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# Maximal Output Admissible Set and Admissible Perturbations Set For Nonlinear Discrete Systems

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**Abstract**

Consider the discrete nonlinear system $x(i+1) = f(x(i)),\ i \geq 0$ and the corresponding output signal $y(i) = Cx(i),\ i \geq 0$. Given a constraint set $\Omega \subset \mathbb{R}^p$, a initial state $x(0)$ is said to be output admissible if the resulting output function satisfies the condition $y(i) \in \Omega$, $\forall i \geq 0$. The set of all possible such initial conditions is the output maximal admissible set $X_\infty$. Contrary to the linear case, the representation of the maximal output admissible set for nonlinear systems is certainly more complex and not available. However, we restrain in this paper to the theoretical and algorithmic characterization of the set $X_\infty^M = X_\infty \cap B(0, M)$ where $B(0, M) = \{x \in \mathbb{R}^n / \|x\| \leq M\}$ with M is as large as we desire it. The maximal output admissible set for discrete delayed systems is also considered. As direct application of obtained results, we propose a technique that allows to determine, among all the perturbations susceptible to infect the initial state of a discrete nonlinear system, those which are relatively tolerable.

**Keywords:** Discrete nonlinear systems, asymptotic lyapunov stability, pointwise-in-time constraints, discrete delayed systems, nonlinear disturbed systems.

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# 1 Introduction

Output admissible sets have important applications in the analysis and design of closed-loop systems with state and control constraints. Although, the theory of output admissible sets has been extensively treated (see[2-11]), in most of the works devoted to its study, the problem for nonlinear systems is not considered hence it's applicability is severely limited.
The aim of this work is to present a contribution to the study of the maximal output admissible sets $X_\infty$ for discrete nonlinear systems. More precisely, the objective is to characterize the initial conditions of an uncontrolled nonlinear discrete system whose resulting trajectory satisfies a specified pointwise-in-time constraint. Such problem have important applications, to illustrate that we consider the following example. A nonlinear controlled discrete time system is described by

$$\begin{cases} x(i+1) & = & F(x(i)) + G(u_i), \quad i \in \mathbb{N} \\ x(0) & & \text{is given in } \mathbb{R}^n \end{cases}$$

where $(u_i)_i$ is the feedback control given by

$$u_i = H(x(i))$$

$F, G$ and $H$ are supposed to be nonlinear appropriate functions. In addition, there may be physical constraints on the state variable if the constraints are violated by an action $u_k$ serious consequences may happen, see ([1] ,[6]). By appropriate choice of matrices $C$ and a set $\Omega$, the constraints above may be summarized by the set inclusion

$$Cx(i) \in \Omega, \quad \forall i \in \mathbb{N} \tag{1}$$

and it is desired to obtain a safe set of initial conditions $x(0)$, i.e. a set $X_\infty$ such that $x(0) \in X_\infty$ implies (1), the problem can be state equivalently as a problem involving an unforced nonlinear discrete systems with output constraints. Specifically, given a continuous nonlinear function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $f(0) = 0$, a $p \times n$ matrix $C$ and a constraint set $\Omega \subset \mathbb{R}^p$.
We have to determine, for a given initial state $x(0)$ if the system

$$\begin{cases} x(i+1) & = & f(x(i)), \quad i \in \mathbb{N} \\ x(0) & \in & \mathbb{R}^n \end{cases} \tag{2}$$

with the output signal

$$y(i) = Cx(i), \quad i \in \mathbb{N} \tag{3}$$

satisfies the output constraint

$$y(i) \in \Omega. \tag{4}$$

An initial condition $x(0)$ is output admissible if the resulting output function (3) satisfied the constraint (4). The set of all such initial state is the maximal output admissible set $X_\infty$.
In the case of a linear system (see [5], [17]) and linear delayed system [13], the maximal output set has completely determined, algorithm based on the mathematical programming, have allowed a numerical simulation of the maximal set. In the nonlinear case, which is the object of this paper, we have not been able to characterize the set $X_\infty$; however, we propose a theoretical and algorithmic characterization of the set $X_\infty^M = X_\infty \cap B(0, M)$ where $B(0, M)$ is the ball of center 0 and radius $M$.
The fact to restrict to the set $X_\infty^M$ does not reduce the importance of the work and this for the next reason. Given an initial state $x(0) \in \mathbb{R}^n$, we wonders if $x(0)$ is output admissible or no. To answer to this question we firstly determines the set $X_\infty^r = X_\infty \cap B(0, r)$ where $r$ is a real that verifies $\|x(0)\| \leq r$ and we verifies if $x(0) \in X_\infty^r$ or no. The numerical simulations are presented and the case of discrete delayed systems is also studied .
The real process are often affected disturbances and it is necessary to consider then in the project of the control. Unfortunately, in many case, non information about the disturbances in deterministic or statistic sense.To better avoid damages being able to be caused by such disturbances on the evolution of a system, it's very important to characterize (under some hypothesis) the set of this disturbances (see [14] [4]). The case of the disturbances which infect the initial state for linear system has considered in [**?**]. Motives by theory developed for maximal output set, in the second part of this work, we consider the nonlinear disturbed discrete system described by

$$\begin{cases} x^d(i+1) &=& f(x^d(i)), \quad i \in \mathbb{N} \\ x^d(0) &=& x(0) + d \end{cases} \tag{5}$$

here $(x^d(i))$ is the disturbed state of system, $d$ is a perturbation that infect the initial state $x(0)$. The corresponding output perturbation is supposed to be

$$y^d(i) = Cx^d(i). \tag{6}$$

The disturbance $d$ being unavoidable, we use technique developed in the first part to determine all perturbations $d$ that are $\epsilon$-tolerable, i.e., the perturbations such that

$$\|y^d(i) - y(i)\| \leq \epsilon, \quad \forall i \geq 0$$

where

$$y(i) = Cx(i), \quad i \geq 0$$

and $(x(i))_{i\geq 0}$ is the uninfected state given by

$$\begin{cases} x(i+1) &= f(x(i)), \quad i \in \mathbb{N} \\ x(0). \end{cases}$$

# 2 Preliminary results

Consider the uncontrolled nonlinear discrete system described by

$$\begin{cases} x(i+1) &= f(x(i)), \quad i \in \mathbb{N} \\ x(0) &= x_0 \in \mathbb{R}^n \end{cases} \tag{7}$$

the corresponding output is

$$y(i) = Cx(i), \quad i \in \mathbb{N} \tag{8}$$

where the state variable $x(i)$ is in $\mathbb{R}^n$, $f$ is a continuous nonlinear function that verify $f(0) = 0$ and the observation variable $y(i) \in \mathbb{R}^p$, satisfies the output constraint

$$y(i) \in \Omega, \quad \forall i \in \mathbb{N} \tag{9}$$

where $C$ is a $p \times n$ real matrix.
An initial condition $x(0) \in \mathbb{R}^n$ is output admissible if $x(0) \in B(0, M)$ and the resulting output function (8) satisfies (9). The set of all such initial conditions is the maximal output admissible set $X_\infty^M$.
We proof that under hypothesis on $f$, the maximal output admissible set is determined by a finite number of functional inequalities and leads to algorithmic procedures for the computation of $X_\infty^M$.
The system (8) can be equivalently rewritten in the form

$$y(i) = Cf^i(x_0), \quad \text{for all } \ i \in \mathbb{N} \tag{10}$$

The set of all output admissible initial states is formally given by

$$X_\infty^M = \{x_0 \in B(0, M) \cap \mathbb{R}^n \ / \ Cf^i(x_0) \in \Omega, \ \ \forall i \in \mathbb{N}\} \tag{11}$$

We assume hereafter that $0 \in \text{int } \Omega$. This assumption is satisfied in any reasonable application and has nice consequences. Imposing special condition on $f$ we have a nonempty maximal admissible set $X_\infty^M$ which contains the origin, indeed

**Proposition 2.1** **(i)** *The closure propertie of* $\Omega$ *is inherited by* $X_\infty^M$.

**(ii)** *If* $f$ *is asymptotically lyapunov stable (i.e.,* $\forall I \in \mathbb{N}\ \exists\delta > 0$ *such that* $\|x(I) - x'(I)\| < \delta$ *then* $\lim_{i \to \infty} \|f^i(x(I)) - f^i(x'(I))\| = 0$*) and* $0 \in int\ \Omega$ *then,* $0 \in int\ X_\infty^M$.

**Proof**.
It is easily to verify the closure of $X_\infty^M$ from his definition and continuity of $f$. The assumption of asymptotic lyapunov stability implies that there exists a constant $\delta > 0$ such that for all $x_0 \in B(0,\delta)$, $\lim_{i\to\infty} \|f^i(x_0)\| = 0$. which implies that for all $x_0 \in B(0,\delta)$ and $\eta > 0$ there exists $i_0 > 0$, such that for all $i \geq i_0$, $f^i(x_0) \in B(0,\eta)$, we deduce that

$$\forall x_0 \in B(0,\delta) \text{ we have } Cf^i(x_0) \in B(0,\eta\|C\|), \quad \forall i \geq i_0. \tag{12}$$

Since $0 \in$ int $\Omega$, we have

$$\exists \rho > 0 \quad \text{such that} \quad B(0,\rho) \subset \Omega \tag{13}$$

if we pose $\eta = \frac{\rho}{\|C\|}$ then

$$\exists \delta > 0 \text{ such that } x_0 \in B(0,\delta) \Longrightarrow Cf^i(x_0) \in B(0,\rho) \subset \Omega, \ \forall i \geq i_0$$

and using the continuity of $f$ and the condition $f(0) = 0$ we deduce that

$$\exists \delta' > 0 \text{ such that } x_0 \in B(0,\delta') \Longrightarrow Cf^i(x_0) \in \Omega \ \ 0 \leq i < i_0$$

we choose $\alpha = \inf(\delta, \delta', M)$ we obtain,

$$\text{for every } x_0 \in B(0,\alpha) \Longrightarrow y(i) = Cf^i(x_0) \in \Omega, \ \forall i \in \mathbb{N}$$

thus $B(0,\alpha) \subset X_\infty^M$, consequently $0 \in Int\ X_\infty^M$ ∎

# 3 Characterization of the maximal output admissible set

In order to characterize the maximal output admissible set given formally by (11), we define for each integer $k$ the set

$$X_k^M = \{x_0 \in B(0,M) \cap \mathbb{R}^n \ / \ Cf^i(x_0) \in \Omega, \quad \forall i = 0,\dots,k\}$$

**Definition 3.1** *The set $X_\infty^M$ is finitely determined if there exists an integer $k$ such that $X_\infty^M$ is nonempty and $X_\infty^M = X_k^M$.*

**Remark 3.1** **(i)** *Obviously, the set $X_k^M$ satisfies the following condition: for $k_1, k_2 \in \mathbb{N}$ such that $k_1 \le k_2$, we have.*

$$X_\infty^M \subset X_{k_2}^M \subset X_{k_1}^M$$

**(ii)** *Suppose that $X_\infty^M$ is finitely determined and let $k_0$ be the smallest $k$ such that $X_k^M = X_{k+1}^M$, then $X_\infty^M = X_{k_0}^M = X_k^M$ for all $k \ge k_0$.*

**Proposition 3.1** **(i)** *If $X_\infty^M$ is finitely determined then there exists an integer $k$ such that $X_k^M$ is nonempty and $X_k^M = X_{k+1}^M$.*

**(ii)** *If $f(B(0,M)) \subset B(0,M)$ and $X_k^M = X_{k+1}^M$ for some integer $k$ then $X_\infty^M$ is finitely determined.*

**Proof**.
(i) If $X_\infty^M = X_k^M$ for some $k \in \mathbb{N}$, then $X_k^M$ is nonempty and obviously $X_k^M = X_{k+1}^M$
(ii) Suppose that $f(B(0,M)) \subset B(0,M)$ and there exists a integer $k$ such that $X_k^M$ is nonempty and $X_k^M = X_{k+1}^M$ then

$$x_0 \in X_k^M \Longrightarrow x_0 \in X_{k+1}^M \Longrightarrow f(x_0) \in X_k^M$$

and by iteration

$$x_0 \in X_k^M \Longrightarrow f^j(x_0) \in X_k^M, \quad \forall j \in \mathbb{N}$$

hence $X_k^M \subset X_\infty^M$, we apply remark 3.1 to deduce that $X_k^M = X_\infty^M$ ■

As a natural consequence of the previous proposition, we shall give in section 4 an algorithm which allows to determine the smallest integer $k^*$ such that $X_\infty^M = X_{k^*}^M$.
It is desirable to have simple condition which assure the finite determination of $X_\infty^M$. Our main results in this direction is the following theorems.

**Theorem 3.1** *Suppose the following assumptions hold*

1. *$f$ is continuous, $f(0) = 0$, $f(B(0,M)) \subset B(0,M)$ and $f$ is asymptotically lyapunov stable.*

2. *$0 \in int\Omega$.*

3. $f(\lambda x) = g(\lambda)f(x), \quad \forall x \in \mathbb{R}^n, \quad \forall \lambda \in \mathbb{R}$ *where* $g$ *is a real function which verify* $g(0) = 0$ *and the sequence* $(g^k(\lambda))_{k\geq 0}$ *is bounded* $\forall \lambda \in \mathbb{R}$, ($g^k = \underbrace{g \circ g \circ \ldots \circ g}_{k-times}$).

*then* $X^M_\infty$ *is finitely determined.*

**Proof**.
First case: $M \leq \delta$, then we apply equation (12) for $\eta = \frac{\rho}{\|C\|}$ we obtain

$$\text{if } x_0 \in B(0, M) \text{ then } Cf^{i_0}(x_0) \in B(0, \rho) \subset \Omega \tag{14}$$

Second case: $M > \delta$, by hypothesis 3 of theorem we have
$Cf^k x_0 = Cf^k(\frac{M}{\delta}\frac{\delta}{M}x_0) = g^k(\frac{M}{\delta})Cf^k(\frac{\delta}{M}x_0)$. Since $(g^k(\frac{M}{\delta}))_{k\geq 0}$ is bounded by some constant $M'$ then using equation (12) and $\eta = \frac{\rho}{\|C\|M'}$ we deduce that $\forall x_0 \in B(0, M)\ \exists i_0$ such that $Cf^k(x_0) \in B(0, \rho), \quad \forall k \geq i_0$. In particular we obtain equation (14).
Then if $x_0 \in X^M_{i_0-1}$, we have $x_0 \in B(0, M)$ and $Cf^k(x_0) \in \Omega, \quad \forall k \in \{0, \ldots, i_0 - 1\}$, by equation(14) we deduce that $x_0 \in X^M_{i_0}$. Consequently $X^M_{i_0-1} = X^M_{i_0}$ and we deduce from proposition 2.1 that the maximal admissible set is finitely determined. ■

**Theorem 3.2** *Suppose the following assumptions hold*

1. $\|f(x)\| \leq \eta\|x\|, \forall x \in \mathbb{R}^n$ *and* $\eta \in ]0, 1[$.

2. $0 \in int\Omega$.

*then* $X^M_\infty$ *is finitely determined.*

**Proof**.
It apparent from hypothesis 1 that

$$\|Cf^i(x_0)\| \leq \|C\|\eta^i\|x_0\|, \quad \forall i\mathbb{N},$$

then for $x_0 \in B(0, M)$ there exists a $k \in \mathbb{N}^*$ such that

$$\|Cf^k(x_0)\| \leq \rho \tag{15}$$

then, if $x_0 \in X^M_{k-1}$ we have $x_0 \in B(0, M)$ and $Cf^i(x_0) \in \Omega$ for $i \in \{0, \ldots, k - 1\}$. Using (13) and by equation (15) we have $Cf^i(x_0) \in \Omega$ for $i \in \{0, \ldots, k\}$. Consequently $x_0 \in X^M_k$. This result imply that $X^M_k = X^M_{k-1}$. ■

## 4 Algorithmic determination

The proposition 3.1 suggests the following conceptual algorithm for determining the output admissible index $k^*$, consequently the characterization of the set $X_\infty^M$.

Algorithm I

step 1 : Set $k = 0$
step 2 : If $X_k^M = X_{k+1}^M$ then set $k^* = k$ and stop,
else continue.
step 3 : Replace $k$ by $k+1$ and return to step 2.

Clearly, the algorithm I will produce $k_0$ and $X_\infty^M$ if and only if $X_\infty^M$ is finitely determined. There appears to be no finite algorithmic procedure for showing that $X_\infty^M$ is not finitely determined.
Algorithm I is not practical because it does not describe how the test $X_k^M = X_{k+1}^M$ is implemented. The difficulty can be overcome if we intruded in $\mathbb{R}^n$ the following norm

$$\|x\| = \max_{1\leq i\leq n}|x_i|, \quad \forall x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n.$$

and if $\Omega$ is defined by:

$$\Omega = \{y \in \mathbb{R}^p; \;\; h_i(y) \leq 0, \;\; i = 1, \ldots, s\} \tag{16}$$

where $h_i : \mathbb{R}^p \longrightarrow \mathbb{R}$ are a given function. Such a sets have many importance in a practical view. In this case, for every integer $k$, $X_k^M$ is given by

$$X_k^M = \{x_0 \in B(0, M); \;\; h_j(Cf^i x_0) \leq 0, \quad j = 1, \ldots, s; \; i = 0, \ldots, k\}$$

On the other hand

$$\begin{aligned} X_{k+1}^M &= \{x_0 \in X_k^M; \;\; Cf^{k+1}(x_0) \in \Omega\} \\ &= \{x_0 \in X_k^M; \;\; h_j(Cf^{k+1}(x_0)) \leq 0, \quad \text{for } j = 1, \ldots, s\} \end{aligned}$$

Now, since $X_{k+1}^M \subset X_k^M$ for every integer $k$, then

$$\begin{aligned} X_{k+1}^M = X_k^M &\iff x_0 \in X_k^M; \;\; h_j(Cf^{k+1}(x_0)) \leq 0, \;\; \forall j = 1, \ldots, s \\ &\iff \sup_{x_0 \in X_k^M} h_j(Cf^{k+1}(x_0)) \leq 0, \;\; \forall j = 1, \ldots, s \\ &\iff \sup_{\underbrace{h_j(Cf^l(x_0)) \leq 0; \; g_r(x) \leq 0}_{j\in\{1,\ldots,s\},\, l\in\{0,\ldots,k\},\, r\in\{1,\ldots,2n\}}} h_i(Cf^{k+1}x_0) \leq 0, \quad \forall i \in \{1, \ldots, s\}. \end{aligned}$$

with $g_l : \mathbb{R}^n \longrightarrow \mathbb{R}$ is described for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ by

$$\begin{cases} g_{2r-1}(x) &= x_r - M, \quad for\ r \in \{1, 2, \ldots, n\} \\ g_{2r}(x) &= -x_r - M, \quad for\ r \in \{1, 2, \ldots, n\} \end{cases}.$$

Consequently the test $X_k^M = X_{k+1}^M$ leads to a set of mathematical programming problems, and algorithm I can be rewritten of practical manner under the form

## Algorithm II

step 1 : Set $k = 0$;
step 2 : For $i = 1, \ldots, s$, do :
Maximize $J_i(x) = h_i(Cf^{k+1}(x_0))$

$$\begin{cases} h_r(CA^l x) \leq 0, \quad g_j(x) \leq 0 \\ r = 1, \ldots, s, \ j = 1, 2, \ldots, 2n, \\ l = 1, \ldots, k. \end{cases}$$

Let $J_i^*$ be the maximum value of $J_i(x)$.
If $J_i^* \leq 0$, for $i = 1, \ldots, s$ then set $k^* := k$ and stop.
Else continue.
step 3 : Replace $k$ by $k + 1$ and return to step 2.

Assumptions of theorems 3.1 and 3.2 are sufficient but not necessary. If these conditions are not verified, then Algorithm II can also be used for every $\Omega$ given by (16). If the Algorithm converge then $X_\infty^M$ is finitely determined, else it is not. To illustrate this work we give some examples.

**Example 1:** Let $f$, $C$, $\Omega$, and M given by

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} 0.4|x| - 0.1y \\ -0.2|x| + 0.5y \end{pmatrix}$$

$C = (-1, 0.2)$, $\Omega = [-0.5, 5]$ and $M = 10$. Then, we use algorithm II to establish that $k^* = 2$ and we have

$$\begin{aligned} X_\infty^M = \{ & \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 / \ |x| \leq 10, \ \ |y| \leq 10, \ -0.5 \leq -x + 0.2y \leq 5, \\ & -0.5 \leq -0.44|x| + 0.11|y| \leq 5, \\ & -0.5 \leq |0.176|x| - 0.02y| - 0.04|x| + 0.1y \leq 5\} \end{aligned}$$

The following figure gives a representation of Maximal output set $X_\infty^M$ corresponding to example 1.

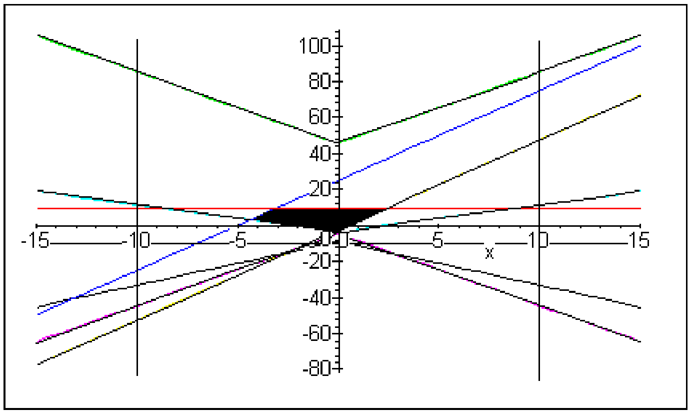


Figure 1: The set $X_\infty^M$ corresponding to example 1

**Example 2:** For $f$, $C$, $\Omega$, and M given by

$$\begin{array}{cccc} f & : & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2, \\ & & \begin{pmatrix} x \\ y \end{pmatrix} & \longrightarrow & \begin{pmatrix} \frac{1}{4}arctan(\frac{x}{2}) \\ \frac{y}{2} \end{pmatrix} \end{array}$$

$C = (1, 2),\ \Omega = [-0.5, 0.5]$ and $M = 1$, we have $k^* = 1$ and

$$X_\infty^M = \{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \ / \ |x| \leq 1,\ |y| \leq 2,\ |x + 2y| \leq \frac{1}{2},\ |\frac{1}{4}arctan(\frac{x}{2}) + y| \leq \frac{1}{2}\}$$

Figure 2 give the representation of Maximal output set $X_\infty^M$ corresponding to example 2.

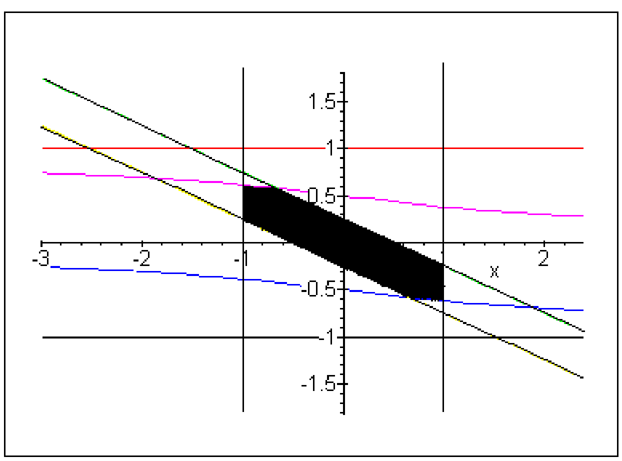


Figure 2: The set $X_\infty^M$ corresponding to example 2

# 5 Maximal output admissible sets for nonlinear discrete delayed Systems

Consider the uncontrolled nonlinear discrete delayed system described by

$$\begin{cases} x(i+1) &= f(x(i),\dots,x(i-r)),\ i\in\mathbb{N} \\ x(0) &= x_0 \\ x(s) &= \alpha_s,\ -r\le s\le -1 \end{cases} \tag{17}$$

the corresponding output is

$$y(i)=\sum_{j=0}^{t} C_j x(i-j), \quad i\in\mathbb{N} \tag{18}$$

where the state variable $x(i)$ is in $\mathbb{R}^n$, $f$ is a continuous nonlinear function that verify $f(0)=0$, $r$ and $t$ are integers such that $t\le r$ and the observation variable $y(i)\in\mathbb{R}^p$, satisfies the output constraint

$$y(i)\in\Omega, \quad \forall i\in\mathbb{N} \tag{19}$$

where $C_j$ are a $p\times n$ real matrices.
An initial condition $\alpha=(x_0,\alpha_{-1},\dots,\alpha_{-r})\in\mathbb{R}^{n(r+1)}$ is output admissible if the resulting output function (18) satisfies (19). The set of all such initial conditions is the maximal output admissible set $X_\infty^M$.
we proof that under hypothesis on $f$, the maximal output admissible set is finitely determined by a finite number of functional inequalities and leads to algorithmic procedures for the computation of $X_\infty^M$.
Define the state variable $\xi(i)$ by

$$\xi(i)=(x(i),x(i-1),\dots,x(i-r))^\top$$

then we easily verify that $(\xi(i))_{i\ge 0}$ is the solution of the following difference equation

$$\begin{cases} \xi(i+1) &= F(\xi(i)) \\ \xi(0) &= \alpha \end{cases}$$

where $F:\mathbb{R}^{n(r+1)}\longrightarrow\mathbb{R}^{n(r+1)}$ is given by

$$F(y_0,\dots,y_r)=(f(y_0,\dots,y_r),y_0,y_1,\dots,y_{r-1})^\top$$

If we define the matrix $\tilde{C}$ by

$$\tilde{C}=(C_0|\dots|C_t|\underbrace{0_{p\times n}|\dots|0_{p\times n}}_{(r-t)-times})\in\mathcal{L}(\mathbb{R}^{n(r+1)},\mathbb{R}^p)$$

then the output function $y(i)$ are described in terms of the variable $\xi(i)$ as follows

$$y(i) = \tilde{C}\xi(i).$$

Thus, the set of all output admissible initial states is formally given by

$$X_\infty^M = \{\alpha \in B(0, M) \cap \mathbb{R}^{n(r+1)} \ / \ \tilde{C}F^i(\alpha) \in \Omega, \ \ \forall i \in \mathbb{N}\}. \tag{20}$$

In order to characterize the maximal output sets given formally by (20), we define for each integer $k$ the set

$$X_k^M = \{\alpha \in B(0, M) \cap \mathbb{R}^{n(r+1)} \ / \ \tilde{C}F^i(\alpha) \in \Omega, \quad \forall i \in \{0, 1, \dots, k\}\}.$$

On the other hand

$$F^i(\alpha) = (f(F^{i-1}(\alpha)), \dots, f(F^{i-r-1}(\alpha)))^\top, \ \ \forall i > r \tag{21}$$

if $\forall x = (x_0, x_1, \dots, x_r) \in \mathbb{R}^{n(r+1)}$

$$f(a_0x_0, a_1x_1, \dots, a_rx_r) = g(a_0, a_1, \dots, a_r)f(x) \tag{22}$$

where $g$ is a real function which verify $g(0) = 0$, then we have

$$F^i(\lambda x) = A_i(\lambda)F^i(x)$$

where

$$A_i(\lambda) = \begin{pmatrix} (\phi^i(\lambda, \lambda, \dots, \lambda))_1 & 0 & \dots & 0 \\ 0 & (\phi^i(\lambda, \lambda, \dots, \lambda))_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & (\phi^i(\lambda, \lambda, \dots, \lambda))_{r+1} \end{pmatrix}$$

and

$$\phi(\lambda_0, \dots, \lambda_r) = (g(\lambda_0, \dots, \lambda_r), \lambda_0, \dots, \lambda_{r-1})^\top$$

Using equation (21), we have the following result

**Theorem 5.1** *Suppose the following assumptions hold*

1. $f$ *is continuous,* $f(0) = 0$, $f(B(0, M)) \subset B(0, M)$ *and* $f$ *satisfied equation (22).*

2. $F$ *is asymptotically lyapunov stable*

3. $0 \in int\Omega$.

4. $(\|A_i(\lambda)\|)_{i\geq 0}$ *is bounded* $\forall\lambda \in \mathbb{R}$.

*then* $X_\infty^M$ *described by equation (20) is finitely determined.*

and similarly to theorem 3.2, we have

**Theorem 5.2** *Suppose the following assumptions hold*

1. $0 \in int\Omega$.
2. $\|F^i(x)\| \leq (Cste)\eta^i\|x\|, \quad \eta \in ]0,1[, \forall i \in \mathbb{N}$.

*then* $X_\infty^M$ *is finitely determined,i.e., there exists* $k \in \mathbb{N}$ *such that* $X_\infty^M = X_k^M$

We determine the output admissible index $k^*$ using the algorithm II by making the assignments $C \longrightarrow \tilde{C}, \quad f \longrightarrow F,$ and $\Omega \longrightarrow \Omega$

# 6 Application to Perturbed Systems

This section is devoted to the characterization of admissible disturbances for the nonlinear discrete infected system described by

$$\begin{cases} x^d(i+1) &= f(x^d(i)), \quad i \in \mathbb{N} \\ x^d(0) &= x_0 + d \in \mathbb{R}^n \end{cases} \tag{23}$$

the corresponding output function is

$$y^d(i) = Cx^d(i), \quad i \in \mathbb{N} \tag{24}$$

where $f$ is a continuous nonlinear function, the observation variable $y^d(i) \in \mathbb{R}^p$, and $C$ is a $p \times n$ real matrix, $x^d(i) \in \mathbb{R}^n$ is the state variable and $d \in \mathbb{R}^n$ represents a unavoidable disturbances which enters the system because of its connections with the environment. The output signal corresponding to $d = 0$ is simply denoted by $(y(i))_{i\geq 0}$ i.e.,

$$y(i) = Cx(i), \quad i \in \mathbb{N} \tag{25}$$

where $(x(i))_{i\geq 0}$ is the uninfected state given by

$$\begin{cases} x(i+1) &= f(x(i)), \quad i \in \mathbb{N} \\ x(0) &= x_0 \in \mathbb{R}^n \end{cases} \tag{26}$$

It is reasonable to agree that a source $d$ is tolerable if, for every integer $i$, the subsequent output variable $y^d(i)$ remains in a neighborhood of the uninfected output $y(i)$, this requires from us to introduce the index of admissibility

$\epsilon(\epsilon > 0)$. We say that the source $d$ is $\epsilon$-admissible if $\|y^d(i) - y(i)\| \leq \epsilon$ for all integer $i \geq 0$.
Motivated by practical consideration, we suppose that the disturbances $d$ susceptible of infecting the initial state of system satisfied $\varphi(d) = (x_0 + d, x_0) \in B(0, M) = \{x \in \mathbb{R}^{2n} / \|x\| \leq M\}$.
The purpose of this section is to characterize, under certain hypothesis, The set $S^M(\epsilon)$ of all source $d$ such that $\varphi(d) \in B(0, M)$ which are $\epsilon$-admissible. We call $S^M(\epsilon)$ the $\epsilon$-admissible set. We proof that under certain hypothesis on $f$, the $\epsilon$-admissible set is finitely determined and leads to algorithmic procedures for the computation of $S^M(\epsilon)$. The set of all $\epsilon$-admissible set is formally given by

$$S^M(\epsilon) = \{d \in \mathbb{R}^n \ /\varphi(d) \in B(0, M), \ \|Cf^i(x_0 + d) - Cf^i(x_0)\| \leq \epsilon, \ \ \forall i \in \mathbb{N}\} \tag{27}$$

The set $S^M(\epsilon)$ is derived from an infinite number of inequalities and it is difficult to characterize. However we propose some sufficient conditions which assure $S^M(\epsilon)$ to be finitely accessible, i.e., there exists an integer $k$ such that $S^M(\epsilon) = S^M_k(\epsilon)$ where

$$S^M_k(\epsilon) = \{d \in \mathbb{R}^n \ /\varphi(d) \in B(0, M), \ \|Cf^i(x_0+d) - Cf^i(x_0)\| \leq \epsilon, \ \ \forall i = 0, \dots, k\}$$

Let us define the functionals $L$ and $F$

$$\begin{array}{rccc} L: & \mathbb{R}^n \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & (a, b) & \longrightarrow & a - b \end{array}$$

$$\begin{array}{rccc} F: & \mathbb{R}^n \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \times \mathbb{R}^n \\ & (x, y) & \longrightarrow & (f(x), f(y)) \end{array}$$

by above notations we can easily establish that the set $S^M(\epsilon)$ can be rewriting as follows

$$S^M(\epsilon) = \{d \in \mathbb{R}^n \ / \ \varphi(d) \in B(0, M), \ \ \|CLF^i\varphi(d)\| \leq \epsilon, \ \ \forall i \in \mathbb{N}\}$$

moreover

$$S^M(\epsilon) = \{d \in \mathbb{R}^n \ / \ \varphi(d) \in H^M(\epsilon)\}$$

where

$$H^M(\epsilon) = \{\xi \in B(0, M) \ / \ \|CLF^i\xi\| \leq \epsilon, \ \ \forall i \in \mathbb{N}\}.$$

For every $k \in \mathbb{N}$, we define the set $H^M_k(\epsilon)$ by

$$H^M_k(\epsilon) = \{\xi \in B(0, M) \ / \ \|CLF^i\xi\| \leq \epsilon, \ \ \forall i \in \{0, \dots, k\},$$

$H^M(\epsilon)$ is said to be finitely accessible if there exists $k \in \mathbb{N}$ such that $H^M(\epsilon) = H_k^M(\epsilon)$, we note $k^*$ the smallest integer such that $H^M(\epsilon) = H_{k^*}^M(\epsilon)$.
Obviously, the set $H_k^M(\epsilon)$ satisfies the following condition:

$$H^M(\epsilon) \subset H_{k_2}^M(\epsilon) \subset H_{k_1}^M(\epsilon),\ \forall k_1, k_2 \in \mathbb{N} \text{ such that } k_1 \leq k_2.$$

We use the result established in proposition 3.1, to give a sufficient conditions to assure that the set $S^M(\epsilon)$ contains the origin and a properties to characterize finitely the set $S^M(\epsilon)$.

**Proposition 6.1** **(i)** *The set $S^M(\epsilon)$ is closed and if $f$ is asymptotically lyapunov stable then, $0 \in int\ S^M(\epsilon)$.*

**(ii)** *If $H^M(\epsilon)$ is finitely determined then there exists an integer $k$ such that $H_k^M(\epsilon)$ is nonempty and $H_k^M(\epsilon) = H_{k+1}^M(\epsilon)$.*

**(iii)** *If $f(B(0,M)) \subset B(0,M)$ and $H_k^M(\epsilon) = H_{k+1}^M(\epsilon)$ for some integer $k \in \mathbb{N}$ then $H^M(\epsilon)$ is finitely determined.*

Suppose that $H^M(\epsilon)$ is finitely determined and let $k_0$ be the smallest $k$ such that $H_k^M(\epsilon) = H_{k+1}^M(\epsilon)$, then $H^M(\epsilon) = H_{k_0}^M(\epsilon) = H_k^M(\epsilon)$ for all $k \geq k_0$.
We apply the result established in theorem 3.1 and 3.2 to give a sufficient conditions which make $H^M(\epsilon)$ accessible and we deduce the following results

**Theorem 6.1** *Suppose the following assumptions hold*

1. *$f$ is continuous, $f(B(0,M)) \subset B(0,M)$ and asymptotically lyapunov stable.*

2. *$f(\lambda x) = g(\lambda)f(x),\ \ \forall x \in \mathbb{R}^n,\ \ \forall \lambda \in \mathbb{R}$ where $g$ is a real function which verify $g(0) = 0$ and the sequence $(g^k(\lambda))_{k\geq 0}$ is bounded for all $\lambda \in \mathbb{R}$, $(g^k = \underbrace{g \circ g \circ \ldots \circ g}_{k-times})$.*

*then $H^M(\epsilon)$ is finitely determined.*

**Theorem 6.2** *If $\|f(x)\| \leq \eta\|x\|,\ \forall x \in \mathbb{R}^n$ and $\eta \in ]0,1[$, then $H^M(\epsilon)$ is finitely determined.*

The proposition 6.1 we suggest the following conceptual algorithm for determining the output admissible index $k_0$, such that $H_{k^*}^M(\epsilon) = H^M(\epsilon)$ and consequently the characterization of the set $S^M(\epsilon)$ by

$$S^M(\epsilon) = S_{k^*}^M(\epsilon) = \varphi^{-1}(H_{k^*}^M(\epsilon)).$$

The set $H_k^M(\epsilon)$ is described by

$$H_k^M(\epsilon) = \left\{ \begin{array}{l} \xi \in \mathbb{R}^{2n} \ / \ g_l(\xi) \leq 0 \text{ and } h_j(CLF^i(\xi) \leq 0 \text{ for} \\ l = 1, 2, \ldots, 4n, \ j = 1, 2, \ldots, 2p \text{ and } i = 1, 2, \ldots, k \end{array} \right\}.$$

with $g_l : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$, $h_j : \mathbb{R}^p \longrightarrow \mathbb{R}$, are described for all $x = (x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n}$ and $y = (y_1, \ldots, y_p)$ by

$$\left\{ \begin{array}{lcl} g_{2r-1}(x) & = & x_r - M, \ \text{ for } r \in \{1, 2, \ldots, 2n\} \\ g_{2r}(x) & = & -x_r - M, \ \text{ for } r \in \{1, 2, \ldots, 2n\} \end{array} \right.$$

$$\left\{ \begin{array}{lcl} h_{2r-1}(y) & = & y_r - \epsilon, \ \text{ for } r \in \{1, 2, \ldots, p\} \\ h_{2r}(y) & = & -y_r - \epsilon, \ \text{ for } r \in \{1, 2, \ldots, p\} \end{array} \right.$$

we deduce from

$$H_k^M(\epsilon) = H_{k+1}^M(\epsilon) \Longrightarrow H_k^M(\epsilon) \subset H_{k+1}^M(\epsilon)$$

that

$$H_k^M(\epsilon) = H_{k+1}^M(\epsilon) \Longrightarrow \forall \xi \in H_k^M(\epsilon), \ \ h_j(CLF^{k+1}(\xi) \leq 0, \ \ \forall j \in \{1, 2, \ldots, 2p\}$$

equivalently to

$$\sup_{\xi \in H_k^M(\epsilon)} h_j(CLF^{k+1}(\xi)) \leq 0, \ \ \forall j \in \{1, \ldots, 2p\}.$$

Consequently the test $H_k^M(\epsilon) = H_{k+1}^M(\epsilon)$ leads to a set of mathematical programming problems, and we can give an practical algorithm under the following form.

## Algorithm III

step 1 : Set $k = 0$;
step 2 : For $i = 1, \ldots, 2p$, do :Maximize
$J_i(x) = h_i(CLF^{k+1}(x))$
$$\left\{ \begin{array}{l} h_i(CLF^l(x) \leq 0, \ \ g_j(x) \leq 0 \\ i = 1, \ldots, 2p, \ j = 1, 2, \ldots, 4n, \ l = 1, \ldots, k. \end{array} \right.$$
Let $J_i^*$ be the maximum value of $J_i(x)$.
If $J_i^* \leq 0$, for $i = 1, \ldots, 2p$ then set $k^* := k$ and stop.
Else continue.
step 3 : Replace $k$ by $k + 1$ and return to step 2.

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