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Fuzzy Linear Regression Models with Fuzzy Entropy

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Abstract

Fuzzy regression analysis using fuzzy linear models with symmetric triangular fuzzy number coefficient has been introduced by Tanaka et al. The goal of this regression is to find the coefficient of a proposed model for all given input-output data sets. In this paper, we propose a new

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method for computation of fuzzy regression. The method is constructed on the basis of minimizing the fuzzy entropy of predicted values. The advantage of the propose approach depend on the entropy's properties to rectify previous problems in fuzzy linear regression with crisp input and fuzzy output. To compare the performance of the proposed approach with the other methods, an example is presented.

Keywords: Fuzzy numbers; Fuzzy linear regression; Fuzzy entropy; Mathematical programming

1 Introduction

Fuzzy linear regression (FLR) was introduced by Tanaka et al. [19] to model situations in which the practitioner cannot accurately measure the dependent variable. Since the introduction of fuzzy linear regression, the literature dealing with fuzzy linear regression has grown rapidly. In general, there are two approaches in fuzzy regression analysis: linear programming-based method [6,10-12,16,18,19] and fuzzy least squares method [2,4,9,13].The first method is based on minimizing fuzziness as an optimal criterion. The second method used least-square of errors as a fitting criterion. The advantage of first approach is its simplicity in programming and computation, while that the degree of fuzziness between the observed and predicted values is minimized by using fuzzy least squares method. In the original fuzzy regression method [19] with fuzzy output, crisp input and fuzzy parameters, the objective function was minimized the sum of the widths of the fuzzy regression coefficients, subject to, in the constraints any Y_i must have at least h membership in the observed dependent variable fuzzy set $\tilde{Y}_i$, and must have at least h membership in the predicted fuzzy set $\hat{\tilde{Y}}_i$ for $i = 1, 2, ..., n$. Note that n is number of observations and h is the threshold value used to measure degree of fit. Tanaka, [15] modified the original fuzzy regression method such that the objective function was to minimize the sum of the widths of the predicted fuzzy numbers subject to original constraint. The aim of this paper is to introduce a new objective function to predict the parameters of a fuzzy linear regression model by applying entropy with the same constraints in [19] .This model improves the problems in previous methods because of the entropy's properties. The paper is organized as follows. In section 2, some elementary properties of fuzzy numbers and fuzzy linear regression are described. The propose method is presented in section 3. A numerical example is illustrated to compare the proposed model with previous ones, in section 4. conclusions are drawn in section 5.

2 Preliminaries

A fuzzy number $\tilde{A}$ is a convex normalized fuzzy subset of the real line R with an upper semi-continuous membership function of bounded support [4].

2.1 Definition

A symmetric fuzzy number $\tilde{A}$, denoted by $\tilde{A} = (a, c)_L$ with membership function $\mu_{\tilde{A}}(x)$ is defined as $\mu_{\tilde{A}}(x) = L((x-a)/c), \;\; c > 0$, Where a and c are the center and spread of $\tilde{A}$ and $L(x)$ is a shape function of fuzzy numbers such that:
$i)$ $L(x) = L(-x)$,
$ii)$ $L(0) = 1, L(1) = 0$,
$iii)$ L is strictly decreasing on $[0, \infty)$,
$iv)$ L is invertible on $[0, 1]$.

The set of all symmetric fuzzy numbers is denoted by $F_L(R)$. If $L(x) = 1 - |x|$, then the fuzzy number is symmetric triangular fuzzy number.

2.2 Definition

Suppose $\tilde{A} = (a, c)_L$ is a symmetric fuzzy number and $\lambda \in R$, then $\lambda\tilde{A} = (\lambda a, |\lambda| c)_L$. The formulation of fuzzy linear regression function has been introduced by Tanaka et al. [15, 18, 19]. There are n independent non-fuzzy variables $X_i, \quad i = 1, 2, ..., n$ while the dependent variable is a symmetric fuzzy number. For i-th observation, the vector of independent variable $X_i = (X_{i0}, X_{i1}, ..., X_{ip})^T, i = 1, ..., n, \;\; n \geqslant p+1,$, where $X_{i0} = 1$, we have a fuzzy number $\tilde{Y}_i = (\bar{y}_i, e_i)_L$. The objective is, to estimate a (FLR) model, is expressed as follows:

$$\hat{\tilde{Y}}_i = \tilde{A}_0 X_{i0} + \tilde{A}_1 X_{i1} + ... + \tilde{A}_p X_{ip} = \tilde{A} X_i, \;\; i = 1, 2, ..., n \qquad (1)$$

In (1), $\tilde{A} = (\tilde{A}_0, \tilde{A}_1, ..., \tilde{A}_p)$ is a vector of fuzzy parameters where $\tilde{A}_j = (a_j, c_j)_L$ is a symmetric fuzzy number for $j = 0, 1, ..., p$. From definition 2.2, we have $\hat{\tilde{Y}}_i = (ax_i, c|x_i|)_L$.

Theorem 2.1. Given fuzzy parameter $\tilde{A}_j = (a_j, c_j)_L$, the fuzzy linear function in (1) is obtained as the following membership:

$$\mu_{\tilde{\hat{Y}}} = \begin{cases} 1 - \frac{|y - x^t a|}{c^t |x|} & x \neq 0 \\ 1 & x = 0, y = 0 \\ 0 & x = 0, y \neq 0 \end{cases} \quad (2)$$

Where $|x| = (|x_1|, ..., |x_n|)^T$.

proof:[19].

Let $\tilde{Y}_i \in F_L(R)$ and $\mu_{\tilde{Y}_i}(y) \geqslant h, \;\; 0 \leqslant h \leqslant 1,$, where the h-level sets of $\tilde{Y}_i$ is $[\bar{y}_i - (1-h)e_i, \bar{y}_i + (1-h)e_i]$. Here, h represents the minimum degree of certainty acceptable and we will refer to this interval as $h-$certain observed interval. Similarly, the predicted interval corresponding to a specific set of x_i values $(x_{i0}, x_{i1}, ..., x_{ip})$ is as

$$[\sum_{j=0}^{p} (a_j - (1-h)c_j)|x_{ij}|, \;\; \sum_{j=0}^{p} (a_j + (1-h)c_j)|x_{ij}|].$$

We will refer to this as $h-$certain predicted interval.

Tanaka in [19], proposed the following linear programming (LP) formulation to estimate $\tilde{A}_j, \;\; j = 0, 1, ..., p,$

$$Minimize : \sum_{j=0}^{p} c_j \quad (3)$$

$$\sum_{j=0}^{p} (a_j + (1-h)c_j)|x_{ij}| \geqslant \bar{y}_i + (1-h)e_i, \;\; i = 1, 2, ..., n$$

$$s.t.: \quad \sum_{j=0}^{p}(a_j-(1-h)c_j)|x_{ij}| \leqslant \bar{y}_i-(1-h)e_i, \quad i=1,2,...,n$$

$$a_j \in R, \quad c_j \geqslant 0 \quad j=0,1,...,p.$$

Note that the above LP forces the h-certain predicted intervals to include the h-certain observed intervals. In this model, the constrains guarantee the support of the predicted values from the model (3) includes the support of the observed values. There have been a few criticisms of Tanaka et al.'s approach. One of shortcoming is, the solution is depended on x_j-scale and many c_js is vanished, [7]. To rectify this problem, sum of half-width of regression coefficients in (3) is replaced by sum of half-width of the predicted intervals, can be used as the objective function, [16],

$$Minimize: \quad \sum_{i=1}^{n}\sum_{j=0}^{p} c_j|x_{ij}| \tag{4}$$

Another problem is, if we replace x_i by $(x_j-\bar{x}_i)$, then the predicted function will be very different, [1]. Some articles have proposed major changes to Tanaka et al.'s approach. The authors of [14] and [17] suggested, first find the centers, a_j, then solve LP with these a_j's. In [6, 13, 16] the objective function was changed according to improve mentioned problems. In this paper, we propose a new objective function that is discussed in section 3 to rectify above problems.

3 The proposed approach

In this section, we propose the fuzzy entropy to measure the fuzziness of the predicted values as an objective function. In fuzzy entropy both of a_j and c_j are influence on the objective function to rectify both of Tanaka's model problems.The other property of entropy is, it isn't necessary the number of observation be more than the number of parameters, [5]. We need some definitions about fuzzy entropy to introduce our approach. Measure of fuzziness, in contrast to fuzzy measure, try to indicate the degree of fuzziness of a fuzzy set, [20]. After introducing Shannon entropy as a measure of information in

[14], De Luca and Termini in [3] consider a measure of fuzziness as a mapping d from the power set $P(X)$ to $[0, \infty]$ that satisfies a number of conditions. Let $\mu_{\tilde{A}}(x)$ be the membership function of the fuzzy set $\tilde{A}$, for $x \in X, X$ is finite, that have the following properties:

i) $d(\tilde{A}) = 0\ if\ is\ a\ crisp\ set\ in$ X
ii) $d(\tilde{A}) = 0$ *assumes a unique maximum if* $\mu_{\tilde{A}}(x) = \frac{1}{2}\ \forall x \in X.$
iii) $d(\tilde{A}) \geqslant d(\tilde{B})$ *if* $\tilde{B}$ *is crisper than* $\tilde{A}$, *i.e. if* $\mu_{\tilde{B}}(x) \leqslant \mu_{\tilde{A}}(x)$ *for* $\mu_{\tilde{B}}(x) \leqslant \frac{1}{2}$ *and* $\mu_{\tilde{B}}(x) \geqslant \mu_{\tilde{A}}(x)$ *for* $\mu_{\tilde{A}}(x) \geqslant \frac{1}{2}$.
iv) $d(\tilde{A}) = d(\tilde{A}^c)$ *where* $\tilde{A}^c$ *is the complement of* $\tilde{A}$, *i.e.,* $\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x)$.

Deluca and Termini in [3] suggested as a measure of fuzziness the "entropy" of a fuzzy set which that defined as follows:

3.1 Definition.

The entropy as a measure of fuzzy set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x))\}$ is defined as:

$$d(\tilde{A}) = H(\tilde{A}) + H(\tilde{A}^c) \tag{5}$$

$$H(\tilde{A}) = -K \sum_{i=1}^{n} \mu_{\tilde{A}}(x_i) \ln \mu_{\tilde{A}}(x_i) \quad \forall x \in X,$$

Where n is the number of elements in the support of $\tilde{A}$ and K is a positive constant. Using Shannon's function $S(x) = -x \ln x - (1 - x) \ln(1 - x)$, Deluca and Termini simplify the expression in definition 3.1 to arrive at the following definition.

3.2 Definition

The entropy d as a measure of fuzziness of a fuzzy set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x))\}$ is defined as
$d(\tilde{A}) = K \sum_{i=1}^{n} S(\mu_{\tilde{A}}(x_i))$.

The objective function of mathematical model: The objective function of the model is to minimize the fuzzy entropy of the predicted values, i.e., minimize the measure of fuzziness of $\hat{\tilde{Y}}_i, i = 1, 2, ..., n$.This can be achieved by minimizing.

$$-K(\sum_{i=1}^{n} \mu_{\hat{\tilde{Y}}_i}(y) \ln \mu_{\hat{\tilde{Y}}_i}(y) + \sum_{i=1}^{n}(1 - \mu_{\hat{\tilde{Y}}_i}(y)) \ln(1 - \mu_{\hat{\tilde{Y}}_i}(y)))$$

where $\mu_{\hat{\tilde{Y}}_i}(y)$ is defined by (2).

The constraints of mathematical model: The problem in the FLR model is to determine fuzzy parameters such that the support of the predicted values from the model includes the support of the observed values.

The resulting mathematical model is as follows:

Minimize

$$d(h) = -K[\sum_{i=1}^{n}(1 - \frac{|y_i - \sum_{j=0}^{p} a_j x_{ij}|}{\sum_{j=0}^{p} c_j |x_{ij}|}) \ln(1 - \frac{|y_i - \sum_{j=0}^{p} a_j x_{ij}|}{\sum_{j=0}^{p} c_j |x_{ij}|}) \tag{6}$$

$$(\frac{|y_i - \sum_{j=0}^{p} a_j x_{ij}|}{\sum_{j=0}^{p} c_j |x_{ij}|}) \ln(\frac{|y_i - \sum_{j=0}^{p} a_j x_{ij}|}{\sum_{j=0}^{p} c_j |x_{ij}|})$$

Subject to: The constraints of model (3).

The model is a mathematical programming model and can be solved by the existing softwares. In the constraints, we must evaluate "h" $(0 \leqslant h \leqslant 1)$ to get h-certain interval. For determining the best "h" we can use the algorithm that suggested by [9]. Since $d(h)$ is non-decreasing, we have following theorem.
Theorem 3.1. In problem (6), if $(h_1 \geqslant h_2)$ then $d(h_1) \geqslant d(h_2)$.

4 Numerical example

In this section, we use an example to compare the proposed method with Tanaka's model. In this example, the independent variable is crisp and the dependent variable is symmetric triangular fuzzy number.

4.1 Example

Tanaka et al. in [16] designed an example to illustrate their regression model. In this example, there are five pairs of $(x_i, (\bar{y}_i, e_i))$ observations as shown in Table 1. By using the proposed method, the FLR model is constructed as:

$$\hat{\hat{Y}}_i = 0.29 + (5.37, 4.79)_L x_i.$$

The predicted values by Tanaka [16] and proposed method were compared with observed values is in Table 1.

Observed values			Predicted values	
i	x_i	$(\bar{y}_i, e_i)$	Tanaka et al.	Proposed method
1	1	(8.0,18)	(5.95,3.85)	(5.66,4.79)
2	2	(6.4,2.2)	(8.05,3.85)	(11.03,9.58)
3	3	(9.5,2.6)	(10.15,3.85)	(16.40,14.37)
4	4	(13.5,2.6)	(12.25,3.85)	(21.77,19.16)
5	5	(13.0,2.4)	(14.35,3.85)	(27.14,23.95)

Table 1: Numerical data and predicted values for example 4.1($h = 0$)

5 Conclusion

In this paper, we proposed a new objective function to estimate the FLR parameters. The advantage of this model is based on; both of center and spread of fuzzy numbers are influence on estimations and it is useful for outliers and influence points in fuzzy regression. Also, this model is useful when the size of samples are not enough.

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