Applied Mathematical Sciences, Vol. 1, 2007, no. 35, 1703 - 1714

Optimal Control of a Production Inventory System with Weibull Distributed Deterioration[1]

A. Al-khedhairi and L. Tadj

King Saud University, College of Science
Department of Statistics and Operations Research
P.O. Box 2455, Riyadh 11451, Saudi Arabia
akhediri@ksu.edu.sa, lotftadj@ksu.edu.sa

Abstract

This paper is concerned with the optimal control of a production inventory system with deteriorating items. It is assumed that the deterioration rate follows the two-parameter Weibull distribution. The continuous-review and periodic-review policies are investigated. In each case, optimality conditions are derived. Also, numerical illustrative examples are presented.

Keywords: Production planning, continuous-review, periodic-review, deterioration, Weibull distribution, optimal control, maximum principle, Lagrange technique

1 Introduction

Application of optimal control theory to management science/operations research problems is a rich research area; see Sethi and Thompson [21]. The problem of interest to us is the production planning problem. We consider a firm that produces a single product, selling some units and stocking the remaining units in a warehouse. Advantages and disadvantages of holding a stock are well-known. High inventory incurs high holding costs and low production costs while low inventory incurs low holding costs and high production costs. Typically, the firm has to balance these costs and find the quantity it should produce in order to keep the total cost at a minimum.

We deal in this paper with the case where units of the product, while in stock, are subject to deterioration. This is a topic that has received a lot of

[1]This research was supported by the College of Science Research Center at King Saud University under project No. Stat/2007/01.

attention; see Goyal and Giri [11]. A large number of theoretical papers make the assumption that the deterioration rate follows the Weibull distribution, see for example [4, 5, 9, 10, 15, 16, 19, 26]. Also, fitting empirical data to mathematical distributions has lead many researchers to use the Weibull distribution to model the deterioration rate. Among the items whose rate of deterioration was assumed to follow the Weibull distribution are refrigerated meats (Andujar and Herrera [1]), roasted and ground coffee (Cardelli and Labuza [3]), pasteurized milk (Duyvesteyn [7]), luncheon meats (Gacula [8]), breakfast cereal (Pickering [14]), cottage cheese (Schmidt and Bouma [20]), cassava flour (Shirose *et al.* [22]), corn seed (Tang *et al.* [23]), frozen foods (Tomasicchio *et al.* [24]), and ice cream (Wittinger and Smith [25]). Besides food stuff, there are many products, such as camera films, drugs, pharmaceuticals, chemicals, electronic components and radioactive substances that deteriorate while in stock. Also, reservoir systems are subject to deterioration in the form of evaporation. Poultry farms and fish ponds witness deterioration in the form of death of the animals (chicken and fish).

Our problem is dynamic and the solution sought is a function of time. So, the problem can be represented as an optimal control problem with one state variable (inventory level) and one control variable (rate of manufacturing). There are a few papers that used an optimal control approach to study dynamic production models. For example, Dobos [6] was interested in the optimal control of reverse logistics systems, where reusable materials returned from the market are remanufactured. Khemlnitsky and Gerchak [12] used an optimal control approach to solve a production system where demand depends on the inventory level. Kiesmüller [13] was interested in the optimal control of recovery systems, where attention is given to recycling and remanufacturing of used products in order to reduce waste. Riddalls and Bennett [17] used an optimal control algorithm to a differential equation model of a production inventory system to cater for batch production costs which, usually, are not modelled in aggregate production problems. Salama [18] considered the optimal control of an unreliable manufacturing system with restarting costs. Zhang *et al.* [27] were concerned with the scheduling of a marketing production system with a demand dependent on the marketing status.

The novelty we will be taking into consideration in this research is that the time to deterioration is a random variable following the two-parameter Weibull distribution. This distribution can be used to model either increasing or decreasing rate of deterioration, according to the choice of the parameters. We note that in the literature, Weibull distributed deterioration rate was considered in optimization models but not in optimal control models. The probability density function for a two-parameter Weibull distribution is given by

$$f(t) = \alpha\beta t^{\beta-1} e^{-\alpha t^{\beta}}, \quad t > 0,$$

where $\alpha > 0$ is the scale parameter and $\beta > 0$ is the shape parameter. The probability distribution function is

$$F(t) = 1 - e^{-\alpha t^{\beta}}, \quad t > 0.$$

The instantaneous rate of deterioration of the on-hand inventory is given by

$$\theta(t) = \frac{f(t)}{1 - F(t)} = \alpha\beta t^{\beta-1}, \quad t > 0.$$

To build our model, we will assume that the demand rate is a general function of time. We will also assume that the firm has set an inventory goal level and a production goal rate. The inventory goal level is a safety stock that the company wants to keep on hand. The production goal rate is the most efficient rate desired by the firm. The objective is to determine the optimal production rate that will keep the inventory level and the production rate as close as possible to the inventory goal level and production goal rate, respectively. Since a review policy by the firm can be either continuous or periodic, we will study both of these policies. In each case, necessary and sufficient optimality conditions are derived. In the case of the continuous-review policy, the main tool in the study of this kind of problems is the Pontryagin maximum principle. In the case of the periodic-review policy, the Lagrangian technique is used.

In the next section we introduce the notation and formally describe the system. In sections 3 and 4 we study the optimal control of the system under a continuous-review and a periodic review, respectively. Illustrative numerical examples are provided in each of these sections.

2 Model Formulation and Notation

Let $T > 0$ represent the length of the planning horizon and consider a firm that manufactures a certain product, selling some and stocking the rest in a warehouse. At any instant of time $t \in [0, T]$, we denote by $I(t)$ the inventory level in the warehouse. The firm has set an inventory goal level (a target level) $\hat{I}$ and a penalty $h \geq 0$ is incurred for the inventory level to deviate from its goal. At any instant of time $t \in [0, T]$, the firm manufactures units of the product at a rate $P(t)$. It has set a production goal rate $\hat{P}$ and a penalty $K > 0$ is incurred for the production rate to deviate from its goal. The production of new units at rate $P(t)$ increases the inventory level while the demand for the product at rate $D(t)$ and deterioration at rate $\theta(t) = \alpha\beta t^{\beta-1}$ decreases the inventory level. The change in the level of inventory in stock is therefore given by the state equation

$$\dot{I}(t) = -\alpha\beta t^{\beta-1} I(t) + P(t) - D(t), \quad \forall t \in [0, T]. \tag{1}$$

We assume that the initial stock $I(0) = I_0$ is known and note that the production goal rate $\hat{P}$ can be computed using the state equation (1) as

$$\hat{P}(t) = D(t) + \alpha\beta t^{\beta-1} \hat{I}.$$

To present the problem as an optimal control problem, we let $I(t)$ represent the *state variable* and $P(t)$ represent the *control variable* which needs to be nonnegative:

$$P(t) \geq 0. \tag{2}$$

Now we look for the optimal production rate, that is the rate that minimizes the performance index

$$J = \frac{1}{2} \int_0^T \left\{ h \left[I(t) - \hat{I} \right]^2 + K \left[P(t) - \hat{P}(t) \right]^2 \right\} dt, \tag{3}$$

subject to constraints (1)-(2).

Now to solve this problem, it is well-known that, historically, there have been two basic inventory systems: the continuous-review system and the periodic-review system. With continuous-review systems, the level of a company's inventory is monitored at all times and the inventory position is constantly adjusted. Management of inventory is an ongoing process. Periodic-review systems, on the other hand, check the inventory level at fixed intervals rather than through continuous monitoring and adjust the inventory position at specific time intervals such as daily, weekly, biweekly, or monthly.

3 Continuous-Review Policy

We first assume that the firm adopts a continuous-review policy. The necessary optimality conditions are derived using Pontryagin maximum principle, see for example Sethi and Thompson [21].

3.1 Analytical Solution. Denoting the adjoint variable by λ, the Hamiltonian is given by

$$H = -\frac{1}{2} \left\{ h \left[I(t) - \hat{I} \right]^2 + K \left[P(t) - \hat{P}(t) \right]^2 \right\} + \lambda(t) \left[P(t) - D(t) - \alpha\beta t^{\beta-1} I(t) \right]. \tag{1}$$

The necessary optimality conditions

$$\frac{\partial H}{\partial P} = 0, \quad \frac{\partial H}{\partial I} = -\dot{\lambda}, \quad \frac{\partial H}{\partial \lambda} = \dot{I},$$

are respectively equivalent to

$$P(t) = \hat{P}(t) + \frac{\lambda(t)}{K}, \tag{2}$$

$$\dot{\lambda}(t) = h\left[I(t) - \hat{I}\right] + \lambda(t)\alpha\beta t^{\beta-1}, \tag{3}$$

and the state equation (1). Substituting expression (2) into the state equation (1) yields

$$\dot{I}(t) = -\alpha\beta t^{\beta-1}I(t) + \hat{P}(t) + \frac{\lambda(t)}{K} - D(t). \tag{4}$$

Note that from (4) we have

$$\frac{\lambda(t)}{K} = \dot{I}(t) + \alpha\beta t^{\beta-1}I(t) - \hat{P}(t) + D(t). \tag{5}$$

Also, differentiating (4), we get

$$\ddot{I}(t) = -\alpha\beta(\beta-1)t^{\beta-2}I(t) - \alpha\beta t^{\beta-1}\dot{I}(t) + \frac{\dot{\lambda}(t)}{K} + \dot{\hat{P}}(t) - \dot{D}(t). \tag{6}$$

Substituting expression (3) into (6) yields

$$\ddot{I}(t) = -\alpha\beta(\beta-1)t^{\beta-2}I(t) + \alpha\beta t^{\beta-1}\left[\frac{\lambda(t)}{K} - \dot{I}(t)\right] + \frac{h}{K}\left[I(t) - \hat{I}\right] + \dot{\hat{P}}(t) - \dot{D}(t). \tag{7}$$

Finally, substitute expression (5) into (7) to obtain

$$\ddot{I}(t) - \left[\frac{h}{K} + \alpha\beta(\alpha\beta - \beta + 1)t^{2(\beta-1)}\right]I(t) = \alpha\beta t^{\beta-1}\left[D(t) - \hat{P}(t)\right] - \frac{\lambda}{K}\hat{I} + \dot{\hat{P}}(t) - \dot{D}(t). \tag{8}$$

Together with the initial condition $I(0) = I_0$ and the terminal condition $\lambda(T) = 0$, this is a boundary value problem that is solved numerically since a closed form solution is not possible.

3.2 Numerical Example. To illustrate, we consider a numerical example where the planning horizon has length $T = 12$ months and the demand rate is a sinusoidal function of time given by $D(t) = 1 + \sin(t)$. The cost parameters are $h = 1$ and $K = 20$. The initial and goal inventory level are $I_0 = 2$ and $\hat{I} = 10$, respectively. Finally, the shape and scale parameters of the Weibull distribution for the deterioration rate are $\alpha = 0.5$ and $\beta = 3$, respectively. The second-order differential equation was solved numerically using version 7.0 of the mathematical package MATLAB. FIGURE 1 (left) shows the convergence of the optimal inventory towards inventory goal level. A similar convergence is observed in FIGURE 1 (right) for the optimal production rate toward the production goal rate. It is always worth investigating the sensitivity of the

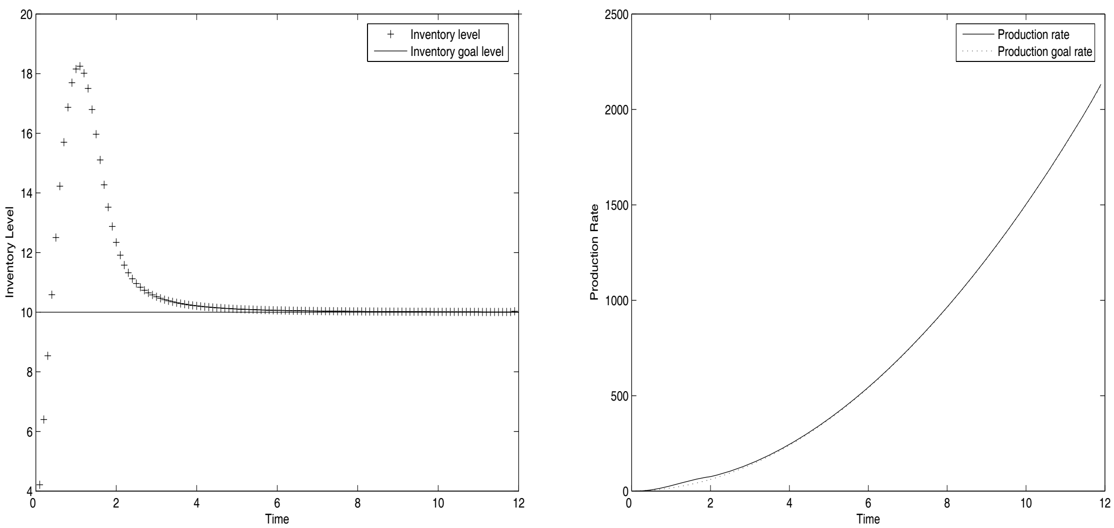

Figure 1: Optimal inventory level (left) and optimal production rate (right).

optimal solution to changes in the system parameters. In our case, the deterioration rate was our main interest and we successively varied the shape and scale parameters of the Weibull distribution and recorded the value of the optimal objective function value. FIGURE 2 shows that the objective function increases as either α or β increases. We also observe that the objective function is a concave function of α but a convex function of β.

4 Periodic-Review Policy

Now suppose the firm adopts a periodic-review policy. Divide the planning horizon $[0, T]$ into N subintervals of equal length and denote respectively by $I(k)$, $P(k)$, $\hat{P}(k)$ and $D(k)$ the inventory level, the production rate, the production goal rate, and the demand rate on each subinterval.

4.1 Model Discretization. We start by discretizing the model. The approximate discrete form of Equation (1) is

$$\frac{I(k+1) - I(k)}{T_s} = P(k) - D(k) - \alpha\beta k^{\beta-1} I(k), \tag{1}$$

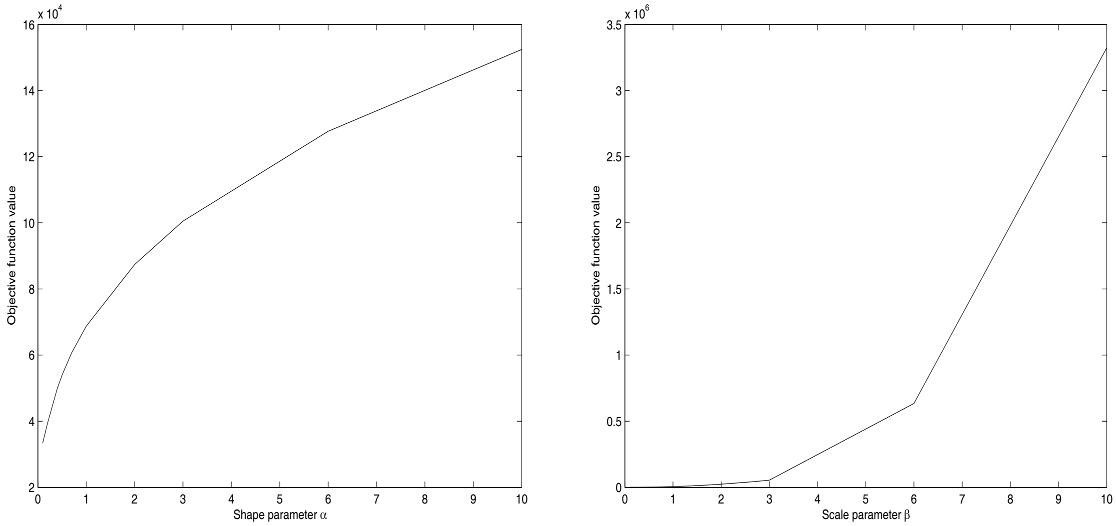

Figure 2: Effect of α (left) and β (right) on the optimal cost.

where T_s is the discretization sampling period. Rearranging the terms in (1) gives

$$I(k+1) = \left[1 - T_s \alpha \beta k^{\beta - 1}\right] I(k) + T_s \left[P(k) - D(k)\right]. \tag{2}$$

Since $\hat{I}$ and $\hat{P}$ satisfy (2), we have

$$\hat{I} = \left[1 - T_s \alpha \beta k^{\beta - 1}\right] \hat{I} + T_s \left[\hat{P}(k) - D(k)\right]. \tag{3}$$

Now introduce the shift operator Δ such that

$$\Delta I(k) = I(k) - \hat{I} \quad \text{and} \quad \Delta P(k) = P(k) - \hat{P}(k).$$

Subtracting expression (3) from (2) yields

$$\Delta I(k+1) = a(k) \Delta I(k) + T_s \Delta P(k), \tag{4}$$

where, to simplify this expression, we let

$$a(k) = 1 - T_s \alpha \beta k^{\beta - 1}.$$

In discrete time, the objective function is

$$J = \frac{1}{2} \sum_{0}^{N} \left[h \Delta I(k)^2 + K \Delta P(k)^2\right]. \tag{5}$$

Therefore, the problem is to determine the production rate $P(k) \geq 0$ that minimizes (5) subject to the constraint (4).

4.2 Analytical Solution. In this case, the problem is nonlinear and can be solved using the Lagrangian technique. To use the Lagrangian technique, introduce the Lagrange multipliers $\lambda(k)$ and the Lagrangian function

$$L = \frac{1}{2}\sum_{0}^{N}\left\{h\Delta I(k)^2 + K\Delta P(k)^2\right\} + \lambda(k+1)\Big[-\Delta I(k+1) + a(k)\Delta I(k) + T_s\Delta P(k)\Big]. \tag{6}$$

The necessary optimality conditions

$$\nabla_{\Delta P(k)}L = 0, \quad \nabla_{\Delta I(k)}L = 0, \quad \nabla_{\lambda(k+1)}L = 0,$$

are respectively equivalent to

$$\Delta P(k) = -\frac{T_s}{K}\lambda(k+1), \tag{7}$$

$$\lambda(k) = h\Delta I(k) + a(k)\lambda(k+1), \tag{8}$$

and the constraint (4). To solve these equations, we use the sweep method of Bryson and Ho [2]. For $k = 0, \cdots, N$, introduce the positive quantities $s(k)$ such that

$$\lambda(k) = s(k)\Delta I(k). \tag{9}$$

We start by determining these quantities. Substituting (9) into (7) yields

$$\Delta P(k) = -\frac{T_s}{K}s(k+1)\Delta I(k+1). \tag{10}$$

Substituting (4) into (10) yields

$$\Delta P(k) = -\frac{T_s}{K}s(k+1)\Big[a(k)\Delta I(k) + T_s\Delta P(k)\Big]. \tag{11}$$

We solve this equation for $\Delta P(k)$ to get

$$\Delta P(k) = -\frac{T_s a(k)s(k+1)}{K + T_s^2 s(k+1)}\Delta I(k). \tag{12}$$

Now, substitute (9) into (8)

$$s(k)\Delta I(k) = h\Delta I(k) + a(k)s(k+1)\Delta I(k+1). \tag{13}$$

Also, substitute (4) into (13)

$$s(k)\Delta I(k) = \Big[h + a(k)^2 s(k+1)\Big]\Delta I(k) + T_s a(k)s(k+1)\Delta P(k), \tag{14}$$

and (12) into (14)

$$s(k)\Delta I(k) = \left[h + \frac{Ka(k)^2 s(k+1)}{K + T_s^2 s(k+1)}\right] \Delta I(k). \tag{15}$$

Hence, we obtain the discrete Ricatti equation

$$s(k) = h + \frac{Ks(k+1)}{K + T_s^2 s(k+1)}\, a(k)^2, \tag{16}$$

which needs to be solved backwards, starting from

$$s(N) = h, \tag{17}$$

since $\Delta P(N) = 0$. Now, we turn to determining $I(k)$. First, substitute (12) into (4)

$$\Delta I(k+1) = a(k)\left[1 - \frac{T_s s(k+1)}{K + T_s^2 s(k+1)}\right] \Delta I(k). \tag{18}$$

Then, starting from $I(0) = I_0$, we compute recursively (forward)

$$I(k+1) = \hat{I} + a(k)\left[1 - \frac{T_s s(k+1)}{K + T_s^2 s(k+1)}\right]\left[I(k) - \hat{I}\right]. \tag{19}$$

To determine $P(k)$, again from (4) we have

$$\Delta P(k) = \frac{1}{T_s}\Big[\Delta I(k+1) - a(k)\Delta I(k)\Big], \tag{20}$$

and for $k = 0, \cdots, N-1$,

$$P(k) = \hat{P}(k) + \frac{1}{T_s}\Big[\Delta I(k+1) - a(k)\Delta I(k)\Big]. \tag{21}$$

Since only a nonnegative production rate is allowed, the optimal production rate is chosen equal to

$$\max\left\{\hat{P}(k) + \frac{1}{T_s}\Big[\Delta I(k+1) - a(k)\Delta I(k)\Big], 0\right\}, \quad k = 0, \cdots, N-1. \tag{22}$$

4.3 Numerical Example. For this numerical example, the planning horizon has length $N = 12$ and the demand rate is $D(t) = 1 + \sin(t)$. The inventory goal level and the initial inventory level are chosen to be $\hat{I} = 10$ and $I_0 = 2$. We also assume the following parameters for the deteriorations rate $\alpha = 0.1$ and $\beta = 3$ and the following penalty costs $h = 1$ and $K = 20$. The first step is to compute the production goal rate from (3) as

$$\hat{P}(k) = \max\left\{D(k) + \alpha\beta k^{\beta-1}\hat{I}, 0\right\}.$$

Next we successively compute the vector s from (16), the optimal inventory level I from (19), and the optimal production rate P from (22). FIGURE 3 (left) shows the optimal inventory level and as can be seen, I converges toward $\hat{I}$. It also shows (right) the optimal production rate P which, as can be seen, converges toward $\hat{P}$.

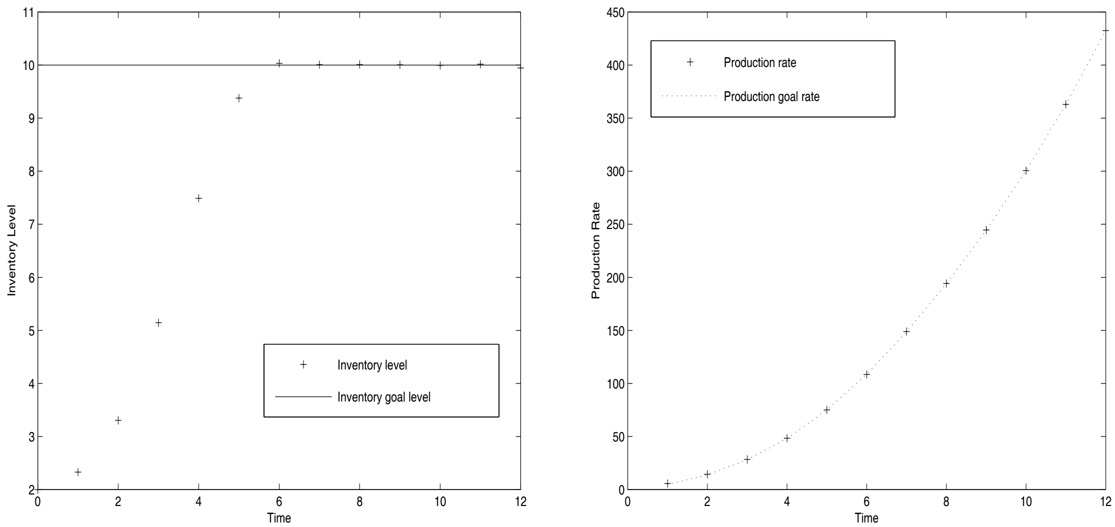

Figure 3: Optimal inventory level (left) and optimal production rate (right).

References

[1] G. Andujar and H. Herrera, The distribution of failure data for meat products, *Proceedings of the European Meeting of Meat Research Workers*, Helsinki, August 2-7 (1987), No. 33, Vol. **II. 8:14**, pp. 396-398.

[2] A.E. Bryson and Y.C. Ho, *Applied Optimal Control*, Washington D.C: Halsted Press (1975).

[3] C. Cardelli and T.P. Labuza, Application of Weibull analysis to the determination of the shelf life of roasted and ground coffee, *Lebensm.-Wiss u.-Technol.* **34** (2001) 273-278.

[4] T. Chakrabarty, B.C. Giri and K.S. Chauduri, An EOQ model for items with Weibull distribution deterioration, shortages and trended demand: an extension of Philip's model, *Computers and Operations Research* **25** (1998) 649-657.

[5] J.M. Chen and S.C. Lin, Optimal replenishment scheduling for inventory items with Weibull distributed deterioration and time-varying demand, *Information and Optimization Sciences* **24** (2003) 1-21.

[6] I. Dobos, Optimal Production-Inventory Strategies for a HMMS-Type Reverse Logistics System, *International Journal of Production Economics* **81-82** (2003) 351-360.

[7] W. Duyvesteyn, Integration of the time-temperature history effect on the shelf life of fluid milk, M.Sc. thesis, University of Minnesota, St. Paul (1997).

[8] M.C. Gacula, The design of experiments for shelf life study, *Journal of Food Science* **40** (1975) 400-403.

[9] S.K. Ghosh and K.S. Chaudhuri, An order-level inventory model for a deteriorating item with Weibull distribution deterioration, time-quadratic demand and shortages, *Advanced Modeling and Optimization* **6** (2004) 21-35.

[10] V.P. Goel and S.P. Aggarwal, Pricing and ordering policy with general Weibull rate of deteriorating inventory, *Indian Journal of Pure and Applied Mathematics* **11** (1980) 618-627.

[11] S.K. Goyal and B.C. Giri, Recent trends in modeling of deteriorating inventory, *European Journal of Operational Research* **134** (2001) 1-16.

[12] E. Khemlnitsky and Y. Gerchak, Optimal Control Approach to Production Systems with Inventory-Level-Dependent Demand, *IIE Transactions on Automatic Control* **47** (2002) 289-292.

[13] G.P. Kiesmüller, Optimal control of a one product recovery system with lead times, *International Journal of Production Economics* **81-82** (2003) 333-340.

[14] S.C. Pickering, Prediction and analysis of shelf life stability of an oat bran cereal with antioxidants, M.Sc. thesis, University of Minnesota, St. Paul (1984).

[15] N.K. Mahapatra and M. Maiti, Decision process for multiobjective, multi-item production-inventory system via interactive fuzzy satisficing technique, *Computers and Mathematics with Applications* **49** (2005) 805-821.

[16] B. Mondal, A.K. Bhunia, Ghosh and M. Maiti, An inventory system of ameliorating items for price dependent demand rate, *Computers and Industrial Engineering* **45** (2003) 443-456.

[17] C.E. Riddalls and S. Bennett, The optimal control of batched production and its effect on demand amplification, International Journal of Production Economics, 72 (2001) 159-168.

[18] Y. Salama, Optimal control of a simple manufacturing system with restarting costs, *Operations Research Letters* **26** (2000) 9-16.

[19] S. Sana and K.S. Chaudhuri, An EOQ model with time-dependent demand, inflation and money value for a warehouse enterpriser, *Advanced Modeling and Optimization* **5** (2003) 135-146.

[20] K. Schmidt and J. Bouma, Estimating shelf life of cottage cheese using hazard analysis, *Journal of Dairy Science* **75** (1992) 2922-2927.

[21] S.P. Sethi and G.L. Thompson, Optimal Control Theory: Applications to Management Science and Economics, 2nd ed., Dordrecht: Kluwer Academic Publishers (2000).

[22] I. Shirose, J.M. Aguirre, and V.L.P. Ferreira, Application of a staggered experimental design to study of the shelf-life of cassava flour enriched with soybean protein extraction residue, *Coletânea do Instituto de Tecnologia de Alimentos* **9** (1978) 313-334.

[23] S. Tang, D.M. TeKrony, D.B. Egli, and P.L. Cornelius, An alternative model to predict corn seed deterioration during storage, *Crop Science* **40** (2000) 463-470.

[24] M. Tomasicchio, R. Andreotti, P. Pirazzoli, and G. Pezzani, Application of the Weibull statistical model to the shelf-life of some frozen foods, *Industria Conserve* **64** (1989) 102-109.

[25] S.A. Wittinger and D.E. Smith, Effect of sweeteners and stabilizers on selected sensory attributes and shelf life of ice cream, *Journal of Food Science* **51** (1986) 1463-1466.

[26] J.W. Wu and W.C. Lee, An EOQ inventory model for items with Weibull distributed deterioration, shortages and time-varying demand, *Information and Optimization Sciences* **24** (2003) 103-122.

[27] Q. Zhang, G.G. Yin, and E.-K. Boukas, Optimal control of a marketing-production system, *IEEE Transactions on Automatic Control* **46** (2001) 416-427.

Received: December 23, 2006