

Factor Demand Modelling: the Theory and the Practice

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Abstract

Since the work of Cobb and Douglas [18], two main innovations have been introduced in applied factor demand analysis, i.e. the use of flexible functional forms and the modelling of dynamics, expectations, and the interrelatedness of the adjustment process. Recently, cointegration theory has provided an additional important contribution, yielding empirical content to the notion of equilibrium employed in economic analysis, encompassing both the idea of centre of gravity relationship, suggested by Classical economists, and the notion of market-clearing position, employed by Neoclassical economist. Also in the light of the most recent generalizations of the concept of cointegration, allowing for economic attractors changing over time, as the evolution of the structural features of the economy proceeds, this paper critically assess the key theoretical and empirical issues in factor demand analysis.

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1 Introduction

Since the seminal work of Cobb and Douglas [18], two major innovative contributions have been provided in the literature on factor demand modelling.

The first contribution, which origins may be traced back to the work of Arrow et al. [5], and continued with Diewert [26], Christensen et al. [16], [17],

and many others¹, aimed at relaxing the a priori assumption of unitarian elasticities of substitution for all the factors characterizing, by construction, the Cobb-Douglas formulation. The CES functional form, introduced by Arrow et al. [5], yield only a partially satisfactory solution, since in this latter formulation the elasticities of substitution still are required to be constant for all the factors, although not necessarily equal to the unity. As shown by Berndt and Christensen [14], imposing a priori assumptions on the elasticity of substitution is equivalent to specify assumptions on the functional structure, both the Cobb-Douglas and CES functions assuming strong separability, and, therefore, the existence of consistent aggregates.² The introduction of the so called “flexible functional forms”, namely the generalized Leontief function (Diewert [26]) and the transcendental logarithmic function (Christensen et al. [16], [17]), yield the solution to the above problem, since flexibility is the property of leaving the substitution pattern of factors to be fully determined by the data.³ Theoretical advances in duality theory, allowing for an easy derivation of factor demand functions from complex cost function specifications, were also employed in the new framework.

On the other hand, the second contribution was concerned with the modelling of dynamics, since the original formulations, as many contributions which followed, were set in a fully static framework. According to Berndt et al. [11], three main generations of dynamic factor demand models can be pointed out, although they could be reduced to just two, on the basis of the theoretical foundation underlying the specification of the dynamic adjustment process. In fact, while in the first and second generation models dynamic specifications are determined empirically (ad hoc), in the third generation models the

¹Other flexible functional forms followed: the generalized CES form by Denny [24], the generalized quadratic function by Lau [61], the nested generalized Leontief form by Fuss [36], the generalized Cobb-Douglas function by Diewert [27] and Magnus [65], the generalized Box-Cox form by Applebaum [4], the Almost Ideal Demand System by Deaton and Muellbauer [21], the Fourier flexible functional form by Gallant [38], the Muntz Szatz demand system by Barnett and Jonas [9], the minflex Laurent generalized Leontief and translog by Barnett [8], [10], the generalized McFadden and generalized Barnett by Diewert and Wales [28], the nonseparable nested CES by Perroni and Rutherford [82].

²The assumption of strong separability implies that the marginal rate of substitution between any two production inputs is independent of the quantity employed of any other factor.

³The basic idea underlying the concept of flexibility is that of second order approximation. According to McFadden [76] and Diewert [26] a function is a second order approximation to a function, at a given point of approximation, if the first- and second-order derivatives of the two functions are equal at the point of approximation, i.e. if arbitrary levels and first- and second-order derivatives can be achieved by the function at the point of approximation. On the other hand, according to Christensen et al. [17], a function is a second-order approximation to a function, at a given point of approximation, if the deviation of the approximation function from the true function consists of terms of the third-order for any value included in a given neighborhood of the approximation point.

specification of the adjustment process is derived from an explicit theoretical framework.

The above two strands of contributions, however, did not work separately, and since the 1970s dynamic models using flexible functional forms have been widely used in the literature.

Sections 2-3 review the static approach to factor demand modelling, based on the application of duality theory and flexible functional forms in empirical analysis. In sections 4-5 the dynamic approach is reviewed and the most recent advances in dynamic factor demand modelling are presented in the light of the developments which have occurred in econometric modelling since the 1980s.

2 Static factor demand modelling and the cost function approach

The principle of duality⁴ in production theory states that, given the production function $Y = f(\mathbf{x})$, under appropriate regularity conditions, it is possible to uniquely derive the corresponding producer's total minimum cost function $C(\mathbf{p}, Y)$ as the solution to the problem of minimizing the cost of producing a specified level of output, that is

$$C(\mathbf{p}, y) = \min_x \{ \mathbf{p}\mathbf{x} : f(\mathbf{x}) \geq Y \}, \quad (1)$$

where \mathbf{x} is a $k \times 1$ vector of input quantities, \mathbf{p} is a $k \times 1$ vector of input prices, and Y is the maximal amount of output which can be produced in a given time period. The converse is also true. Given a cost function, which satisfies the usual regularity conditions, the corresponding production function can be uniquely derived as the solution to the problem of maximizing the produced output level, given a specified level of cost and input quantities, that is

$$f^*(\mathbf{x}) = \max_Y \{ Y : C(\mathbf{p}, Y) \leq \mathbf{p}\mathbf{x}, \mathbf{p} \geq \mathbf{0} \}. \quad (2)$$

The regularity conditions required on the cost function are:

- i) if $\mathbf{p} \geq \mathbf{p}'$, then $C(\mathbf{p}, Y) \geq C(\mathbf{p}', Y)$, $\mathbf{p} > \mathbf{0}$;
- ii) $C(\lambda\mathbf{p}, Y) = \lambda C(\mathbf{p}, Y)$, $\forall \lambda > 0$, $\mathbf{p} > \mathbf{0}$;
- iii) $C(\lambda\mathbf{p}' + (1 - \lambda)\mathbf{p}'', Y) \geq \lambda C(\mathbf{p}', Y) + (1 - \lambda)C(\mathbf{p}'', Y)$, $0 \leq \lambda \leq 1$, $\mathbf{p}', \mathbf{p}'' > \mathbf{0}$;

⁴The theory of production duality was formulated by Hotelling [58], Roy [88], Hicks [56]. Samuelson [91], [92] and Shephard [94] provided the first comprehensive treatment of the subject and proof of the basic duality of production and cost. Uzawa [103], McFadden [72], Hanoch [49], Shephard [95], Denny [23], Diewert [26], Hall [46], Ruys [89], Waddepohl [104], Epstein [33], and Epstein and Denny [32] proposed later extensions.

iv) $C(\mathbf{p}, Y)$ is a continuous function of \mathbf{p} , for $\mathbf{p} > \mathbf{0}$.

Condition i) requires that the function $C(\cdot)$ is nondecreasing in \mathbf{p} ; condition ii) ensures that the function $C(\cdot)$ is positive homogeneous of degree one; condition iii) requires that the function $C(\cdot)$ is a concave function; finally, condition iv) just requires the cost function to be a continuous functions of input prices.

A result of particular importance is Shephard's [94] Lemma: if the cost function $C(\cdot)$ satisfies the requirements stated above, and, in addition, is differentiable with respect to input prices at the point $\mathbf{p}^* > \mathbf{0}$, then

$$\frac{\partial C(\mathbf{p}^*, Y)}{\partial p_i} = x_i(\mathbf{p}^*, Y) \quad \forall i, \quad (3)$$

where $x_i(\mathbf{p}^*, Y)$ is the cost minimizing quantity of input i needed to produce Y units of output given input prices \mathbf{p}^* , that is the derived factor demand for input i .

Since the cost function is a nondecreasing function, it follows that the derived factor demands are nonnegative ($x_i(\mathbf{p}^*, Y) \geq 0$), and homogeneous of degree zero in \mathbf{p}^* .⁵ In addition, from the concavity of the cost function it follows that the matrix of its second derivatives, that is the matrix of first derivatives of the factor demand functions, is a symmetric negative semidefinite matrix⁶; thus, the cross-price effects are symmetric

$$\frac{\partial x_i(\mathbf{p}^*, Y)}{\partial p_j} = \frac{\partial^2 C(\mathbf{p}^*, Y)}{\partial p_j \partial p_i} = \frac{\partial^2 C(\mathbf{p}^*, Y)}{\partial p_i \partial p_j} = \frac{\partial x_j(\mathbf{p}^*, Y)}{\partial p_i}, \quad (4)$$

and the own-price effects are nonpositive, that is, the conditional factor demands are downward sloping

$$\frac{\partial x_i(\mathbf{p}^*, Y)}{\partial p_i} = \frac{\partial^2 C(\mathbf{p}^*, Y)}{\partial^2 p_i} \leq 0. \quad (5)$$

The key implication of duality theory for applied analysis is that it allows for a straightforward derivation of a system of factor demand functions by simple differentiation of a cost function satisfying properties i)-iv). Hence, the problem of recovering the exact functional specifications from the solution of the profit maximization problem does not arise, allowing the use of more general functional forms, as the flexible approximations, without imposing a priori constraints on the pattern of factors substitution.

⁵The homogeneity of degree zero of factor demands follows from the homogeneity of degree one in \mathbf{p}^* of the cost function.

⁶The symmetry and negative semidefiniteness of the first derivatives matrix of the factor demand functions follows only from the hypothesis of cost minimization, since the concavity of the cost function follows solely from the hypothesis of cost minimization: no restrictions on the structure of the technology are involved.

3 The static modelling approach in practice

Separability and substitution are two issues of fundamental importance in empirical analysis. The assumption of separability is necessary to sustain efficient two-stage allocation procedures. On the other hand, the substitution process among the inputs of production or consumption goods is related to the comparative statics properties of the corresponding derived demands. Given the twice-differentiable, strictly quasi-concave homothetic production function $Y = f(\mathbf{x})$, where \mathbf{x} is an n -dimensional input vector characterized by strictly positive marginal products, the input set may be partitioned in q mutually exclusive and exhaustive subsets V_i $i = 1, \dots, q$. Weak separability of the production function with respect to the partition selected requires

$$\frac{\partial(f_i/f_j)}{\partial x_k} = 0 \quad \forall i, j \in V_i, k \notin V_i, \quad (6)$$

where $f_i = \frac{\partial f(\mathbf{x})}{\partial x_i}$ and $f_j = \frac{\partial f(\mathbf{x})}{\partial x_j}$, that is that the marginal rate of substitution between any two production factors is independent of the quantities of the inputs belonging to the other subset. Goldman and Uzawa [41] have proved that weak separability is a necessary and sufficient condition for the production function to be rewritten in the form

$$Y = f(g_i(\mathbf{x}), \dots, g_r(\mathbf{x})), \quad (7)$$

where $g_i(\mathbf{x})$ is a consistent aggregate index of the elements of V_i . Given the duality between the production and cost functions, conditions [6] and [7] may be rewritten in the cost function framework. Given the cost function dual to the production function $Y = f(\mathbf{x})$

$$C(\mathbf{p}, Y)Y = H(Y)C(\mathbf{p}),$$

we have, therefore,

$$\frac{\partial(C_i/C_j)}{\partial p_k} = 0 \quad \forall i, j \in V_i, k \notin V_i, \quad (8)$$

where $C_i = \frac{\partial C(\mathbf{p})}{\partial p_i}$ and $C_j = \frac{\partial C(\mathbf{p})}{\partial p_j}$, and

$$C = C(d_1(\mathbf{p}), \dots, d_r(\mathbf{p})),$$

where $d_i(\mathbf{p})$ is a function of the prices of the elements belonging to V_i only.

As shown by Denny and Fuss [22], if the aggregates $g_i(\mathbf{x})$ ($d_i(\mathbf{p})$) are homothetic in their components, the optimization process can be conducted sequentially, by optimally choosing the composition of each subset in a first stage

and by optimizing among the different aggregates successively. Therefore, the assumption of weak separability has two important practical implications. Firstly, weak separability ensures the existence of aggregator functions, so that the separate analysis of submodels is justified. Secondly, if the aggregates are homothetic in their components, the optimization procedure may be conducted in a multi-stage framework, and the separate analysis of each aggregates at the time is allowed for.

In the literature the cost function approach has been widely employed, not only because of the straightforward derivation of the derived demands from the cost function by Shephard's Lemma, but, consistent with economic theory, also because of the reduction in dimensionality yield by the modelling of factor prices as exogenous variables.

Moreover, linear (in the parameters) specifications can be obtained, often expressed in cost share form, i.e. for a generic cost function $C(\mathbf{p}, Y)$ it is found

$$\frac{\partial \ln C}{\partial \ln p_i} = \frac{p_i}{C} \frac{\partial C}{\partial p_i} = \frac{p_i q_i}{C} = S_i, \quad (9)$$

where $\sum_{i=1}^n p_i q_i = C$. Because the shares must sum to unity, the adding-up condition

$$\sum_{i=1}^n S_i = 1. \quad (10)$$

then holds.

The Allen-Uzawa partial elasticities of substitution (AES) and price elasticities can be easily computed from the cost shares specification. The AES between inputs x_i and x_j yields a measure of the response of the derived demand for input x_i to a price change in x_j , holding output and the other factor prices constant. The AES between inputs x_i and x_j may be written as

$$\sigma_{ij} = \frac{\sum_{h=1}^n f_h x_h |\bar{f}_{ij}|}{x_i x_j |\bar{f}_{ij}|}, \quad (11)$$

where $|\bar{f}|$ is the determinant of the bordered Hessian matrix

$$\bar{f} = \begin{bmatrix} 0 & f_1 & f_2 & \dots & f_n \\ f_1 & f_{11} & f_{12} & \dots & f_{1n} \\ f_2 & f_{21} & f_{22} & \dots & f_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_n & f_{n1} & f_{n2} & \dots & f_{nn} \end{bmatrix},$$

$f_h = \frac{\partial f(\mathbf{x})}{\partial x_h}$, and $|\bar{f}_{ij}|$ is the cofactor of $f_{ij} = \frac{\partial f(\mathbf{x})}{\partial x_i \partial x_j}$ in $|\bar{f}|$.

In terms of the cost function, the Allen-Uzawa partial elasticities of substitution between inputs i and j can be written as

$$\sigma_{ij} = \frac{C C_{ij}}{C_i C_j}, \quad (12)$$

where the i, j subscripts refer to first and second partial derivatives of the cost function $C(\cdot)$ with respect to input prices p_i, p_j . In general, the estimated elasticities vary across observations and depend on the point in the data set at which they are measured. The partial price elasticities

$$\varepsilon_{ij} = \frac{\partial x_i / x_i}{\partial p_j / p_j} \frac{x_j p_j}{\sum_{h=1}^n p_h x_h} \quad (13)$$

may be derived from the elasticities of substitution as follows

$$\varepsilon_{ij} = S_j \sigma_{ij}, \quad (14)$$

with $\sum_{j=1}^n \varepsilon_{ij} = 0$, where S_j is the cost share relative to input j . Lastly, a measure of the returns to scale can be computed via the inverse of the elasticity of total cost with respect to output

$$\mu = 1/\eta_{CY}, \quad (15)$$

where $\eta_{CY} = \frac{\partial \ln C}{\partial \ln Y}$.

The theoretical properties of the cost function place restrictions on the cost share model. As previously noted, a well behaved cost function is characterized by price homogeneity and symmetry, and it is strictly quasi concave, monotonically increasing and continuous in factor prices. The usual practice followed in the literature has been to estimate the system under the hypothesis of price homogeneity and symmetry, checking after estimation whether the maintained regularity conditions were met. In fact, consistency with the assumptions of a cost function monotonically increasing and strictly quasi concave in factor prices may be assessed by checking whether the fitted shares are all positive (monotonicity) and whether the matrix of substitution elasticities is negative semidefinite at each observation (strict quasi concavity).

In many empirical applications flexible functional forms have been found to violate the maintained regularity conditions, curvature and monotonicity in particular, sometimes at many points of the data set. On the one hand, simple forms such as the Cobb Douglas and the CES satisfy the regularity

conditions (curvature and monotonicity) globally, at the cost of no flexibility. On the other hand, full flexibility may constrain the regions over which regularity holds (Guilkey et al. [45]). Wales [105] has interpreted the violation of the maintained regularity conditions as a consequence of the local value of Taylor series expansions, more than of the failure of the optimizing paradigm, since, by construction, a Taylor series expansion ensures that the properties of the approximated function are necessarily preserved only at the point of approximation. In the literature, the Taylor's series expansion has been the most widely linear approximation employed, given its minimality property and simplicity⁷, and the generalized Leontief and the translog functions have been the two most widely specifications used in applied production analysis⁸. As a possible solution to the above mentioned drawbacks, the monotonicity and concavity conditions may be imposed globally or locally. For instance, the imposition of global curvature conditions can be carried out by using Choleski factorization methods or eigenvalue decomposition methods (Talpez et al. [97]; Coelli [19]) or simply by using flexible functional forms satisfying a priori this property (Diewert and Wales [28]). On the other hand, local curvature conditions can be imposed by directly specifying restrictions on the eigenvalues of the Slutsky matrix to be maintained in estimation (Gallant and Gollub [39]), by using Bayesian estimation methods (Chalfant and Wallace [15]; Terrel [98]), or, following Diewert and Wales [28], through indirect estimation of the Slutsky matrix (Ryan and Wales [90], Moschini [77]). While global methods may ensure consistency with theoretical curvature conditions only at the cost of loosing the flexibility property, local methods do not suffer from this latter drawback, allowing the imposition of the curvature properties over a region of values, and not only at a single point. Yet, both procedures do not allow for a proper testing of the hypotheses considered.

Moreover, the adding up condition has relevant consequences for empirical analysis, since it implies that of the n factor shares only $n - 1$ are linearly independent, and that the n th cost share equation may be implicitly derived from the other $n - 1$ shares. In an econometric model the adding-up condition implies that, for each observation, the sum of disturbances across equations must always equal zero. Then, the disturbance variance-covariance matrix is singular and nondiagonal. The singularity problem has usually been handled by dropping one of the share equations from the model and then jointly es-

⁷The comparative statics properties of a cost (production) function at a point sum up to $(n+1)(n+2)/2$, where n is the number of inputs. A necessary and sufficient condition for a functional form to reproduce these economic features, without imposing restrictions across them, is, therefore, to have $(n+1)(n+2)/2$ distinct parameters. A second-order Taylor's expansion meets exactly this condition (Fuss et al. [37]). Barnett [8] has shown, however, that the same restrictions can be met by a Laurent's series expansion.

⁸The translog and the generalized Leontief functional forms may be obtained by a second-order Taylor's expansion in powers of $\ln x_i$ and $\sqrt{x_i}$, respectively.

estimating the remaining $n - 1$ shares by maximum likelihood (ML). The use of ML guarantees that parameter estimates, estimated standard errors, and log likelihood values are invariant to the choice of which equation is deleted (Berndt [12]). Yet, assuming a Gaussian distribution for the data is not appropriate, since neither the zero-one interval restriction for the value of the shares nor the adding-up condition are necessarily respected. A possible solution to this latter problem has been proposed by Fry et al. [35], in the framework of compositional data analysis, by imposing the additive log ratio transformation to the data, i.e.

$$S_i^* = \ln(S_i/S_n) \quad i = 1, \dots, n - 1. \quad (16)$$

If the log ratio shares follow a multivariate normal distribution, then the shares will follow an additive logistic log normal distribution, with the same mean vector and variance covariance matrix. Moreover, reordering the components or changing the denominator used to compute the log ratios is not going to affect ML estimation of the system.

4 Dynamic models of factor demand

Static modelling implicitly assumes that all inputs adjust instantaneously to their long-run equilibrium values. Hence, static models of factor demands cannot describe satisfactorily real economic activity, where the process of adjustment can only be gradual. Actually, the rejection of the economic restrictions (price homogeneity and symmetry) or evidence of misspecification (parameter instability or serially correlated residuals), once the restrictions are imposed, may both be symptoms of neglected dynamics.

Dynamic models of factor demands were introduced with the purpose of dealing with the above problems. Two main approaches to dynamic factor demand modelling may be distinguished, according to whether the dynamic process is set in an explicit theoretical framework or modelled ad hoc. Furthermore, two typologies of ad hoc dynamic models, i.e. the first and second generation (Nadiri and Rosen [78]) models, may be distinguished. With the works of Lucas [63],[64] the third generation of dynamic models was born, and theoretical justifications for the adjustment process were firstly provided. Later on, particularly in the 1980s, new important novelties in both the second and third generations of dynamic factor demand models were introduced, the former (Anderson and Blundell [2], [3]) in strict connection with developments in time series econometric theory and modelling.

4.1 The first and second generation of dynamic factor demand models

The main difference between first and second generations dynamic factor demand models is the recognition of the interrelatedness of factor demands. The first generation of dynamic factor demand models was, in fact, essentially based on the Koyck [60] partial adjustment mechanism, applied to single equations. On the other hand, with the multiple equations partial adjustment model of Nadiri and Rosen [78], from which the second generation of dynamic factor demand models originated, interrelated disequilibria in factor markets were explicitly introduced.⁹ By denoting the vector of n inputs at time t as $\mathbf{x}_t = (x_{1,t}, \dots, x_{n,t})$ and by \mathbf{x}_t^* their long-run equilibrium values, the Nadiri-Rosen model can be written as

$$\mathbf{x}_t - \mathbf{x}_{t-1} = \mathbf{M}(\mathbf{x}_t^* - \mathbf{x}_{t-1}), \quad (17)$$

where \mathbf{M} is a partial adjustment matrix, not necessarily symmetric. A typical equation from (17) is

$$x_{i,t} = \sum_{\substack{i=1 \\ j \neq i}}^n m_{ij}(x_{j,t}^* - x_{j,t-1}) + (1 - m_{ii})x_{i,t-1} + m_{ii}x_{i,t}^* \quad (18)$$

Equation (18) clearly shows how disequilibrium in one factor market ($x_{j,t}^* - x_{j,t-1} \neq 0$) may affect the demand for another factor (i). In this case, the difference between short and long-run elasticities for the generic i th input would depend on all the parameters m_{ij} . In the light of the results of Lucas [63]¹⁰, Nadiri and Rosen [78] interpreted the interrelated partial adjustment model in terms of an approximation to the solution to an optimal control problem in which the firm maximizes its net worth in the presence of adjustment costs. However, as Berndt et al. [11] have observed, the Nadiri-Rosen and Lucas approximations would differ in two important features. Firstly, in the Nadiri-Rosen model the endogenous partial adjustment matrix \mathbf{M} is fixed, while in the Lucas framework is not necessarily constant over time, depending, at least, on

⁹In addition to the Nadiri-Rosen model, a different second generation modelling strategy, distinguishing between variable and quasi-fixed factors, and based on the restricted cost function introduced by Samuelson [92], was proposed by Lau [62] and McFadden [72].

¹⁰Lucas [63] extended the single equation flexible accelerator model to a system of equations in order to interrelate the adjustment process of the quasi-fixed inputs. He showed that when quasi-fixed (x) and variable (v) factors are distinguished, an approximate solution in the neighborhood of the equilibrium value to any standard economic dynamic optimization problem could be written in the partial adjustment form.

the discount rate and on the state of technology; secondly, in the Nadiri-Rosen model the adjustment specification is extended to all of the inputs, while the results of Lucas hold for quasi-fixed inputs only. In both cases, however, static expectations are employed.

4.2 Third generation dynamic factor demand models

The key feature of third generation dynamic factor demand models is the introduction of adjustments costs for quasi-fixed inputs. Eisner and Strotz [31] justified the introduction of adjustment lags on the basis of the fact that the more rapidly is the addition of one unit of capital and the more costly accumulation is. Lucas [63] generalized the results of Eisner and Strotz [31] to the multiple quasi-fixed inputs case, in order to consider interrelated factor markets disequilibria. In his model, adjustment cost functions are defined relative to net investment in quasi-fixed factors.¹¹ Differently, in Gould [42] and Lucas [64] the adjustment cost functions are defined relative to gross investment, accounting therefore for replacement costs as well. In all of the above models the adjustment cost functions are supposed to be strictly convex and quadratic. Finally, Treadway [99] introduced adjustment costs by an additively separable technical constraint. In this latter specification rental and adjustment costs are both expressed in terms of net investment but are considered separately. All of the successive applications have made reference to the specification typologies outlined above. In particular, the usual practice has been to express the adjustment cost in terms of net investment ($\dot{\mathbf{x}}$), including in the production (cost) function ($Y = f(x, \dot{x}, v, t)$). Moreover, in general static expectations have been used, and the latter is the main weakness of third generation dynamic factor demand models.¹²

In a standard third generation dynamic factor demand model the firm is assumed to minimize the present value of a stream of future costs at time t

$$C(0) = \int_0^{\infty} e^{-rt} \left(\sum_j \hat{p}_j v_j + \sum_i \hat{a}_i z_i \right) dt, \quad (19)$$

¹¹In Lucas [63] adjustment cost functions, representing the sum of purchase costs and internal installation costs of net accumulation, are explicitly introduced in the form $C(\dot{\mathbf{x}}) = \mathbf{q}\dot{\mathbf{x}}$, where \mathbf{q} is a vector of parameters and $\dot{\mathbf{x}}$ is the vector of net investments in the quasi-fixed factors. The adjustment cost functions are then added to the variable factors hiring costs to obtain a total cost function.

¹²A possible solution to this problem was indicated by Morrison [75] for the case of a quasi-fixed factor. The dynamic optimization problem is solved in two stages. In the first stage the structural model of intertemporal cost minimization is solved under unspecified expectations; in the second stage, expectations are specified, and the solutions previously derived are corrected appropriately.

given the production function $F(x, \dot{x}, v, Y, t) = 0$ and the entire future paths of output and input prices, where r is the firm's after tax discount rate, $z_i = \dot{x}(t) + d_i x_i(t-1)$ is the gross addition to the stock of the i th quasi-fixed input x_i , and $\hat{a}_i = \hat{u}_i / (r + d_i)$ is its asset price (\hat{u}_t is the rental price, and d_i is the depreciation rate). The firm chooses the time paths of the control variables $\mathbf{v}(t)$, $\dot{x}(t)$ and the state variable $\mathbf{x}(t)$ to minimize costs, given the exogenous known sequences of output demands $\{Y_{j,t+s}\}_{s=0}$, variable factor prices $\{\hat{p}_{j,t+s}\}_{s=0}$, quasi-fixed factor prices $\{\hat{a}_{j,t+s}\}_{s=0}$, and any initial $\mathbf{x}(0)$, with $\mathbf{v}(t)$, $\mathbf{x}(t) > 0$. Internal costs of adjustment are represented by the presence of net investment (\dot{x}) in the production function, with $\frac{\partial f}{\partial |\dot{x}|} > 0$ and $\frac{\partial^2 f}{\partial |\dot{x}|^2} < 0$. In this way, current production is affected by the accumulation or decumulation of the quasi-fixed factor: the greater the rate of accumulation and the greater is the marginal loss of productive ability. A version of the above problem, in which adjustment costs are related to gross rather than net investment, has also been proposed by Berndt and Watkins [13].

The general solution to this problem, that is the factor demand equations for the quasi-fixed and variable inputs, may be written as

$$\dot{\mathbf{x}} = \mathbf{M}^*(\mathbf{x}^*, r)(\mathbf{x}^* - \mathbf{x}) \quad (20)$$

for the quasi-fixed factors, and

$$\mathbf{v}^*(t) = v(\mathbf{p}(t), \mathbf{x}^*(t), Y(t), t) \quad (21)$$

for the variable factors.

As pointed out by Treadway [101], the matrix of adjustment parameters $\mathbf{M}^*(\mathbf{x}^*, r)$, in general, is dependent on the discount rate r and on the equilibrium value \mathbf{x}^* , and, consequently, on the exogenous variables which determine it. Therefore, empirical implementations which treat the matrix of adjustment parameters as fixed would be likely to employ misspecified models.¹³ Moreover, the interest rate invariant path (*RIP*) assumption would seem theoretically weak, since intuitively one would expect a firm faced with given actual and desired stocks to adjust more rapidly for low rates of interests than for high ones. Also the assumption of constant (relative) prices can be criticized. This latter assumption could in fact be acceptable in the case of an economy in the

¹³Treadway [101] showed that an optimal solution characterized by an interest rate invariant path (*RIP*) is a necessary and sufficient condition for the solution to the problem of the firm to be characterized by a flexible accelerator path (*FAP*), that is $RIP \Leftrightarrow FAP$. In this case, the linear adjustment paths are globally optimal. The constant flexible accelerator path (*CFAP*), in addition to the independence from the discount factor, would require that the adjustment matrix be also independent of the exogenous variables determining the equilibrium value.

steady state or on the balanced growth path, since when the economy is not in such a state a firm's investment plans are bound to be revised. Finally, concerning the modelling of expectations, by relying on the assumption that, under rational expectations, agents derive predictions of the future which differ from actual outcomes only by random and independent random processes, two different approaches have been proposed. The first approach consists of predicting the exogenous variables influencing the equilibrium level using a vector autoregressive (VAR) model. The rationality of the expectations process is then attained by requiring that all of the relevant information is considered in forming expectations, that is, a white noise error term. On the other hand, in the second approach the expectations formation process is not explicit, and consistency with rational expectations is achieved by employing an instrumental variable estimator to achieve error orthogonality, i.e. a system composed of the Euler equations (for the quasi-fixed factors) and the variable factors demand equations (obtained via Shephard's Lemma) is estimated by system methods (Pindyck and Rotemberg [84], [85]).

Recent contributions to third generation dynamic factor demand modelling have been given by Epstein [33] and Epstein and Denny [32]. In this latter works duality theory has been generalized to the dynamic case and estimable factor demand functions are derived from a dynamic version of Shephard's lemma. Interestingly, the actual features of the adjustment mechanism are found to depend on the form of the value function. Moreover, by modelling the time path of the exogenous variables (output and prices) by an autoregressive process, rational expectations are empirically implemented. While the main advantage of the Epstein approach is that increasing the number of inputs modelled as quasi-fixed neither results in model intractability, nor constrains the interrelatedness of the adjustment process, a possibly large number of parameters is required to ensure full flexibility.¹⁴

5 Recent developments in second generation dynamic factor demand models

The main developments in second generation factor demand modelling originated from the works of Anderson and Blundell [2], [3], reflecting the key changes which have taken place in econometrics since the 1980s, namely the development of the general to specific approach to econometric modelling and cointegration theory. Anderson and Blundell [2], [3] have proposed an error correction model (ECM) for the long-run share relationships, in which the

¹⁴Flexibility in the dynamic context requires that the value function may assume, at the point of approximation, any given set of theoretically consistent values for the value function and all its first- and second-order derivatives. See Epstein [33].

restrictions suggested by economic theory are supposed to hold only in the long-run. On the other hand, in the short-run the factors of production are left to adjust freely towards the target levels. The error correction model not only generalizes the partial adjustment mechanism, as Hendry and von Hurgen Sternberg [50] have shown, but it is also theoretically founded, since, as Nickell [79] has shown, the ECM mechanism can be derived as the solution to a dynamic optimization problem, in presence of costs of adjustment, under rational expectations.

5.1 The error correction factor demand model

Following Anderson and Blundell [2], [3], given the long-run share relationship in system form

$$\mathbf{S}_t = \boldsymbol{\beta}\mathbf{x}_t + \mathbf{u}_t \quad t = 1, \dots, T, \quad (22)$$

where \mathbf{S}_t is an n -dimensional vector of cost factor shares, \mathbf{x}_t is an m -dimensional vector of exogenous regressors (factor prices plus an intercept term), $\boldsymbol{\beta}$ is a $n \times m$ matrix of coefficients, and \mathbf{u}_t is the n -dimensional vector of residuals, a general dynamic version of the model can be written as

$$\mathbf{B}^*(L)\mathbf{S}_t = \Gamma^*(L)\mathbf{x}_t + \mathbf{u}_t, \quad (23)$$

where $\mathbf{B}^*(L)$ and $\Gamma^*(L)$ are polynomial matrices in the lag operator, and reparameterized in the observationally equivalent error correction form

$$\Delta\mathbf{S}_t = -\mathbf{B}(L)\Delta\mathbf{S}_t + \Gamma(L)\Delta\tilde{\mathbf{x}}_t - \mathbf{K}[\mathbf{S}_{t-1} - \mathbf{A}\mathbf{x}_{t-1}] + \mathbf{u}_t \quad t = 1, \dots, T, \quad (24)$$

where $\mathbf{B}(L) = \sum_{i=1}^{p-1} \left(\sum_{j=0}^i \mathbf{B}_j^* \right) L^j$, $p > 1$, $\Gamma(L) = \sum_{i=0}^{q-1} \left(\sum_{j=0}^i \tilde{\Gamma}_j^* \right) L^j$, $q > 1$, $\mathbf{K} = \sum_{j=0}^p \mathbf{B}_j^* = \mathbf{I} + \mathbf{B}_1^* + \mathbf{B}_2^* + \dots + \mathbf{B}_p^*$, $\tilde{\Gamma}_j^*$ is Γ_j^* with the first column deleted and $\tilde{\mathbf{x}}_t$ is \mathbf{x}_t with the first element (the intercept) deleted. The adding-up condition implies the following restrictions:

$$\mathbf{i}'\mathbf{B}_j = m_j\mathbf{i}' \quad j = 1, \dots, p-1 \quad (25)$$

$$\mathbf{i}'\Gamma_j = \mathbf{0} \quad j = 1, \dots, q-1 \quad (26)$$

$$\mathbf{i}'\mathbf{K} = k\mathbf{i}' \quad (27)$$

$$\mathbf{i}'\boldsymbol{\beta} = [1 \ 0 \ \dots \ 0], \quad (28)$$

where \mathbf{i} is a column vector composed of ones, \mathbf{B}_j is the j th coefficient matrix in $\mathbf{B}(L)$, Γ_j is the j th coefficient matrix in $\Gamma(L)$ and m_j and k are constants.

The ECM formulation shows that the adjustment process of the involved economic variables is not just a short-run phenomenon, but it concerns the recovery of the long-run equilibrium as well. While it cannot be expected that a long-run equilibrium state continuously holds, yet if some economic variables are related by a long-run relationships, then the system will tend to move towards the long-run equilibrium over time. The error correction model actually capture this convergence process, pointing to a progressive correction of the equilibrium errors, i.e. the gap between the actual values and the values predicted on the basis of the long-run relationship. In fact, as shown in [24], the adjustment in the dependent variables at time t is not a function of the level of the explanatory variables but of the deviation of the explanatory variables \mathbf{x} from the equilibrium relationship with the dependent variables \mathbf{S} at the previous time period $t - 1$. As for the static case, the shares system in [24] may be estimated after one of the collinear dynamic share equations is deleted. Once the restrictions of price homogeneity and symmetry have been imposed on the long-run specifications, and the short-run dynamics selected, the FIML estimator can be used to estimate the model. Anderson and Blundell [2], [3] have suggested testing the economic restrictions within an appropriate dynamic specification, since this allows to account for the actual properties of economic data. The selection of the dynamic specification is then a particularly delicate step. Yet, within the Anderson and Blundell error correction framework, the modelling of dynamics is constrained by the adding-up restrictions. Moreover, the short-run dynamics parameters are not identified¹⁵, and adding the dynamic cost function to the system of dynamic share equations, as for instance proposed by Urga [102], does not solve the problem (Skjerpen [96]). Applications of the above approach to the UK manufacturing sector have been provided by Mc Avinchey [67] and Holly and Smith [57].

5.2 General to specific modelling and the selection of dynamic specification

The general to specific approach assumes that the data generation process (DGP) is unknown, and given a certain economic phenomenon to be investi-

¹⁵The identification problem refers to the determination of the conditions under which a unique correspondence between reduced and structural forms of a model can be established, or, said in another way, of the conditions under which an estimate of the parameters of the structure may be retrieved by the estimate of the reduced form.

gated, economic theory is called upon to suggest the relevant variables to be considered, but not the exact specification of the model. Econometric modelling is then called for to provide an adequate simplification of the unknown DGP.¹⁶

The starting point of the analysis is the joint probability distribution of all the sample data, both on the dependent and independent variables, that is the unknown DGP

$$\prod_{t=1}^T D(\mathbf{z}_t | \mathbf{Z}_{t-1}; \boldsymbol{\theta}), \quad (29)$$

where \mathbf{z}_t is the vector of all the variables at time t , $\mathbf{Z}_{t-1} = (\mathbf{z}_1, \dots, \mathbf{z}_{t-1})'$, and $\boldsymbol{\theta}$ is the vector of parameters of the joint density function $D(\cdot)$. Then, econometric modelling is employed to obtain a simplified version of the DGP by imposing a set of restrictions. The outcome of this process is a set of estimated equations. Following Gilbert [40], the simplification process involves the following stages:

- (1) marginalization of the DGP with respect to the variables that are not relevant for the determination of the variables of current interest;
- (2) conditioning of the endogenous variables on the (weakly) exogenous variables;
- (3) selecting a suitable and simple representations of the conditioned marginalized DGP;
- (4) replacing the unknown parameters in this representation by estimated values.

Economic theory would then be employed in steps (1) to (3) to derive the simplified DGP, or conditional model

$$\prod_{t=1}^T B(\mathbf{y}_t | \mathbf{Y}_{t-1}, \mathbf{X}_t; \boldsymbol{\beta}), \quad (30)$$

from the joint probability distribution by the relationship

$$\prod_{t=1}^T D(\mathbf{z}_t | \mathbf{Z}_{t-1}; \boldsymbol{\theta}) = \prod_{t=1}^T A(\mathbf{w}_t | \mathbf{Z}_t; \boldsymbol{\alpha}) B(\mathbf{y}_t | \mathbf{Y}_{t-1}, \mathbf{X}_t; \boldsymbol{\beta}) C(\mathbf{x}_t | \mathbf{Y}_{t-1}, \mathbf{X}_{t-1}; \boldsymbol{\gamma}), \quad (31)$$

¹⁶The general to specific methodology can be considered as a direct reaction against the specific to general practice followed by classical econometrics and its exact specification assumption. The approach has been developed by Sargan [93] and some of his students at the London School of Economics. The most recent contributions are due to the work of Hendry [55].

where $\prod_{t=1}^T A(\mathbf{w}_t|\mathbf{Z}_t; \boldsymbol{\alpha})$ specifies the determination of the variables of no interest, $C(\mathbf{x}_t|\mathbf{Y}_{t-1}, \mathbf{X}_{t-1}; \boldsymbol{\gamma})$ relates the exogenous variables to lagged endogenous and exogenous variables, \mathbf{y}_t is the vector of the endogenous variables of interest at time t , $\mathbf{Y}_{t-1} = (\mathbf{y}_1, \dots, \mathbf{y}_{t-1})'$ is the vector of lagged endogenous variables, $\mathbf{X}_{t-1} = (\mathbf{x}_1, \dots, \mathbf{x}_{t-1})'$ is the vector of (at least) weakly exogenous regressors, and $\boldsymbol{\beta}$ is the vector of parameters of the simplified joint density function $B(\cdot)$.

Estimation theory is then employed in the last step to provide numerical values to the unknown parameters.

Within this approach economic theory is associated with long-run equilibrium relationships, while economic data are supposed to be generated by a disequilibrium process. Hence, equilibrium economic theory is viewed as an inadequate starting point for applied analysis, and lags of the variables already considered in the equilibrium relationships are added to model short-run dynamics. The starting point of econometric modelling is then a very general model, intentionally overparameterized. With reference to the cost shares system, given the equilibrium share relationships in system form

$$\mathbf{S}_t = \boldsymbol{\beta}\mathbf{x}_t + \mathbf{u}_t \quad t = 1, \dots, T, \quad (32)$$

the general model would then be specified in the vector autoregressive distributed lag (VADL) form

$$\mathbf{B}^*(L)\mathbf{S}_t = \mathbf{G}^*(L)\mathbf{x}_t + \mathbf{u}_t \quad t = 1, \dots, T, \quad (33)$$

where $\mathbf{B}^*(L)$ and $\mathbf{G}^*(L)$ are polynomial matrices in the lag operator. Through a data based simplification procedure, made operational as a set of sequential statistical tests, the above model is reduced to a more parsimonious form, satisfying a set of model acceptance criteria. The criteria that the simplified model must meet to be an adequate simplification of the DGP are the following: a) data admissibility, i.e. it is required to be logically possible that the data have been generated by the specification chosen; b) consistency with theory, i.e. the selected model must be consistent with at least one economic theory; c) weak exogeneity of the regressors, i.e. conditioning has to be valid, since variables which violate weak exogeneity should be modelled jointly;¹⁷ d)

¹⁷As Engle et al. [30] have shown, weak exogeneity of the regressors with respect to the parameters of interest is a necessary and sufficient condition for consistent estimation, and a necessary, not always sufficient, condition for hypotheses testing; strong exogeneity, that is weak exogeneity along with Granger-noncausality of the dependent variable, is required for valid forecasting; finally, super exogeneity, that is weak exogeneity plus independence of the parameter of interest by the process generating the regressors, is the relevant concept for policy analysis.

parameter constancy, i.e. the parameters must have the same values within and outside the sample for the model to be useful for forecasting or for policy simulation; e) data coherency, i.e. it is required that the actual residuals be consistent with a white noise process and, hence, not predictable from their own past; f) encompassing of all the rival models, i.e. the chosen model has to be capable of explaining not only the data, but also the results of other specifications used to account for the same data.

Hendry [55]’s general to specific econometric methodology has bridged structural classical econometrics and the non structural VAR approach by unifying, in the general to specific framework, the main advances occurred in time series econometrics since the 1980s. In particular, the link with cointegration theory is very strict, given the relationship which has been shown to exist between cointegrated variables and the error correction representation¹⁸ and the conceptual connection between stable long-run relationships and economic equilibria. The concept of equilibrium employed in cointegration, however, is purely statistical, since two or more series are cointegrated if they tend to move together over time, that is if the gap between them assumes bounded values: “An equilibrium state is defined as one in which there is no inherent tendency to change ... we generally consider only stable equilibria ... states to which the system is attracted, other things being equal ... Equilibrium may be general or partial ... Over finite periods of time, the long run or equilibrium relationship may fail to hold, but [it] will eventually hold ... if the equilibrium is stable, and if the system does not experience further shocks from outside. Expressed differently, a long-run equilibrium relationship entails a systematic comovement among economic variables which an economic system exemplifies precisely in the long run.”¹⁹

Despite its statistical formulation, the concept of cointegration has been found particularly relevant to describe long-run economic phenomena, when economic theory proposes forces, such as market forces, which imply that some combination of variables should not diverge from each other at least in the long-run. Hence, the concept of equilibrium embedded in cointegration theory is close to the notion of steady state in dynamical systems, an equilibrium notion much closer to the Classical economists’ idea of centre of gravity relationship than that of market clearing position of Neoclassical economics. Neoclassical economics assumes, in fact, that market forces, if left to operate without restrictions, would lead the economic system towards a general markets clearing position. However, because of market imperfections and shocks affecting con-

¹⁸Granger’s Representation Theorem (Engle and Granger [29]), among other results, proves that a cointegrated system of variables may be represented in the error correction form, and that the existence of an error correction representation for a system of I(1) variables implies cointegration for at least some of them.

¹⁹Banerjee et al. [7], pp.2-3.

tinuously the system, the equilibrium position would never actually be reached. On the other hand, Classical economists thought of the equilibrium state in terms of an attractor for the economic system, not necessarily associated with a market clearing situation. The definition of cointegration is general, and can in principle account for both market-clearing and not market-clearing equilibria.

Concerning the factor shares model, then, if cointegration holds between cost shares and prices, assuming both sets of variables being $I(1)$, and considering the generic i th share, although the points $(\mathbf{x}_t, S_{i,t})$ will move widely around the $\mathbf{x} - S_i$ hyperplane, there will be a tendency for the points to locate around the attractor described by the cointegration relationship

$$S_{i,t} = \beta \mathbf{x}_t + u_{i,t} \quad t = 1, \dots, T. \quad (34)$$

Moreover, according to the Engle and Granger representation theorem, there exists an error correction representation for the involved variables, which can be computed starting from the vector autoregressive (VAR) representation for the $I(1)$ vector stochastic process $\mathbf{z}'_t = [\mathbf{S}'_t \ \mathbf{x}'_t]$, where in \mathbf{S}'_t , to avoid perfect collinearity, only $n - 1$ shares have been considered, i.e.

$$\mathbf{z}_t = \sum_{j=1}^p \mathbf{C}_j \mathbf{z}_{t-1} + \boldsymbol{\varepsilon}_t, \quad (35)$$

from which the error correction (ECM) form can be written as

$$\Delta \mathbf{z}_t = \sum_{j=1}^{p-1} \Gamma_j \Delta \mathbf{z}_{t-j} + \Pi \mathbf{z}_{t-1} + \boldsymbol{\varepsilon}_t, \quad (36)$$

where $\Gamma_j = -\mathbf{I} + \mathbf{C}_1 + \dots + \mathbf{C}_i$, \mathbf{I} is the identity matrix, $\Pi = -(\mathbf{I} - \mathbf{C}_1 - \dots - \mathbf{C}_p)$. From the Granger representation theorem (Engle and Granger [29]), under some general conditions, if the rank of the matrix Π is equal to $r < q$ (q is the total number of variables explained in the model), there is a representation of Π such that

$$\Pi = \boldsymbol{\alpha} \boldsymbol{\gamma}',$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$ are $(q \times r)$ matrices. If the cointegrating matrix $\boldsymbol{\gamma}$, is such that

$\boldsymbol{\gamma}' \mathbf{z}_t \sim I(0)$, where $\mathbf{z}_t \sim I(1)$, the \mathbf{z}_t variables are cointegrated, and the matrix $\boldsymbol{\gamma}' \mathbf{z}_{t-1}$ is a system of r error correction mechanisms. After normalization, the elements of the matrix $\boldsymbol{\gamma}$ may be interpreted as long-run parameters, while the elements of the feedback matrix $\boldsymbol{\alpha}$ as speeds of adjustment (Johansen [59]).

The error correction model in Anderson and Blundell [2], [3], i.e.

$$\Delta \mathbf{S}_t = -\mathbf{B}(L)\Delta \mathbf{S}_t + \Gamma(L)\Delta \tilde{\mathbf{x}}_t - \mathbf{K}[\mathbf{S}_{t-1} - \mathbf{A}\mathbf{x}_{t-1}] + \mathbf{u}_t \quad t = 1, \dots, T, \quad (37)$$

can be derived as special case from the above error correction model, when the cointegration relationships are given by the static share relationships

$$\mathbf{S}_t = \boldsymbol{\beta}\mathbf{x}_t + \mathbf{u}_t \quad t = 1, \dots, T, \quad (38)$$

and the weak exogeneity of factor prices \mathbf{x}_t holds. In this latter case, the dimensionality of the problem can be reduced, since the modelling of the variables of interest can be validly carried out by conditioning with respect to the weakly exogenous regressors. Hence, the number of equations in the system drops to $n - 1$, $\mathbf{K} = \boldsymbol{\alpha}$, $\mathbf{I} - \mathbf{A} = \boldsymbol{\gamma}$.

Therefore, under cointegration, there exists an error correction form for the cost shares, and the existence of an error correction representation for the cost shares is a necessary and sufficient condition for cointegration of the cost shares and prices. Moreover, the error correction equations describe the convergence (disequilibrium) process towards the attractor represented by the static share equations or cointegration relationships.

5.3 Applications of cointegration theory to systems of share equations

In the literature few applications of cointegration theory to factor demand modelling can be found. Atfield [6] has employed the triangular error correction representation of Phillips [83] to model the demand for non-durables for the UK, while Morana [73] has modelled the Italian energy production sector within the Hendry's general to specific error correction framework, and McAvinchey [71] energy demand for Germany and the UK. Moreover, in McAvinchey and Morana [68] the interaction of high and low frequency for cost shares forecasting has been investigated, still within the general to specific framework. On the other hand, Pesaran and Shin [81] have modelled the non-durable expenditure system for the UK within the Johansen [59] framework. Finally, Oniki [80] has modelled the rice production sector in Japan, McAvinchey [70], [69] the demand for alcohol in the UK, and Fanelli and Mazzocchi [34] the demand for meat in Italy.

5.4 Recent advances in equilibrium economic and econometric modelling

According to Robinson [86], economic equilibria would be different from equilibria in physical systems because of three main properties. Firstly, an economic equilibrium contains the germs of its own destruction, since the maximizing behavior of agents make them to seek different equilibrium configurations over time. Secondly, an economic equilibrium shows path dependency, i.e. the equilibrium position where the system settles down is not independent of the transition path towards it. Thirdly, uncertainty about the future makes expectations relevant to understand the transition towards equilibrium.

While on the one hand rational expectations theory has provided with an instrument to consistently model expectations, on the other hand little attention has been paid to the other two properties so far. Moreover, the notion of steady state growth path, introduced to reformulate the equilibrium concept in a dynamic framework, has been found unable to account for at least two essential issues, i.e. the fact that relationships which may exist in the short-run may well be indeterminate in the long-run, and that a disequilibrium may not be instantaneously eliminated. Recently, two new terms have entered economic modelling, that is chaos and hysteresis. As pointed out by Cross [20], hysteresis may help to explain situations in which economic effects persist after their causes are removed. A hysteretic system, in fact, differently from an $I(1)$ process, shows selective rather than global memory, so that only the non-dominated maximum and minimum values of past shocks would affect the present behavior of the system. Hence, a shock, in order to affect a hysteretic system, must have a certain magnitude, i.e. the system is thought of remaining in a given equilibrium position up to the moment in which a new (large enough) shock affects the economy. Moreover, once the system is disturbed from an established position, it would fluctuate up to the moment in which a new attractor emerges. This behavior undermines the admissibility of stable and unique equilibria. On the other hand, non-linear systems, as the chaotic ones, would seem to be particularly suited to reproduce analytically the property of high sensitivity to initial conditions, i.e. the fact that the effects of a shock on the system depend crucially on the time period at which the shock has occurred. Finally, different typologies of equilibria, useful to understand economic behavior, are shown by non linear models, as for instance limit cycle behavior, in addition to multiple equilibria.

Some of the conceptual developments described above have found empirical application, particularly in the framework of cointegration analysis. For instance, Granger and Lee [43] have introduced time-varying cointegration theory to study situations in which it is the attractor of the system and the speed of adjustment which vary over time. Modelling structural change as in

Granger and Lee [43] may be indeed important in order to account for the fact that in the real economic world technology and tastes change over time, and these latter changes are going to affect the equilibrium relationships among the involved variables. Moreover, neglecting structural change could lead to wrongly conclude against the existence of long-run relationships, which, on the other hand, could exist but be non linear.

Previous to the work of Granger and Lee [43], Harvey [51] has pointed out to the usefulness of including structural unobserved components in the error correction term, while Harvey and Scott [53] have introduced time varying seasonal components in the short-run specification of an error correction model. Differently, Hall and O'Sullivan [47] have introduced an error correction model with fixed long-run specification and time-varying speed of adjustment. Moreover, Hall et al. [48] have employed a finite-state Markov process with unknown transition probabilities to describe the stochastic shifts between alternative cointegrating regimes, while Granger and Terasvirta [44] have introduced non linear cointegration to model situations in which the strength of attraction of a long-run equilibrium varies according to certain rules, for instance on the basis of the gap between the actual and the long-run states of a process.

Concerning the applications of time-varying parameter cointegration to factor demand analysis, only few contributions have been provided in the literature so far. For instance, while in Allen [1] an application in the context of cost function estimation has been provided, in Morana [74] time-varying parameter share equation cointegrating relationships have been modelled by allowing stochastically evolving seasonal and trend components. In fact, in the latter paper the long-run structure of the model is specified as

$$\mathbf{S}_t = \boldsymbol{\mu}_t + \boldsymbol{\gamma}_t + \boldsymbol{\beta}\mathbf{x}_t + \mathbf{u}_t \quad t = 1, \dots, T, \quad (39)$$

$$\boldsymbol{\mu}_t = \delta_{T,t-1} + \boldsymbol{\mu}_{t-1} + \boldsymbol{\varepsilon}_t \quad (40)$$

$$\delta_{T,t} = \delta_{T,t-1} \quad (41)$$

$$\boldsymbol{\gamma}_t = -\boldsymbol{\gamma}_{t-1} - \boldsymbol{\gamma}_{t-2} - \boldsymbol{\gamma}_{t-3} + \boldsymbol{\omega}_t \quad (42)$$

$$\boldsymbol{\gamma}_{t-1} = \boldsymbol{\gamma}_{t-2} \quad (43)$$

$$\boldsymbol{\gamma}_{t-2} = \boldsymbol{\gamma}_{t-3}, \quad (44)$$

where the trend component $\boldsymbol{\mu}_t$ is evolving according to a multivariate random walk with drift, $\boldsymbol{\gamma}_t$ is a multivariate stochastic seasonal component, and the innovations vectors \mathbf{u}_t , $\boldsymbol{\varepsilon}_t$, and $\boldsymbol{\omega}_t$ follows multivariate Gaussian distributions. The same specification, albeit not in a cointegration context, has been previously used by Harvey and Marshall [52] in their study on energy demand for the UK. The above specification allows to account for an evolving economic

environment, useful to capture actual dynamics of technical progress, which is unlikely to proceed at a constant pace, as on the other hand assumed when a linear deterministic model is employed. Moreover, by allowing for stochastic seasonality further flexibility is allowed. For instance, with reference to the energy production sector, the latter component may account for the effects of climate change on energy demand. The evolving long-run structure is then

embedded in a standard error correction model framework.

An application of the above modelling strategy has been provided by Mazzocchi et al. [66] to model meat demand in Italy. On the other hand, Deschamps [25] has employed a restricted version of the above model, allowing for time varying intercept components only, to investigate the demand of non-durables for the US. Yet, differently from previous contributions, in this latter paper estimation is carried out by means of Bayesian methods rather than by means of the Kalman filter.

6 Concluding remarks

Since the work of Cobb and Douglas [18], applied factor demand modelling has been the object of relevant innovations, i.e. the introduction of flexible functional forms, and the modelling of dynamics, expectations, and of the interrelatedness of the adjustment process. Recently, cointegration theory has provided an important contribution to the definition of the equilibrium concept, yielding empirical content to this latter notion as well. The concept of equilibrium considered in cointegration theory, i.e. the idea of comovements of economic variables, encompasses both the idea of centre of gravity relationship, suggested by Classical economists, and the notion of market-clearing position, employed by Neoclassical economist. Time-varying parameters long-run relationships may well be important to model how economic attractors change over time, as the evolution of the structural features of the economy proceeds. Hence the concepts of multiple equilibria, hysteresis and chaos, may be the relevant mathematical framework in which economic dynamics may be understood from a theoretical point of view, while non linear and time-varying parameter cointegration may prove to be the appropriate econometric tools for their empirical modelling.

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