The Laplace Order and Ordering

of Reversed Residual Life

I. Elbatal

King Saud University, College of Science
Dept. of Statistics and Operation Research
P.O. Box 2455, Riyadh 11451, Saudi Arabia
ielbatal@ksu.edu.sa

Abstract

The purpose of this paper is to study new stochastic comparisons based on the Laplace transform order of reversed residual life. Some basic properties of the new order are given. We also provide several preservation properties of it under the reliability operations of convolution, mixture and parallel system.

1 Introduction

Suppose X is a non-negative random variable having absolutely continuous distribution $F_X(x)$ with density $f_X(x)$. Then the ordinary Laplace transform of the density function f_X is given by

$$L_X(s) = \int_0^\infty e^{-su} \, f_X(u)d(u), \quad s > 0.$$

The Laplace transform of F_X is defined by

$$L_X^*(s) = \int_0^\infty e^{-su} \, F_X(u)du.$$

It is easy to check that if F_X is continuous then

$$L_X^*(s) = \frac{1}{s}L_X(s), \quad s \geq 0.$$

Throughout this paper X and Y are two random variables with distribution functions F_X and F_Y, respectively. Denote by L_X the Laplace transform of F_X, and by $\overline{F}_X = 1 - F_X$ the corresponding survival function. We use a similar notation for all other distribution functions. Moreover, we will use the term *increasing* in place of *non-decreasing*, and *decreasing* in place of *non-increasing.*

Given two random variables X and Y, X is said to be smaller than Y in the *Laplace transform order* (denoted by $X \leq_{Lt} Y$) if $L_X(s) \geq L_Y(s)$, for all $s > 0$. Clearly, $X \leq_{Lt} Y \Leftrightarrow L_X^*(s) \geq L_Y^*(s)$, for all $s > 0$ if both F_X and F_Y are continuous.

Also, it can be checked that $M \leq_{Lt} N$ is equivalent to $L_M^*(s) \geq L_N^*(s)$ for all $s > 0$, or

$$M \leq_{Lt} N \Leftrightarrow \sum_{n=0}^{\infty} \epsilon^n Q_M(n) \geq \sum_{n=0}^{\infty} \epsilon^n Q_N(n), \quad \epsilon \in (0,1), \qquad (1.1)$$

if M and N are non-negative integer-valued random variables with distribution probabilities $Q_M(n) = P(M < n)$ and $Q_N(n) = P(N < n)$, respectively.

The Laplace transform can be interpreted in several ways when the random variable represents the lifetime of a system or a unit, which yields several applications of Laplace transform order. Applications, properties and interpretations of the *Laplace transform order* can be found in *Alzaid et al.* (1991), *Klefsjo* (1983), *Shaked* and *Wong* (1997) *Denuit* (2001) and *Belzunce et al.* (1999).

For any random variable X, let

$$X_{(t)} = [t - X \mid X < t], \quad t \in \{x : F_X(x) < 1\},$$

denote a random variable whose distribution is the same as the conditional distribution of $t - X$ given that $X < t$. When the random variable X denote the life time ($X \geq 0$, with probability one) of a unit, $X_{(t)}$ is known as reversed residual life (or time since failure or inactivity time, or idle time), (see, for

instance, *Chandra* and *Roy* (2001), *Block et al.* (1998), *Nanda et al.* (2003), *Kayid* and *Ahmad* (2004), *Ahmad et al.* (2005)).

In this case given a random variable $X_{(t)}$, then

$$L^*_{X_{(t)}}(s) = \int_0^t \frac{e^{su} F_X(u) du}{e^{st} \ F_X(t)}, \quad s > 0.$$

In this paper, based on this measure we will deal with a new ordering called Laplace transform ordering of reversed residual life. In *Section* 2, we present definitions, notations and basic properties used throughout the paper. Several preservation properties of stochastic comparisons and some aging notions based on the Laplace transform order of reversed residual lives under the reliability operations of mixture, convolution and parallel system are studied in *Section* 3. Finally, in *Section* 4 we provide a preservation property under monotone transformation of the new order.

2   Definitions, notations and characterization

In the context of reliability theory several orders have been introduced to compare two lifetime distributions. These orders have been found useful for modelling, or the design of better systems. We see below some of these orders (see, *Shaked* and *Shanthikumar* (1994), *Nanda et al.* (2003) and *Kayid and Ahmad* (2004)).

Given a random variable X, with distribution function F_X, $X_t = (X - t \mid X \geq t)$, denotes the additional residual life where $t \in (0, l_X)$ and $l_X = \sup\{t : F_X(t) < 1\}$. Based on the stochastic comparison of X_t, the hazard rate order $(X \leq_{HR} Y)$ can be defined as

$$X_t \leq_{ST} Y_t, \quad \text{for all } t,$$

where the stochastic ordering $(\leq_{ST})$ is defined as

$$X \leq_{ST} Y \ \Leftrightarrow \ \overline{F}_X(t) \leq \overline{F}_Y(t), \ \text{ for all } t.$$

Definition 2.1. Let X and Y be two non-negative random variables. The random variable X is said to be smaller than Y in the

(i) mean residual life order (denoted by $X \leq_{MRL} Y$) if, and only if,

$$\frac{\int_t^\infty \overline{F}_Y(x)dx}{\int_t^\infty \overline{F}_X(x)dx} \text{ increasing in } t \text{ for all } t \in (0, l_X) \cap (0, l_Y);$$

(ii) reversed mean residual life order (denoted by $X \leq_{RMRL} Y$) if, and only if,

$$\frac{\int_0^t F_Y(x)dx}{\int_0^t F_X(x)dx} \text{ increasing in } t \text{ for all } t \in R^+.$$

(iii) increasing concave order (denoted by $X \leq_{icv} Y$) if, and only if,

$$\int_0^x \overline{F}_X(u)du \leq \int_0^x \overline{F}_Y(u)du, \quad \text{for all } x.$$

Let us observe that the $\leq_{MRL}$ order are more informative than the $\leq_{ST}$ order, since it compare the underlying systems at any time t in contrast to the global comparison offered by the orders $\leq_{ST}$. However, it is reasonable to presume that in many realistic situations the random variable is not necessarily related to the future but can also refer to the past. For instance, consider a system whose state is observed only at certain preassigned inspection times. If at time t the system is inspected for the first time and it is found to be 'down', then the failure relies on the past, i.e. on which instant in $(0, t)$ it has failed. It thus seems natural to introduce a notion that is dual to the Laplace transform order of residual lives $\leq_{Lt-rl}$, introduced by *Belzunce et al.* (1999) and furthered by several authors, including *Gao et al.*(2003), *Kayid* and *Ahmad* (2004), in the sense that it refers to past time and not to future time.

Following this idea we introduc a new partial order based on the Laplace transform order of reversed residual life.

Definition 2.2. Let X and Y be two non-negative random variables. The random variable X is said to be smaller than Y in the *Laplace transform order of reversed residual lives* (denoted by $X \leq_{LT-RRL} Y$) if

$$X_{(t)} \geq_{Lt} Y_{(t)} \quad \text{for all } t \in R^+,$$

Observe that, by definition of $\leq_{LT-RRL}$ order, it holds $X \leq_{LT-RRL} Y$, if and only if $L^*_{X_{(t)}}(s) \geq L^*_{Y_{(t)}}(s)$ for all $t, s \geq 0$. Actually, an equivalent condition for $LT - RRL$ order is given in *Ahmad* and *Kayid* (2005), and is the following.

Proposition 2.1. Let X and Y be two continuous non-negative random variables with distribution functions F and G, respectively, then for all $t \geq 0$ and $s > 0$

$$X \leq_{LT-RRL} Y \iff \frac{\int_0^t e^{su} F(u) du}{\int_0^t e^{su} G(u) du} \text{ is decreasing in } t \geq 0. \quad (2.1)$$

On the other hand, in the literature, many non-parametric aging classes of distributions have been defined (cf. *Barlow* and *Proschan* (1975)). In particular, decreasing reversed hazard rate ($DRHR$) and increasing reversed mean residual life ($IRMR$) classes of distributions has been studied by many researchers in the recent past

The following two propositions gives most characterizations of $DRHR$ and $IRMR$ classes.

Proposition 2.2. For $0 < t_1 < t_2$, the following statements are equivalent.

(a) X has decreasing reversed hazard rate;

(b) $X_{(t_1)}$ is smaller than $X_{(t_2)}$ in stochastic ordering;

(c) $X_{(t_1)}$ is smaller than $X_{(t_2)}$ in hazard rate ordering;

(d) the distribution function of X is logconcave.

Proposition 2.3. The random variable X is $IRMR$ if, and only if, any one of the following conditions holds:

(i) $[t - X \mid X < t] \leq_{RMRL} [s - X \mid X < s]$ whenever $t \leq s$.

(ii) $X \leq_{RMR} [t - X \mid < t]$ for all $t > 0$ (when X is a non-positive random variable).

(iii) $X + t \leq_{RMR} X + s$ whenever $t \leq s$.

Next we introduce a new aging class based on the Laplace transform order

Definition 2.3. A non-negative random variable X is said to have increasing reversed residual lives in the Laplace order, denoted by $X \in IRRL_{Lt}$ if

$$X_{(t)} \leq_{Lt} X_{(s)}, \quad \text{for all } s > t \geq 0.$$

The following implications are well known or easy to prove as indicated below:

$$X \leq_{RHR} Y \Rightarrow X \leq_{LT-RRL} Y \Rightarrow X \leq_{RMRL} Y, \qquad (2.2)$$

and

$$DRFR \Rightarrow IRRL_{LT} \Rightarrow IRMR.$$

The implications $X \leq_{RHR} Y \Rightarrow X \leq_{LT-RRL} Y$ follows from *Theorem 3.B.6* of *Shaked* and *Shanthikumar* (1994) and $X \leq_{LT-RRL} Y \Rightarrow X \leq_{RMRL} Y$ follows from (3.B.2) of *Shaked* and *Shanthikumar* (1994) and from *Kayid* and *Ahmad* (2004).

From the definition it could be thought that the new order is equivalent to one of the orders ($\leq_{LT-R}$ and $\leq_{R-LT-R}$) introduced by *Shaked* and *Wong* (1997). However this is not true in general. Recall that $\leq_{RH} \nrightarrow \leq_{R-LT-R}$ and $\leq_{LT-R} \nrightarrow \leq_{ICV}$ (see *Shaked* and *Wong*, 1997), therefore from (2.2) and from $\leq_{RMRL} \longrightarrow \leq_{ICV}$ (see *Kayid* and *Ahmad*, 2004) neither $\leq_{LT-R}$ nor $\leq_{R-LT-R}$ is equivalent to $\leq_{LT-RRL}$.

One may refer to *Shaked* and *Shanthikumar* (1994), and *Muller* and *Stoyan* (2002) for the hazard rate order ($\leq_{HR}$), the reversed hazard rate order ($\leq_{RHR}$), the mean residual life order ($\leq_{MRL}$), and other common used stochastic orders. Also, one can refer to, *Nanda et al.* (2003) for the reversed mean residual life order ($\leq_{RMRL}$) and increasing reversed mean residual life aging class ($IRMR$) and *Shaked* and *Shanthikumar* (1994) for the decreasing reversed failure rate aging class ($DRFR$) and other aging notions.

3   Preservation properties of the LT-RRL order

In this section, we describe some basic properties of the $LT-RRL$ order, which will be used frequently in the sequel, and then give preservation properties of this order under the operations of mixture, convolution, monotone transformation and series system.

3.1 Basic properties

From (1.1) and *Definition* 2.2, we state the next proposition without proof. Its proof follows a similar argument to that of *Proposition* 2.1.of *Ahmad* and *Kayid* (2005).

Proposition 3.1. Let M and N be two nonnegative integer-valued random variables with distribution probabilities $Q_M(n)$ and $Q_N(n)$, respectively. Then

$$\frac{\sum_{k=0}^{k=n} \epsilon^k Q_N(k-1)}{\sum_{k=0}^{k=n} \epsilon^k Q_M(k-1)} \quad \text{is increasing in } n \in \{0, 1, ...\} \text{ for each } \epsilon \in (0, 1).$$

Observe that

$$X \in IRRL_{Lt} \quad \Leftrightarrow \quad X_{(t)} \leq_{Lt-RRL} X_{(t')} \quad \text{for all } t' > t \geq 0.$$

From (2.1), we easily obtain the following useful proposition.

Proposition 3.2. Let X be a non-negative continuous random variable with distribution function F, and denote

$$\phi(t) = \int_{-\infty}^{t} e^{su} F(u) \ du \quad \text{for all } t.$$

Then

$$X \in IRRL_{LT} \ \Leftrightarrow \ \phi(t) \text{ is logconcave in } t \in R^+ \text{ for all } s \geq 0.$$

Special attention should be paid to the logconcavity and logconvexity of a non-negative function ζ. The logconcavity of $\zeta(t)$ in $t \in R$ is equivalent to that $\zeta(x - y)$ is TP_2 in $(x, y) \in R^2$, whereas the logconvexity of $\zeta(t)$ in $t \in R$ dose not in general imply that $\zeta(x + y)$ is TP_2 in $(x, y) \in R^2$. The definition of TP_2 is given in the paragraph before *Theorem* 3.1. However, if $\zeta(t)$ is logconvex [logconcave] in $t \in R^+$, then $\zeta(x + y)$ [$\zeta(x + 1/y]$ is TP_2 in $(x, y) \in R^2$. These facts will be used in the sequel.

Proposition 3.3. Let X be a non-negative random variable. Then for any non-negative random variable Y independent of X

$$X \in IRRL_{LT} \ \Rightarrow \ X \leq_{LT-RRL} X + Y.$$

Proof. Suppose that $X \in IRRL_{LT}$ and Y is any non-negative random variable, and fix $s \geq 0$. Let F_W denote the distribution function of a random variable W, and denote

$$\phi_W(t) = \int_{-\infty}^{t} e^{su} F_W(u) du, \quad \text{for all } t \in R^+.$$

Then it follows from *Proposition* 3.2 that $\phi_X(t)$ is log-concave in $t \in R$; that is $\phi(t-v)/\phi_X(t)$ is increasing in $t \in R$ for each $v \geq 0$. Also, by Fubini's Theorem, we have

$$
\begin{aligned}
\phi_{X+Y}(t) &= \int_{-\infty}^{t} e^{su} \left(\int_{0}^{\infty} F_X(u-v) dF_Y(v) \right) du \\
&= \int_{0}^{\infty} \left(\int_{-\infty}^{t} e^{su} F_X(u-v) du \right) dF_Y(v) \\
&= \int_{0}^{\infty} \phi_X(t-v) \, e^{sv} \, dF_Y(v).
\end{aligned}
$$

Therefore,

$$\frac{\phi_{X+Y}(t)}{\phi_X(t)} = \int_{0}^{\infty} e^{sv} \frac{\phi_X(t-v)}{\phi_X(t)} dv \quad \text{is increasing in } t \in R.$$

Thus, by *Proposition* 2.1, we get $X \leq_{LT-RRL} X + Y$.

3.2 Mixture and convolution

The following result shows that the $(LT - RRL)$ order is preserved under convolutions, when appropriate assumptions are satisfied.

Theorem 3.1. Let X_1, X_2 and Y be three non-negative random variables, where Y is independent of both X_1 and X_2, and let Y have density g. If $X_1 \leq_{LT-RRL} X_2$ and g is log-concave then $X_1 + Y \leq_{LT-RRL} X_2 + Y$.

Proof. First we note that, for fixed $s \geq 0$ and $i = 1, 2$,

$$
\begin{aligned}
\Phi(i,t) &= \int_{0}^{\infty} e^{-sv} F_{X_i+Y}(t-v) dv \\
&= \int_{0}^{\infty} e^{-sv} \int_{0}^{\infty} F_i(t-v-u) dF_Y(u) dv
\end{aligned}
$$

$$= \int_0^\infty e^{-sv} \int_0^t F_i(z-v) \, f_Y(t-z) \, dzdv$$

$$= \int_0^\infty f_Y(t-z) \int_0^t e^{-sv} F_i(z-v) dvdz$$

$$= \int_{-\infty}^t f_Y(t-z) \, \psi(i,z) \, dz.$$

As shown in *Proposition* 2.1, the assertion follows if we prove that $\Phi(i,t)$ is TP_2 in (i,t) (see *Joag-Dev et al.*, 1995). By the assumption $X_1 \leq_{LT-RRL} Y_1$ we can say that $\psi(i,z)$ is TP_2 in (i,z). Moreover, since Y has logconcave density, $f_Y(t-z)$ is TP_2 in (t,z). Therefore by the basic composition formula (*Karlin*, 1968) it follows that $\Phi(i,t)$ is TP_2 in (i,t). This completes the proof.

Corollary 3.1. If $X_1 \leq_{LT-RRL} Y_2$ and $X_2 \leq_{LT-RRL} Y_2$ where X_1 is independent of X_2 and Y_1 is independent of Y_2, then the following statements hold:

(*i*) If X_1 and Y_2 have log-concave densities, then $X_1 + X_2 \leq_{LT-RRL} Y_1 + Y_2$.

(*ii*) If X_2 and Y_1 have log-concave densities, then $X_1 + X_2 \leq_{LT-RRL} Y_1 + Y_2$.

Proof. The following chain of inequalities, which establish (*i*), follows by *Theorem* 3.1:

$$X_1 + X_2 \leq_{LT-RRL} X_1 + Y_2 \leq_{LT-RRL} Y_1 + Y_2.$$

The proof of (*ii*) is similar.

Repeated application of *Theorem* 3.1, using the closure property of log-concaves under convolution, yields the following result.

Theorem 3.2. If X_1, X_2, ... and Y_1, Y_2, ... are sequences of independent random variables with $X_i \leq_{LR-RRL} Y_i$ and X_i, Y_i have log-concave densities for all i, then

$$\sum_{i=1}^n X_i \leq_{LR-RRL} \sum_{i=1}^n Y_i, \quad (n = 1, 2, ...).$$

Proof. We shall prove the theorem by induction. Clearly, the result is true for $n = 1$. Assume that the result is true for $p = n - 1$, i.e.,

$$\sum_{i=1}^{n-1} X_i \leq_{LR-RRL} \sum_{i=1}^{n-1} Y_i. \tag{4.1}$$

Note that each of the two sides of (4.1) has a log-concave density (see, e.g., *Karlin*, 1968, page128). Appealing to *Corollary 4.1*, the result follows.

Let now $X(\theta)$ be a random variable having distribution function F_θ, and let Θ_i be a random variable having distribution G_i, for $i = 1, 2$, and support R^+. The following is a closure of $LT - RRL$ order under mixture.

Theorem 3.3. Let $X(\theta), \theta \in R^+$ be a family of random variables independent of Θ_1 and Θ_2. If $\Theta_1 \leq_{lr} \Theta_2$ and if $X(\theta_1) \leq_{LT-RRL} X(\theta_2)$ whenever $\theta_1 \leq \theta_2$, then $X(\Theta_1) \leq_{LT-RRL} X(\Theta_2)$

Proof. Let F_i be the distribution function of $X(\Theta_i)$, with $i = 1, 2$. We known that

$$F_i(x) = \int_0^\infty F_\theta(x) dG_i(\theta).$$

Again, because of *Proposition* 2.1, we should prove that $\Phi(i, t) = \int_0^\infty e^{-sx} F_i(t - x) dx$ is TP_2 in (i, t). But actually

$$\begin{aligned}
\Phi(i, t) &= \int_0^\infty e^{-sx} F_i(t - x) dx \\[2mm]
&= \int_0^\infty e^{-sx} \int_0^\infty F_\theta(t - x) dG_i(\theta) dx \\[2mm]
&= \int_0^\infty g_i(\theta) \int_0^\infty e^{-sx} F_\theta(t - x) \, dx d(\theta) \\[2mm]
&= \int_0^\infty g_i(\theta) \, \psi(\theta, t) d\theta.
\end{aligned}$$

By assumption $X(\theta_1) \leq_{LT-RRL} X(\theta_2)$ whenever $\theta_1 \leq \theta_2$, we have that $\psi(\theta, t)$ is TP_2 in (θ, t), while from assumption $\Theta_1 \leq_{lr} \Theta_2$ follows that $g_i(\theta)$ is TP_2 in (i, θ). Thus again the assertion follows from the basic composition formula.

Suppose that X_i, $i = 1, ..., n$ be a collection of independent random variables. Suppose that F_i is the distributions functions of X_i. Let $\underline{\alpha} = (\alpha_1, ..., \alpha_n)$ and $\underline{\beta} = (\beta_1, ..., \beta_n)$ be two probability vectors. Let X and Y be two random variables having the respective distribution functions F and G defined by

$$F(x) = \sum_{i=1}^n \alpha_i F_i(x) \quad \text{and} \quad G(x) = \sum_{i=1}^n \beta_i F_i(x). \tag{4.2}$$

The following result gives conditions under which X and Y are comparable with respect to the $LT - RRL$ order. One may refer to *Ahmed* and *Kayid* (2004) and *Kayid* and *Ahmad* (2004) for a similar preservation property of the mean residual life order ($\leq_{MRL}$), the Laplace transform of residual life order ($\leq_{LT-RL}$) and mean inactivity time order ($\leq_{MIT}$), respectively. Definition, properties and applications of $\leq_{MRL}$ order and $\leq_{LT-RL}$ order can be found, for instance, in *Shaked* and *Shanthikumar* (1994), *Belzunce et al.* (1999) and *Gao et al.* (2003).

Corollary 3.2. Let $X_1, ..., X_n$ be a collection of independent random variables with corresponding distribution functions $F_1, ..., F_n$, such that $X_1 \leq_{LT-RRL} X_2 \leq_{LT-RRL} ... \leq_{LT-RRL} X_n$ and let $\underline{\alpha} = (\alpha_1, ..., \alpha_n)$ and $\underline{\beta} = (\beta_1, ..., \beta_n)$ such that $\underline{\alpha} \leq_{DLR} \underline{\beta}$. Let X and Y have distribution functions F and G defined in (4.2). Then $X \leq_{LT-RRL} Y$.

To demonstrate the usefulness of the above results in recognizing $(LT - RRL)$-ordered random variables, we consider the following examples.

Example 3.1. Let $X_{\underline{\lambda}}$ denote the convolution of n exponential distributions with parameters $\lambda_1, ..., \lambda_n$ respectively. Assume without loss of generality that $\lambda_1 \leq ... \leq \lambda_n$. Since exponential densities are log-concave, *Theorem* 3.3 implies that $X_{\underline{\lambda}} \leq_{LT-RRL} Y_{\underline{\mu}}$ whenever $\lambda_i \geq \mu_i$ for $i = 1, ..., n$.

Example 3.2. Let $X_i \sim Exp(\lambda_i)$, $i = 1, ..., n$ be independent random variables. Let X and Y be $\underline{\alpha}$ and $\underline{\beta}$ mixtures of X_i's. An application of *Theorem* 4.3, immediately $X \leq_{LT-RRL} Y$ for every two probability vector $\underline{\alpha}$ and $\underline{\beta}$ such that $\underline{\alpha} \leq_{DLR} \underline{\beta}$.

Another application of *Theorem* 3.3 is contained in following example.

Example 3.3. Let $X_{\underline{\lambda}}$ and $X_{\underline{\mu}}$ be as given in *Example* 3.1. For $0 \leq q \leq p \leq 1$ and $p + q = 1$, we have

$$pX_{\underline{\lambda}} + qX_{\underline{\mu}} \leq_{LT-RRL} qX_{\underline{\lambda}} + pX_{\underline{\mu}}.$$

3.3 Parallel system

Preservation properties of an order under parallel and/or series systems are of importance in reliability theory. The next result gives conditions under

which the $Lt - RRL$ order is closed under parallel systems. The conditions here are the same as those given in Theorem 5.3(ii) of *Alzaid* et al. (1991) (see also *Theorem* 3.B.7(b) of *Shaked* and *Shanthikumar* (1994)) for the Laplace transform order.

We first recall the definition of completely monotone ($c.m$) functions. A function $h : R_+ \to R$ is said to be completely monotone if all its derivatives $h^{(n)}$ exist and satisfy $h^{(0)} \equiv h \geq 0$, $h^{(1)} \leq 0$, $h^{(2)} \geq 0, ...$ More formally, h is $c.m$. if $(-1)^n h^{(n)}(x) \geq 0$ for all $x > 0$ and $n = 0, 1, 2, ...$.

Theorem 3.4. Let the independent non-negative random variables $X_1, X_2,...,X_n$, $Y_1, Y_2, ..., Y_n$ have the distribution functions $F_1, F_2, ..., F_n$, $G_1, G_2, ..., G_n$, respectively. If $X_i \leq_{LT-RRL} Y_i$, $i = 1, 2, ..., n$, and F_i and G_i are completely monotone then

$$\max\{X_1, ..., X_n\} \leq_{LT-RRL} \max\{Y_1, ..., Y_n\}.$$

Proof. Denote $T_n = \max\{X_1, ..., X_n\}$ and $W_n = \max\{Y_1, ..., Y_n\}$. Note that, for the maximum, it holds

$$[t - T_n \mid T_n < t] = \max\{[t - X_i \mid X_i < t], \ i = 1, ..., n\},$$

and

$$[t - W_n \mid W_n < t] = \max\{[t - Y_i \mid Y_i < t], \ i = 1, ..., n\},$$

for all $t \geq 0$. Let $F_{i|t}$ and $G_{i|t}$ are $c.m$. if F_i and G_i are $c.m$.

Fix $t \geq 0$. By the assumption $X_i \leq_{LT-RRL} Y_i$, we have

$$[t - X_i \mid X_i < t] \leq_{LT} [t - Y_i \mid Y_i < t].$$

Then, by *Theorem* 3.B.7(b) of *Shaked* and *Shanthikumar* (1994), we have

$$\max\{[t - X_i \mid X_i < t], \ i = 1, ..., n\} \leq_{LT} \max\{[t - Y_i \mid Y_i < t], \ i = 1, ..., n\}.$$

Thus,

$$[t - T_n \mid T_n < t] \leq_{LT} [t - W_n \mid W_n < t] \quad \text{for all } t \geq 0,$$

implying that $T_n \leq_{LT-RRL} W_n$. This completes the proof.

Unfortunately, the proof above does not hold for more general coherent systems.

4 Monotone transformation

We first recall the definition of completely monotone (*c.m.*) functions. A function $h : R_+ \to R$ is said to be completely monotone if all its derivatives $h^{(n)}$ exist and satisfy $h^{(0)} \equiv h \geq 0$, $h^{(1)} \leq 0$, $h^{(2)} \geq 0, ...$ More formally, h is *c.m.* if $(-1)^n h^{(n)}(x) \geq 0$ for all $x > 0$ and $n = 0, 1, 2, ...$.

The next theorem states that the $Lt - RRL$ order is preserved under *c.m.* transformations. One may refer to *Alzaid et al.* (1991) for a similar preservation property of the $\leq_{LT}$ order.

Theorem 4.1. Let X and Y be two nonnegative random variables. Then $X \leq_{LT-RRL} Y$ if and only if $h(X) \leq_{LT-RRL} h(y)$ for all non-negative function h with a *c.m.* derivative.

Proof. We give the proof of the necessity only. Let h be any nonnegative function with a *c.m.* derivative, and suppose that $X \leq_{LT-RRL} Y$. We have to prove that

$$[h(X) \mid h(X) < t] \geq_{Lt} [h(Y) \mid h(Y) < t] \quad \text{for all } t \in R^+. \tag{4.1}$$

From the assumption, it follows that $X_{h^{-1}(t)} \geq_{LT} Y_{h^{-1}(t)}$ or, equivalently,

$$\left[X \mid X < h^{-1}(t)\right] \geq_{LT} \left[Y \mid Y < h^{-1}(t)\right] \quad \text{for all } t \in R^+. \tag{4.2}$$

Here the inverse h^{-1} of h is taken to be the right continuous version of it defined by $h^{-1}(u) = sup\{x : h(x) \leq u\}$ for $u \in R..$ From the definition of h^{-1} and the continuity of h, it is easy to check that $x < h^{-1}(t)$ if and only if $h(x) < t$. Thus (4.2) can be rewritten as

$$[X \mid h(X) < t] \geq_{LT} [Y \mid h(Y) < t] \quad \text{for each } t \in R^+.$$

By *Corollary* 3.2 of *Alzaid et al.* (1991), we get that

$$h\left([X \mid h(X) < t]\right) \geq_{LT} h\left([Y \mid h(Y) < t]\right) \quad \text{for each } t \in R^+.$$

implying (4.1). This completes the proof.

The following theorem characterizes the $LT - RRL$ order in terms of the reversed mean residual life $\leq_{RMRL}$ order.

Theorem 4.2. Let X and Y be two nonnegative random variables. Then

$X \leq_{LT-RRL} Y$ if and only if $h(X) \leq_{RMRL} h(Y)$ for each nonnegative function h with a *c.m.* derivative.

Proof. <u>*Necessity*</u> : Suppose that $X \leq_{LT-RRL} Y$ and that h is any non-negative function with a *c.m.* derivative. Then, by *Theorem* 3.1, we get $h(X) \leq_{LT-RRL} h(Y)$. Since the order $\leq_{LT-RRL}$ is stronger than the order $\leq_{RMRL}$ (see *Kayid* and *Ahmad,* 2004) it follows that $h(X) \leq_{RMRL} h(Y)$.

<u>*Sufficiency:*</u> Let h be a nonnegative function with a *c.m.* derivative, and fix $s > 0$. Since $\eta(x) = 1 - e^{-sx}$ is nonnegative with a *c.m.* derivative, then it is easy to see that $\eta(h(x))$ is also nonnegative with a *c.m.* derivative. So, by the assumption, we have that $\eta(h(X)) \leq_{RMRL} \eta(h(Y))$; that is,

$$E[\eta(h(X)) \mid \eta(h(X)) < t] \geq E[\eta(h(Y)) \mid \eta(h(Y)) < t] \quad \text{for all } t \in R^+,$$

or, equivalently,

$$E[\eta(h(X)) \mid h(X) < t] \geq E[\eta(h(Y)) \mid h(Y) < t] \quad \text{for all } t \in R^+. \tag{3.3}$$

By the definition of the order $\leq_{LT}$, it follows from (3.3) that

$$[h(X) \mid h(X) < t] \geq_{Lt} [h(Y) \mid h(Y) < t] \quad \text{for all } t \in R^+.$$

implying $h(X) \leq_{LT-RRL} h(Y)$. Therefore, the desired result follows from *Theorem* 4.1.

ACKNOLEDGMENT

REFERENCES

Alzaid,A., Kim, J.S. and Proschan, F. (1991). Laplace ordering and its applications. *J. Appl. Probab.,* **28**, 116-130.

Ahmed, A. and Kayid, M. (2004). Preservation properties for the Laplace transform ordering of residual lives. *Stat. Papers,* **45,** 583-590.

Ahmad, I. A. and Kayid, M. (2005). Characterizations of the RHR and MIT

orderings and the $DRHR$ and $IMIT$ classes of life distributions. *Prob. Eng. Inf. Sci.,* **19**, 447–461.

Ahmad, I. A., Kayid, M. and Pellerey, F. (2005). Further results involving the MIT order and the IMIT class. *Prob. Eng. Inf. Sci.,* **19**, 377-395.

Barlow, R.E. and Proschan, F. (1975). *Statistical Theory of Reliability and Life Testing: Probability Models*, Holt, Rinehart, and Winston, New York.

Belzunce, F., Ortega, E. and Ruiz, J. M. (1999). The Laplace order and ordering of residual lives. *Stat. Probab Lett,* **42**, 145-156.

Block, H.W., Savits, T. H. and Singh, H. (1998). The reversed hazard rate function. *Prob. Eng. Inf. Sci.,* **12**, 69-90.

Chandra, N. K. and Roy, D. (2001) Some results on reversed hazard rate. *Prob. Eng. Inf. Sci.,* **15**, 95-102.

Denuit, M. (2001). Laplace transform ordering of actuarial quantities. *Insurance: Math Economics* **29**, 83-102.

Gao, X., Belzune, F., Hu, T. and Pellerey, F. (2003). Developments of some preservation properties of the Laplace transform order of residual lives. *Technical report, Dept. of Statistics and Finance, University of Science and Technology of China.*

Joag-Dev, K., Kochar, S. and Proschan, F. (1995). A general composition theorem and its applications to certain partial orderings of distributions. *Stat. Probab. Lett.,* **22**, 111-119.

Karlin, S. (1968). *Total Positivity*, Stanford University Press, Stanford, CA.

Kebir, Y. (1994). Order-preserving shock models. *Prob. Eng. Inf. Sci.,* **8**, 125-134.

Kayid, M. and Ahmad, I. A. (2004). On the mean inactivity time ordering with reliability applications. *Probability in the Engineering and Informational Sciences,* **18**, 395-409.

Klefsjo, B. (1983). A useful ageing property based on the Laplace transform. *J. Appl. Probab.,* **20,** 615-626.

Nanda, A. K., Singh, H., Misra, N. and Paul, P.(2003). Reliability properties

of reversed residual lifetime. *Communications in Statistics- Theory & Methods,* **32,** 2031-2041.

Muller, A. and Stoyan, D. (2002). *Comparison Methods for Stochastic Models and Risks.* John Wiley & Sons, Ltd., West Sussex.

Shaked, M. and Shanthikumar, J.G. (1994). *Stochastic Orders and Their Applications.* Academic Press, New York.

Shaked, M. and Wong, T. (1995). Preservation of stochastic orderings under random mapping by point processes. *Prob. Eng. Inf. Sci.,* **9**, 563-580.