boldsymbol{T} \cdot .. \cdot \boldsymbol{c}) = (\bar{\nabla} \times \boldsymbol{T}) \cdot .. \cdot \boldsymbol{c}$. At last Eq.(14) becomes

$$\boldsymbol{c}_0 \cdot \left(\iint_A \bar{\bar{\nabla}} \times \boldsymbol{T} dA - \oint_C d\boldsymbol{s} \cdot \hat{\boldsymbol{L}} \times \boldsymbol{T} + \iint_A 2K d\boldsymbol{A} \times \boldsymbol{T} \right) \cdot\cdot \boldsymbol{c} = 0 \tag{15}$$

Since this is true for all constant $\boldsymbol{c}_0$ and $\boldsymbol{c}$, it follows

$$\iint_A \bar{\bar{\nabla}} \times \boldsymbol{T} dA = \oint_C d\boldsymbol{s} \cdot \hat{\boldsymbol{L}} \times \boldsymbol{T} - \iint_A 2K d\boldsymbol{A} \times \boldsymbol{T} \tag{16}$$

Here $\bar{\bar{\nabla}} \times \boldsymbol{T}$ is the second curl of the tensor $\boldsymbol{T}$. Eq.(16) is the second curl theorem for a tensor field. This theorem shows the conservation between a tensor field and its second curl on curved surfaces.

3.4 The generalized second circulation theorems

Apply Eq.(8) to tensor $\boldsymbol{n} \times \boldsymbol{T}$. Note $\bar{\bar{\nabla}} \times \boldsymbol{n} = \boldsymbol{0}$, $\bar{\bar{\nabla}} \cdot \left(\boldsymbol{n} \times \boldsymbol{T} \right) = \boldsymbol{n} \cdot \left(\bar{\bar{\nabla}} \times \boldsymbol{T} \right)$ and $d\boldsymbol{A} \cdot \left(\boldsymbol{n} \times \boldsymbol{T} \right) = 0$. Thus Eq.(8) may be transformed into

$$\iint_A d\boldsymbol{A} \cdot \left(\bar{\bar{\nabla}} \times \boldsymbol{T} \right) = \oint_C d\boldsymbol{s} \cdot \hat{\boldsymbol{L}} \cdot \left(\boldsymbol{n} \times \boldsymbol{T} \right) \tag{17}$$

Because of the relation $d\boldsymbol{s} \cdot \hat{\boldsymbol{L}} \cdot \left(\boldsymbol{n} \times \boldsymbol{T} \right) = d\boldsymbol{r} \cdot \boldsymbol{L} \cdot \boldsymbol{T}$, the following integral transformation may keep valid

$$\iint_A d\boldsymbol{A}\cdot\left(\bar{\bar{\nabla}}\times\boldsymbol{T}\right)=\oint_C d\boldsymbol{r}\cdot\boldsymbol{L}\cdot\boldsymbol{T} \tag{18}$$

If $\oint_C d\boldsymbol{r}\cdot\boldsymbol{T}$ is called the first circulation of the tensor $\boldsymbol{T}$, then $\oint_C d\boldsymbol{r}\cdot\boldsymbol{L}\cdot\boldsymbol{T}$ may be named the second circulation of the tensor $\boldsymbol{T}$. Eq.(18) is the second circulation theorem for a tensor field. This theorem reveals the conservation between the second curl and the second circulation of a tensor field on curved surfaces.

One can prove the relation $\boldsymbol{n}\cdot\left(\bar{\bar{\nabla}}\times\boldsymbol{T}\right)=\left(\boldsymbol{n}\times\bar{\bar{\nabla}}\right)\cdot\boldsymbol{T}$. Thus Eq.(18) may be rewritten as

$$\iint_A \left(d\boldsymbol{A}\times\bar{\bar{\nabla}}\right)\cdot\boldsymbol{T}=\oint_C d\boldsymbol{r}\cdot\boldsymbol{L}\cdot\boldsymbol{T} \tag{19}$$

In sections 3.2 and 3.3, Eq.(12) and Eq.(16) are derived on the basis of the second divergence theorem in Eq.(8). Similarly, the same procedures may also be used here, and the following two equations may be derived on the basis of the second circulation theorem in Eq.(19):

$$\iint_A \left(d\boldsymbol{A}\times\bar{\bar{\nabla}}\right)\boldsymbol{T}=\oint_C d\boldsymbol{r}\cdot\boldsymbol{L}\boldsymbol{T} \tag{20}$$

$$\iint_A \left(d\boldsymbol{A}\times\bar{\bar{\nabla}}\right)\times\boldsymbol{T}=\oint_C d\boldsymbol{r}\cdot\boldsymbol{L}\times\boldsymbol{T} \tag{21}$$

Eq.(19) ~ Eq.(21) may be termed the “generalized second circulation theorems”.

3.5 Unified formulations of the above theorems

For convenience, Eq.(8), Eq.(12) and Eq.(16) may be unified as follows

$$\iint_A \bar{\nabla} \otimes \boldsymbol{T} dA = \oint_C d\boldsymbol{s} \cdot \hat{\boldsymbol{L}} \otimes \boldsymbol{T} - \iint_A 2K d\boldsymbol{A} \otimes \boldsymbol{T} \qquad (22)$$

If the symbol “$\otimes$” is replaced by “$\cdot$”, then Eq.(8) will be deduced. If “$\otimes$” is eliminated, Eq.(12) will be derived. If “$\otimes$” is replaced by “$\times$”, Eq.(16) will be obtained. In short, Eq.(22) is the unified expression of the second divergence theorem, the second gradient theorem and the second curl theorem for a tensor on curved surfaces.

Similarly, the generalized second circulation theorems in Eq.(19) ~ Eq.(21) may be unified as

$$\iint_A \left(d\boldsymbol{A} \times \bar{\nabla} \right) \otimes \boldsymbol{T} = \oint_C d\boldsymbol{r} \cdot \boldsymbol{L} \otimes \boldsymbol{T} \qquad (23)$$

4. Comparisons between Two Categories of Integral Theorems for Tensor Fields

The first divergence of the tensor $\boldsymbol{T}$ can be demonstrated to be

$$\nabla \cdot \boldsymbol{T} = \frac{1}{\sqrt{g}} \frac{\partial\left(\sqrt{g}\, g^{ij} \boldsymbol{R}_i\right)}{\partial u^j} - 2H\boldsymbol{S} \tag{24}$$

Similar to section 3, the first category of integral theorems for the tensor field may be expressed as:

$$\iint_A \nabla \otimes \boldsymbol{T} dA = \oint_C d\boldsymbol{s} \otimes \boldsymbol{T} - \iint_A 2H d\boldsymbol{A} \otimes \boldsymbol{T} \tag{25}$$

$$\iint_A \left(d\boldsymbol{A} \times \nabla\right) \otimes \boldsymbol{T} = \oint_C d\boldsymbol{r} \otimes \boldsymbol{T} \tag{26}$$

It's interesting to note that the analytical structures of the two categories of integral theorems are completely symmetric. If $\overline{\overline{\nabla}}$, $d\boldsymbol{s} \cdot \hat{\boldsymbol{L}}$, K and $d\boldsymbol{r} \cdot \boldsymbol{L}$ are substituted by ∇, $d\boldsymbol{s}$, H and $d\boldsymbol{r}$ respectively, then the second category in Eq.(22) and Eq.(23) will degenerate to the first one in Eq.(25) and Eq.(26) respectively, and vise versa.

If the tensor $\boldsymbol{T}$ is replaced by vector $\boldsymbol{v}$ or scalar φ, then Eq.(25), Eq.(26) will degenerate to the first category of integral theorems for vector or scalar field in conventional geometry; and Eq.(22), Eq.(23) will degenerate to the second category of integral theorems for vector or scalar field (Y.J. Yin, 2005).

The concepts related to the right-hand sides of Eq.(23) and Eq.(26), i.e. the second circulation and the first one, need to be further explored and compared. In fluid mechanics, it has been proven that the lift efficiency of a large-scale wing is mainly attributed to the first circulation $\oint_C d\boldsymbol{r}\cdot\boldsymbol{v}$ generated by the flow on the wing's surface. The larger the first circulation is, the larger the lift force is. Now an interesting idea may be stimulated: The first circulation $\oint_C d\boldsymbol{r}\cdot\boldsymbol{v}$ is the origin for lift force under large-scale flow. Similarly, is it possible that the second circulation $\oint_C d\boldsymbol{r}\cdot\boldsymbol{L}\cdot\boldsymbol{v}$ is the origin for a new lift force under small-scale flow? If this is true, the existence of new lift force mechanism for insects may become possible.

5. Conservation Laws about the Mean Curvature and Gauss Curvature

From Eq.(25) and Eq.(22), some interesting results may be deduced. Replace "$\otimes$" by "$\cdot$" and $\boldsymbol{T}$ by $\boldsymbol{r}$, where $\boldsymbol{r}$ is the position vector of a point on the curved surface. Because $\nabla\cdot\boldsymbol{r} = 2$ and $\overline{\nabla}\cdot\boldsymbol{r} = 2H$, one has

$$\iint_A 2dA = \oint_C d\boldsymbol{s}\cdot\boldsymbol{r} - \iint_A 2Hd\boldsymbol{A}\cdot\boldsymbol{r} \tag{27}$$

$$\iint_A 2HdA = \oint_C d\boldsymbol{s}\cdot\hat{\boldsymbol{L}}\cdot\boldsymbol{r} - \iint_A 2Kd\boldsymbol{A}\cdot\boldsymbol{r} \tag{28}$$

Eq.(28) has appeared in . If the curve C converges to a point, then on the smooth and closed surface Eq.(27) and Eq.(28) will become

$$\oiint_A dA = -\oiint_A H d\boldsymbol{A}\cdot\boldsymbol{r} \tag{29}$$

$$\oiint_A H dA = -\oiint_A K d\boldsymbol{A}\cdot\boldsymbol{r} \tag{30}$$

Eq.(29) and Eq.(30) are exactly the Minkowski integral formulas in differential geometry.

Let $\boldsymbol{T} = \boldsymbol{G}$ in Eq.(25). Note $\nabla\cdot\boldsymbol{G} = 2H\boldsymbol{n}$, $d\boldsymbol{s}\cdot\boldsymbol{G} = d\boldsymbol{s}$ and $d\boldsymbol{A}\cdot\boldsymbol{G} = \boldsymbol{0}$. One has

$$\oint_C d\boldsymbol{s} = \iint_A 2H d\boldsymbol{A} \tag{31}$$

Eq.(31) is an important integral theorem about the mean curvature in conventional geometry. Similarly, let $\boldsymbol{T} = \boldsymbol{G}$ in Eq.(22). Note $\overline{\nabla}\cdot\boldsymbol{G} = 2K\boldsymbol{n}$, $d\boldsymbol{s}\cdot\hat{\boldsymbol{L}}\cdot\boldsymbol{G} = d\boldsymbol{s}\cdot\hat{\boldsymbol{L}}$ and $d\boldsymbol{A}\cdot\boldsymbol{G} = \boldsymbol{0}$. Eq.(22) becomes

$$\oint_C d\boldsymbol{s}\cdot\hat{\boldsymbol{L}} = \iint_A 2K d\boldsymbol{A} \tag{32}$$

Eq.(32) is an important integral theorem about the Gauss curvature (Y.J. Yin, 2005). This equation can also be obtained from Eq.(25) if one lets $\boldsymbol{T} = \hat{\boldsymbol{L}}$ and replaces "$\otimes$" by "$\cdot$". Because of the relation $ds\cdot\hat{\boldsymbol{L}} = \left(k_n\boldsymbol{m} + \tau_g\boldsymbol{t}\right)ds$ on curve C, Eq.(32) may be further expressed as:

$$\oint_C \left(k_n\boldsymbol{m} + \tau_g\boldsymbol{t}\right)ds = \iint_A 2Kd\boldsymbol{A} \tag{33}$$

Here $k_n = \frac{d\boldsymbol{t}}{ds}\cdot\boldsymbol{n}$ and $\tau_g = \frac{d\boldsymbol{n}}{ds}\cdot\boldsymbol{m}$ are respectively the normal curvature and geodesic torsion of the curve C.

In differential geometry, there is another integral theorem about Gauss curvature K, i.e. the famous Gauss-Bonnet (local) integral theorem (M.P. Carmo, 1976):

$$2\pi - \oint_C k_g ds = \iint_A KdA \tag{34}$$

Here $k_g = \frac{d\boldsymbol{t}}{ds}\cdot\boldsymbol{m}$ is the geodesic curvature of the curve C. Although Eq.(33) and Eq.(34) are all related to K, they are very different: The former is a vector integral theorem, while the latter is a scalar one. The former represents the conserved characteristics of vector $K\boldsymbol{n}$, while the latter reflects the conserved properties of scalar K. The former is concerned in the normal curvature k_n and the geodesic torsion τ_g, while the latter is connected with the geodesic curvature

k_g .

There are many applications of the Gauss-Bonnet integral theorem, both in mathematics and in other scientific disciplines. Fortunately, the new integral theorem in Eq.(33) also shows its powers in bionano sciences and carbon nano sciences (Y.J. Yin and J. Yin, 2006, Y.J. Yin et al., 2006). Here a simple example is displayed. For a smooth and closed surface, the left-hand sides of Eq.(31) and Eq.(33) will vanish (Y.J. Yin, 2005):

$$\oint\!\!\oint_A H d\boldsymbol{A} = \boldsymbol{0} \tag{35}$$

$$\oint\!\!\oint_A K d\boldsymbol{A} = \boldsymbol{0} \tag{36}$$

Eq.(35) and Eq.(36) are also Minkowski integral formulas. In cell biology, smooth and closed cell membranes or vesicles will obey Eq.(35) and Eq.(36).

If one lets the curve C converge smoothly and tangentially to a point outside the surface, then a closed surface with a singular point may be formed (Fig.2a). In this case Eq.(33) may be deduced as

$$\oint\!\!\oint_A K d\boldsymbol{A} = -\pi \boldsymbol{m} \tag{37}$$

Here $\boldsymbol{m}$ characterizes the direction of the singular point. If the number of singular points on the surface is n ($n = 2$ in Fig.2b), then Eq.(37) may be further extended as:

$$\oiint_A K d\boldsymbol{A} = -\pi \sum_{i=1}^{n} \boldsymbol{m}_i \tag{38}$$

In cell biology, cells or vesicles with sharp protuberances always occur. Geometrically such cells or vesicles may be idealized as closed surfaces with singular points and Eq.(38) should be satisfied.

6. Conclusions

For tensor fields there are the second category of integral theorems on curved surfaces, including the second divergence theorem, the second gradient theorem, the second curl theorem and the second circulation theorem. Analytically the first and second categories of integral theorems have symmetric structures. Geometrically, the first category is mainly controlled by the first fundamental tensor, while the second category is mainly dominated by the second fundamental tensor. This implies that the former specially is applicable to structures with large scale, while the latter may play dominant roles in structures with small scale such as cell membranes whose spaces are intensively bended. Besides, from these integral theorems the Minkowski integral formulas are deduced easily and cells with singular points are depicted perfectly.

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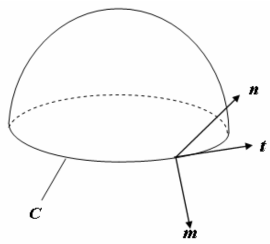

Fig.1 Schematic of the curved surface with unit vectors $\boldsymbol{m}$, $\boldsymbol{t}$ and $\boldsymbol{n}$ at its boundary

(a) One singular point (b) Two singular points

Fig.2. Cells or vesicles with one or two singular points

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