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A Note on the Exponential Stability of Nonlinear Difference Systems

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Abstract

One of the directions arising from applications of difference equations is linked with qualitative investigation of their solutions. The analysis of numerical methods applied to autonomous linear problems is well-developed. The direct analysis of nonlinear and nonautonomous problems is less well understood and is dependent on the availability of suitable general theorems on the behavior of solutions to difference equations. In this study, we show that under conditions relating A and B, if the zero solution of the system $x_{n+1} = (I + A_n)x_n + f_n,\ x_{n_0} = x_0$, $n_0 \leq n < \infty$ is exponentially stable, then the zero solution of the system $y_{n+1} = (I + B_n)y_n + g_n,\ y_{n_0} = y_0$, $n_0 \leq n < \infty$ has the same property.

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1 Introduction

The idea of using difference equations to approximate solutions of differential equations originated in 1769 with Euler's polygonal method, for which the proof of convergence was given by Cauchy around 1840. During the 1950's, several ecologists used simple nonlinear difference equations, including the logistic equation, to study the change in populations from one year (or season) to the next with the emphasis on the stability of the iteration. The excitement of these discoveries attracted the attention of researchers who attempted to apply the results to fields from economics to medicine.

Conti [2] defined two $n \times n$ matrix functions A and B on $[0, \infty]$ to be $t_\infty-$similar if there is $n \times n$ matrix function S defined on $[0, \infty]$ such that S'

is continious, S and S^{-1} are bounded on $[0, \infty]$, and

$$\int_0^\infty \|S' + SB - AS\| \, dt < \infty.$$

Conti has shown that $t_\infty-$similarity preserves exponential stability; that is, if the system $x' = A(t)x$ has exponential stability property and B is $t_\infty-$similar to A, then so does the system $y' = B(t)y$.

In this paper, we give an analogy of this result in [2] for systems of difference equations.

For convenience, we first list all special notation used throughout the rest of this note:

$\mathbf{F}$: the set of all nonnegative integer;

$\mathcal{F}^{k\times k}$: the set of all $k \times k$ matrix functions(with real or complex entries) defined on $\mathbf{F}$;

I : $k \times k$ identity matrix

$A_n, I + A_n, B$ and $B + I_n$: $k \times k$ nonsingular matrix functions(with real or complex entries) defined on $\mathbf{F}$;

$x(n) = x_n, y(n) = y_n, f(n, x_n) = f_n$ and $g(n, y_n) = g_n$: $k \times 1$ vector functions (with real or complex entries) defined on $\mathbf{F}$;

$\mathcal{L}$: the set of all $k \times k$ bounded matrices in $\mathcal{F}^{k\times k}$;

$\Im$: the set of all $k \times k$ matrix functions F_m in $\mathcal{F}^{k\times k}$ such that $\sum\limits_{m=0}^{\infty} F_m$ exists;

$\Re$: the set of all $k \times k$ matrix functions F_m in $\mathcal{F}^{k\times k}$ such that $\sum\limits_{m=0}^{\infty} \|F_m\| < \infty$.

This paper is organized as follows. Section 2 introduces basic concepts and principal results needed in this paper. We are interested in relating exponential stability properties of two $k \times k$ systems of difference equations

$$x_{n+1} = (I + A_n)x_n + f_n, \; x_{n_0} = x_0, \; n_0 \le m \le n < \infty, \tag{1}$$

and

$$y_{n+1} = (I + B_n)y_n + g_n, \; y_{n_0} = y_0, \; n_0 \le m \le n < \infty, \tag{2}$$

without actually computing solutions of them in Section 3.

2 Preliminary Notes

Definition 1 *The system*

$$x_{n+1} = A_n x_n, \; x_{n_0} = x_0, \; n \ge n_0 \ge 0 \tag{3}$$

is homogeneous linear difference system which is nonautonomous or time-variant. The corresponding nonhomogeneous system is given by

$$y_{n+1} = A_n y_n + g_n, \ y_{n_0} = y_0, \quad n \geq n_0 \geq 0. \tag{4}$$

([1, p.120])

Theorem 1 *There exists a uniqe solution x_n of Eq.* (3), *with $x_{n_0} = x_0$ such that*

$$x_n = \left[\prod_{i=n_0}^{n-1} A_i \right] x_0$$

where

$$\prod_{i=n_0}^{n-1} A_i = \begin{cases} A_{n-1} A_{n-2} \ldots A_{n_0} & n > n_0 \\ I & n = n_0 \end{cases} .$$

([1, p.120])

Let X_n be $k \times k$ matrix whose columns are solutions of Eq. (3). Hence, X_n satisfies the matrix difference equation

$$X_{n+1} = A_n X_n, \ X_{n_0} = X_0, \ n \geq n_0 \geq 0. \tag{5}$$

([1, p.121])

Definition 2 *If X_n is a solution of Eq.* (5), *then it is said to be a fundamental matrix for the system* (3). *([1, p.121])*

Note that if X_n is a fundamental matrix of the system (3) and C is any nonsingular matrix, then $X_n C$ is also a fundamental matrix of the system (3). Thus there are infinitely many fundamental matrices for the system (3). However, there is one fundamental matrix that we already know, namely,

$$X_0 = I, \ X_n = A_{n-1} A_{n-2} \ldots A_{n_0}, \ n \geq n_0 \geq 0. \tag{6}$$

Theorem 2 *(Variation of Constants Formula) Let X_n as in* (6). *Then, the uniqe solution of the system* (4) *is given by*

$$y_n = X_n X_0^{-1} y_0 + \sum_{r=n_0}^{n-1} X_n X_{r+1}^{-1} g_r. \tag{7}$$

([1, p.124])

While the fundamental idea of exponential stability is widely understood, there remains some latitude in definitions among authors and so for the sake of clarity, we give definition of exponential satability of the system (1) here.

Definition 3 *For all x_n solutions of the system* (4), *if there exist $\delta > 0$, $\mu_A > 0$, $\eta \in (0,1)$ such that $\|x_n\| \leq \mu_A \|x_0\| \eta^{n-n_0}$, whenever $\|x_0\| \leq \delta_0$, then the zero solution of nonlinear System* (4) *is exponentially stable.*

Assumption There is an $S_n \in \mathcal{L}$ such that

$$F_n^{(0)} = \Delta S_n + S_{n+1} B_n - A_n S_n$$

defines an element of $\Im$. Either $F_n^{(0)} \in \Re$ or there is a positive integer p such that the $n \times n$ matrix functions $F_n^{(1)}, \ \ldots, \ F_n^{(p)}$ defined by

$$Q_n^{(r)} = \sum_{k=n}^{\infty} F_k^{(r-1)}$$

and

$$F_n^{(r)} = Q_{n+1}^{(r)} B_n - A_n Q_n^{(r)}$$

are in $\Im$, and $F_n^{(p)} \in \Re$.

Theorem 3 *Suppose that Assumption holds. Define*

$$\Gamma_n^{(0)} = I, \ \Gamma_n^{(r)} = I + S_n^{(-1)} \sum_{l=1}^{r} Q_n^{(l)}, \ 1 \leq r \leq p.$$

Then

$$\Gamma_j^{(p)} Y_j = S_j^{-1} X_j \left[X_i^{-1} S_i \Gamma_i^{(p)} Y_i + \sum_{m=i}^{j-1} X_{m+1}^{-1} F_m^{(p)} Y_m \right], \ 0 \leq i \leq j. \qquad (8)$$

(Trench [3])

The symbol *o (small oh) is one of the main tools of approximating functions and is widely used in all branches of science. Now, we shall give definition of the symbol o (small oh) for functions defined on the real or complex numbers.*

Definition 4 *Let $f(t)$ and $g(t)$ be two functions defined on the real or complex numbers. If $\lim_{t\to\infty} \frac{f(t)}{g(t)} = 0$, then we say that*

$$f(t) = o(g(t)), \quad (t \to \infty).$$

([1, p.305])

A series usually is defined as a pair of sequences: the sequence of terms of the series: $a_0,\ a_1,\ a_2, ...$; and the sequence of partial sums $S_0,\ S_1,\ S_2, ...$ where $S_N = \sum_{n=0}^{N} a_n$. The notation $\sum_{n=0}^{\infty} a_n$ represents then a priori this pair of sequences, which is always well defined, but which may or may not converge. In the case of convergence, i.e., if the sequence of partial sums S_N has a limit, the notation is also used to denote the limit of this sequence. A series $\sum_{n=0}^{\infty} a_n$ is said to converge absolutely if the series of absolute values $\sum_{n=0}^{\infty} |a_n|$ converges. In this case, the original series, and all reorderings of it, converge, and converge towards the same sum. For a sequence of numbers $a_1, a_2, a_3, ...$, the infinite product $\prod_{n=1}^{\infty} a_n$ is defined to be the limit of the partial products $a_1 a_2 ... a_n$ as n goes to infinity. The product is said to converge when the limit exists and is not zero. Otherwise the product is said to diverge. A series $\prod_{n=1}^{\infty} a_n$ is said to converge absolutely if the series of absolute values $\prod_{n=1}^{\infty} |a_n|$ converges.

Theorem 4 *The product* $\prod_{n=1}^{\infty}(1 + a_n)$ *with positive terms* a_n *is convergent if and only if the series* $\sum_{n=0}^{\infty} a_n$ *converges.([5, p.219])*

3 Main Results

Theorem 5 *Let Assumption holds and assume that* $g_n = o(\|y\|)$ *as* $\|y\| \to 0$. *If the zero solution of nonlinear System* (1) *is exponentially stable, then the zero solution of nonlinear System* (2) *is also.*

Proof. Since $I + A_n$ and $I + B_n$ are invertible for every $n \geq n_0$, this guarantees that $X, Y \in M^{k \times k}$ defined by

$$X_o = I, X_n = (I + A_{n-1})(I + A_{n-2})...(I + A_0)$$

$$Y_o = I, Y_n = (I + B_{n-1})(I + B_{n-2})...(I + B_0)$$

are fundamental matrices for the systems

$$x_{n+1} = (I + A_n)x_n,\ x_{n_0} = x_0,\ n_0 \leq n < \infty,$$

and

$$y_{n+1} = (I + B_n)y_n,\ y_{n_0} = y_0,\ n_0 \leq n < \infty$$

, respectively. If m is a fixed nonnegative integer, then the solutions of the systems (1) and (2) satisfy

$$x_n = X_n X_m^{-1} x_m, \quad \text{and} \quad y_n = Y_n Y_m^{-1} y_m \ , \quad n \geq 0$$

respectively. By the Variation of Constants Formula (7), the solution of Eq. (2) is given by

$$y_n = Y_n Y_{n_0}^{-1} y_0 + \sum_{r=n_0}^{n-1} Y_n Y_{r+1}^{-1} g_r. \tag{9}$$

From Definition 5 and our hypothesis on the system (1), there exist $\delta > 0$, $\mu_A > 0$, $\eta \in (0, 1)$ such that $\|x_n\| \leq \mu_A \|x_0\| \eta^{n-n_0}$, whenever $\|x_0\| \leq \delta_0$. This guarantees that there exist a fixed noninteger m integer such that $\|x_m\| \leq \delta_1$, $n_0 \leq m \leq n < \infty$. Thus we have

$$\begin{aligned} \left\|X_n X_m^{-1}\right\| &= \sup_{\|\xi\| \leq 1} \left\|X_n X_m^{-1} \xi\right\| \\ &= \frac{1}{\delta_1} \sup_{\|x_m\| \leq 1} \left\|X_n X_m^{-1} x_m\right\| \\ &\leq \frac{1}{\delta_1} \mu_A \|x_0\| \eta^{n-n_0} \\ &\leq \frac{\delta_0}{\delta_1} \mu_A \eta^{n-n_0}. \end{aligned} \tag{10}$$

Letting $\kappa_A = \frac{\delta_0}{\delta_1} \mu_A$, then from (10) we get

$$\left\|X_n X_m^{-1}\right\| \leq \kappa_A \eta^{n-n_0}, \ n_0 \leq m \leq n < \infty. \tag{11}$$

Now, we will show that there are positive constants μ_B and η such that

$$\left\|Y_n Y_m^{-1}\right\| \leq \mu_B \eta^{n-n_0}, \ n_0 \leq m \leq n < \infty. \tag{12}$$

Since $\lim\limits_{m \to \infty} Q_m^{(r)} = 0$, $1 \leq r \leq p$, it follows that $(\Gamma_n^{(p)})^{-1}$ exists and is bounded for i sufficiently large, say $n \geq n_0 \geq 0$. Hence, we apply matrix norm to either sides of the Eq.(8) to get

$$\begin{aligned} \left\|Y_n Y_m^{-1}\right\| \leq \left\|(\Gamma_n^{(p)})^{-1}\right\| \Big[&\left\|S_n^{-1}\right\| \left\|X_n X_m^{-1}\right\| \|S_m\| \left\|\Gamma_m^{(p)}\right\| \\ &+ \sum_{k=m}^{n-1} \left\|S_n^{-1}\right\| \left\|X_n X_{k+1}^{-1}\right\| \left\|F_k^{(p)}\right\| \left\|Y_k Y_m^{-1}\right\| \Big] \end{aligned} \tag{13}$$

, with $n_0 \le m \le n < \infty$. Since S_n, S_n^{-1}, $\Gamma_n^{(p)}$ and $(\Gamma_n^{(p)})^{-1}$ are bounded, so (11) and (13) implies that there are constants α and β such that

$$\eta^{m-n} \left\| Y_n Y_m^{-1} \right\| \le \alpha + \beta \sum_{k=m}^{n-1} \eta^{m-k-1} \left\| F_k^{(p)} \right\| \left\| Y_k Y_m^{-1} \right\|, \quad n_0 \le m \le n < \infty. \tag{14}$$

Let $u_{m,n} = \alpha + \beta \sum\limits_{k=m}^{n-1} \eta^{m-k-1} \left\| F_k^{(p)} \right\| \|Y_k Y_m^{-1}\|$, $n_0 \le m \le n < \infty$. Thus we have

$$u_{m,n+1} - u_{m,n} \le \beta \eta^{-1} \left\| F_n^{(p)} \right\| u_{m,n}, \; n \ge m$$

or

$$u_{m,n+1} \le (1 + \beta \eta^{-1} \left\| F_n^{(p)} \right\|) u_{m,n}, \; n \ge m \; .$$

By a simple iteration, it is easy to see that

$$u_{m,n} \le \alpha \prod_{k=m}^{n-1} (1 + \beta \eta^{-1} \left\| F_k^{(p)} \right\|). \tag{15}$$

or

$$u_{m,n} \le \alpha \prod_{k=0}^{\infty} (1 + \beta \eta^{-1} \left\| F_k^{(p)} \right\|). \tag{16}$$

Since $F_n^{(p)} \in \Re$, from Theorem 8, we say that the product $\prod\limits_{k=0}^{\infty} (1 + \beta \eta^{-1} \left\| F_k^{(p)} \right\|)$ converges. Thus, (14) and (16) imply (12) with

$$\mu_B = \alpha \prod_{k=0}^{\infty} (1 + \beta \eta^{-1} \left\| F_k^{(p)} \right\|).$$

Note that μ_B is independent of n_0. From (9), we obtain

$$\|y_n\| \le \mu_B \eta^{n-n_0} \|y_o\| + \mu_B \eta^{-1} \sum_{j=n_0}^{n-1} \eta^{(n-j)} \|g(n, y_j)\|, \quad n_0 \le m \le n < \infty. \tag{17}$$

We say that $g(n, y_j) = o(y_j)$ ("small oh of y_j") as $\|y_j\| \to 0$ if, given $\varepsilon > 0$, there is $\delta > 0$ such that $\|g(n, y_j)\| \le \varepsilon \|y_j\|$ whenever $\|y_j\| < \delta$.So as long as $\|y_j\| < \delta$, Equation (17) becomes

$$\eta^{-n} \|y_n\| \le \mu_B \left[\eta^{-n_0} \|y_o\| + \sum_{j=n_0}^{n-1} \varepsilon \eta^{-j-1} \|y_j\| \right], \quad n_0 \le m \le n < \infty. \tag{18}$$

Thus, from (18) we have

$$\eta^{-n-1}\|y_{n+1}\| - \eta^{-n}\|y_n\| \le \mu_B \varepsilon \eta^{-1} \eta^{-n}\|y_n\|, \ \ n_0 \le m \le n < \infty.$$

or

$$\eta^{-n-1}\|y_{n+1}\| \le \eta^{-n}\left[1+\mu_B\varepsilon\eta^{-1}\right]\|y_n\|, \ \ n_0 \le m \le n < \infty. \tag{19}$$

By a simple iteration, from (19) we deduce that

$$\eta^{-n}\|y_n\| \le \prod_{j=n_0}^{n-1}\left[1+\mu_B\varepsilon\eta^{-1}\right]\eta^{-n_0}\|y_0\|, \ \ n_0 \le m \le n < \infty. \tag{20}$$

Thus, from (20), we obtain

$$\|y_n\| \le \eta^{n-n_0}\|y_0\|\left[1+\mu_B\varepsilon\eta^{-1}\right]^{n-n_0}, \ \ n_0 \le m \le n < \infty.$$

or

$$\|y_n\| \le \|y_0\|\left(\eta+\mu_B\varepsilon\right)^{n-n_0}, \ \ n_0 \le m \le n < \infty. \tag{21}$$

If we choose $\varepsilon < \frac{1-\eta}{\mu_B}$, then $\eta + \mu_B\varepsilon < 1$. Thus $\|y_n\| < \|y_0\| < \delta$ for all $n_0 \le m \le n < \infty$. Thereby, (18) holds and consequently, by virtue of the Ineq. (21), we obtain that the zero solution of the nonlinear System (2) is exponentially stable. ■

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